Computer Algebra Independent Integration Tests

Summer 2023 edition with Rubi V 4.17.3

1-Algebraic-functions/1.2-Trinomial-products/1.2.2-Quartic/42-1.2.2.5-P-x-a+b-x^2+c-x^4-^p

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December 9, 2023 Compiled on December 9, 2023 at 4:14am

Contents

CHAPTER 1

Introduction

This report gives the result of running the computer algebra independent integration test.The download section in on the main webpage contains links to download the problems in plain text format used for all CAS systems. The number of integrals in this report is [111]. This is test number [42].

1.1 Listing of CAS systems tested

The following are the CAS systems tested:

1. Mathematica 13.3.1 (August 16, 2023) on windows 10.
2. Rubi 4.17.3 (Sept 25, 2023) on Mathematica 13.3.1 on windows 10
3. Maple 2023.1 (July, 12, 2023) on windows 10.
4. Maxima 5.47 (June 1, 2023) using Lisp SBCL 2.3.0 on Linux via sagemath 10.1 (Aug 20, 2023).
5. FriCAS 1.3.9 (July 8, 2023) based on sbcl 2.3.0 on Linux via sagemath 10.1 (Aug 20, 2023).
6. Giac/Xcas 1.9.0-57 (June 26, 2023) on Linux via sagemath 10.1 (Aug 20, 2023).
7. Sympy 1.12 (May 10, 2023) Using Python 3.11.3 on Linux.
8. Mupad using Matlab 2021a with Symbolic Math Toolbox Version 8.7 on windows 10.

Maxima and Fricas and Giac are called using Sagemath. This was done using Sagemath `integrate` command by changing the name of the algorithm to use the different CAS systems.

Sympy was run directly in Python not via sagemath.

1.2 Results

Important note: A number of problems in this test suite have no antiderivative in closed form. This means the antiderivative of these integrals can not be expressed in terms of elementary, special functions or `Hypergeometric2F1` functions. `RootSum` and `RootOf` are not allowed. If a CAS returns the above integral unevaluated within the time limit, then the result is counted as passed and assigned an A grade.

However, if CAS times out, then it is assigned an F grade even if the integral is not integrable, as this implies CAS could not determine that the integral is not integrable in the time limit.

If a CAS returns an antiderivative to such an integral, it is assigned an A grade automatically and this special result is listed in the introduction section of each individual test report to make it easy to identify as this can be important result to investigate.

The results given in in the table below reflects the above.

System	% solved	% Failed
Rubi	100.00 (111)	0.00 (0)
Mathematica	100.00 (111)	0.00 (0)
Maple	100.00 (111)	0.00 (0)
Mupad	95.50 (106)	4.50 (5)
Giac	95.50 (106)	4.50 (5)
Fricas	83.78 (93)	16.22 (18)
Maxima	74.77 (83)	25.23 (28)
Sympy	40.54 (45)	59.46 (66)

Table 1.1: Percentage solved for each CAS

The table below gives additional break down of the grading of quality of the antiderivatives generated by each CAS. The grading is given using the letters A,B,C and F with A being the best quality. The grading is accomplished by comparing the antiderivative generated with the optimal antiderivatives included in the test suite. The following table describes the meaning of these grades.

grade	description
A	Integral was solved and antiderivative is optimal in quality and leaf size.
B	Integral was solved and antiderivative is optimal in quality but leaf size is larger than twice the optimal antiderivatives leaf size.
C	Integral was solved and antiderivative is non-optimal in quality. This can be due to one or more of the following reasons 1. antiderivative contains a hypergeometric function and the optimal antiderivative does not. 2. antiderivative contains a special function and the optimal antiderivative does not. 3. antiderivative contains the imaginary unit and the optimal antiderivative does not.
F	Integral was not solved. Either the integral was returned unevaluated within the time limit, or it timed out, or CAS hanged or crashed or an exception was raised.

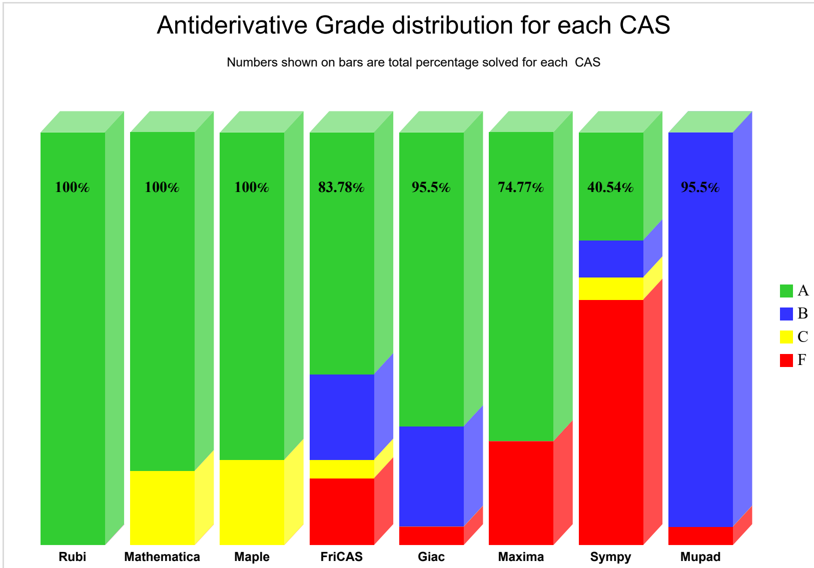
Table 1.2: Description of grading applied to integration result

Grading is implemented for all CAS systems. Based on the above, the following table summarizes the grading for this test suite.

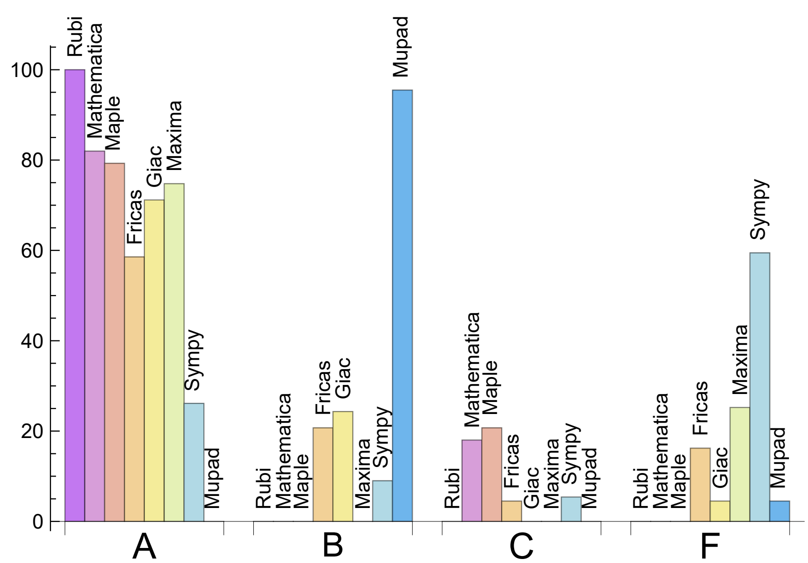
System	% A grade	% B grade	% C grade	% F grade
Rubi	100.000	0.000	0.000	0.000
Mathematica	81.982	0.000	18.018	0.000
Maple	79.279	0.000	20.721	0.000
Maxima	74.775	0.000	0.000	25.225
Giac	71.171	24.324	0.000	4.505
Fricas	58.559	20.721	4.505	16.216
Sympy	26.126	9.009	5.405	59.459
Mupad	0.000	95.495	0.000	4.505

Table 1.3: Antiderivative Grade distribution of each CAS

The following is a Bar chart illustration of the data in the above table.



The figure below compares the grades of the CAS systems.



The following table shows the distribution of the different types of failures for each CAS. There are 3 types failures. The first is when CAS returns the input within the time limit, which means it could not solve it. This is the typical failure and given as **F**.

The second failure is due to time out. CAS could not solve the integral within the 3 minutes time limit which is assigned. This is assigned **F(-1)**.

The third is due to an exception generated, indicated as **F(-2)**. This most likely indicates an interface problem between sagemath and the CAS (applicable only to FriCAS, Maxima and Giac) or it could be an indication of an internal error in the CAS itself. This type of error requires more investigation to determine the cause.

System	Number failed	Percentage normal failure	Percentage timeout failure	Percentage exception failure
Rubi	0	0.00	0.00	0.00
Mathematica	0	0.00	0.00	0.00
Maple	0	0.00	0.00	0.00
Mupad	5	0.00	100.00	0.00
Giac	5	100.00	0.00	0.00
Fricas	18	0.00	100.00	0.00
Maxima	28	100.00	0.00	0.00
Sympy	66	12.12	87.88	0.00

Table 1.4: Failure statistics for each CAS

1.3 Time and leaf size Performance

The table below summarizes the performance of each CAS system in terms of time used and leaf size of results.

Mean size is the average leaf size produced by the CAS (before any normalization). The Normalized mean is relative to the mean size of the optimal anti-derivative given in the input files.

For example, if CAS has **Normalized mean** of 3, then the mean size of its leaf size is 3 times as large as the mean size of the optimal leaf size.

Median size is value of leaf size where half the values are larger than this and half are smaller (before any normalization). i.e. The Middle value.

Similarly the **Normalized median** is relative to the median leaf size of the optimal.

For example, if a CAS has Normalized median of 1.2, then its median is 1.2 as large as the median leaf size of the optimal.

System	Mean time (sec)
Maxima	0.21
Rubi	0.59
Giac	0.76
Maple	0.93
Mathematica	1.35
Mupad	6.07
Fricas	7.07
Sympy	10.70

Table 1.5: Time performance for each CAS

System	Mean size	Normalized mean	Median size	Normalized median
Maxima	100.89	0.88	88.00	0.87
Maple	197.41	0.90	107.00	0.93
Rubi	220.25	1.02	147.00	1.00
Mathematica	260.32	1.08	146.00	1.02
Sympy	620.93	6.32	122.00	1.19
Giac	1853.60	3.90	124.00	0.98
Mupad	5695.50	9.58	132.00	1.00
Fricas	61061.85	246.12	141.00	1.27

Table 1.6: Leaf size performance for each CAS

1.4 Performance based on number of rules Rubi used

This section shows how each CAS performed based on the number of rules Rubi needed to solve the same integral. One diagram is given for each CAS.

On the y axis is the percentage solved which Rubi itself needed the number of rules given the x axis. These plots show that as more rules are needed then most CAS system percentage of solving decreases which indicates the integral is becoming more complicated to solve.

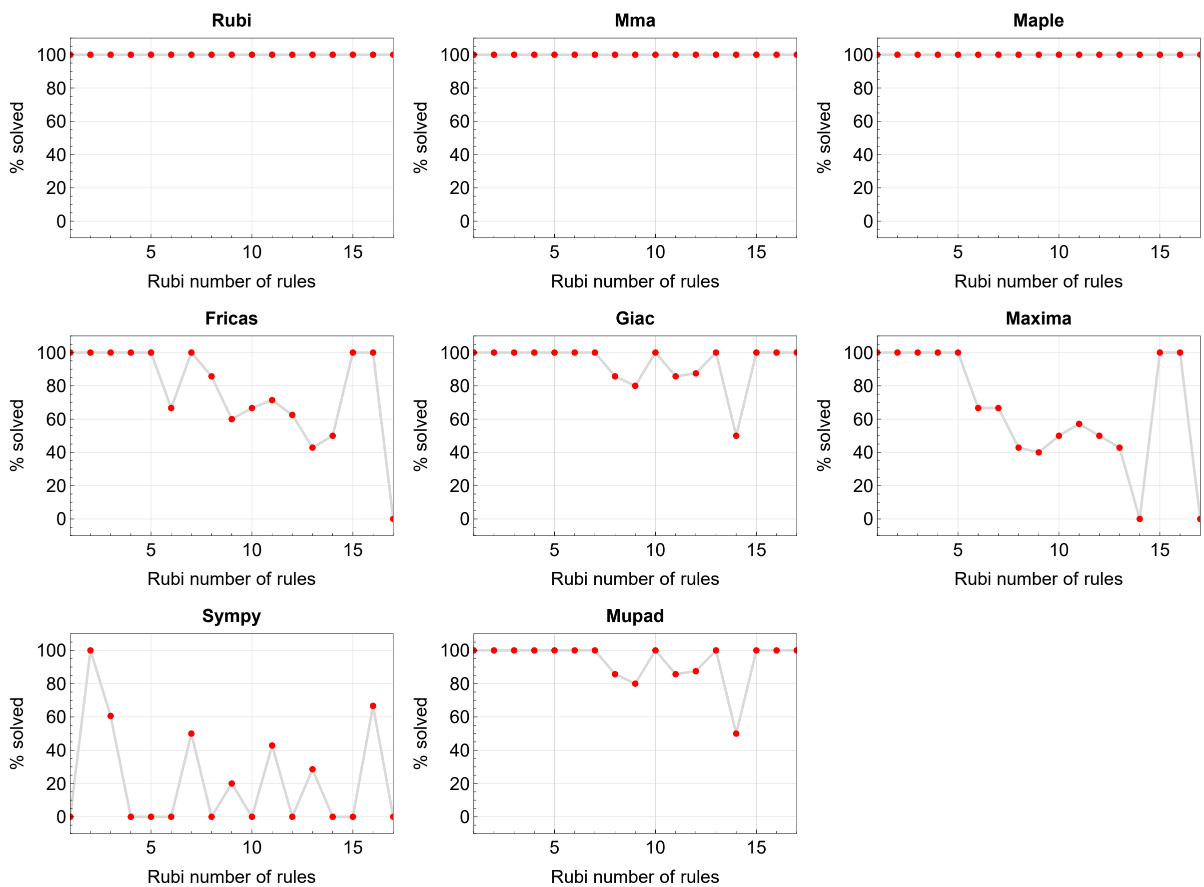


Figure 1.1: Solving statistics per number of Rubi rules used

1.5 Performance based on number of steps Rubi used

This section shows how each CAS performed based on the number of steps Rubi needed to solve the same integral. Note that the number of steps Rubi needed can be much higher than the number of rules, as the same rule could be used more than once.

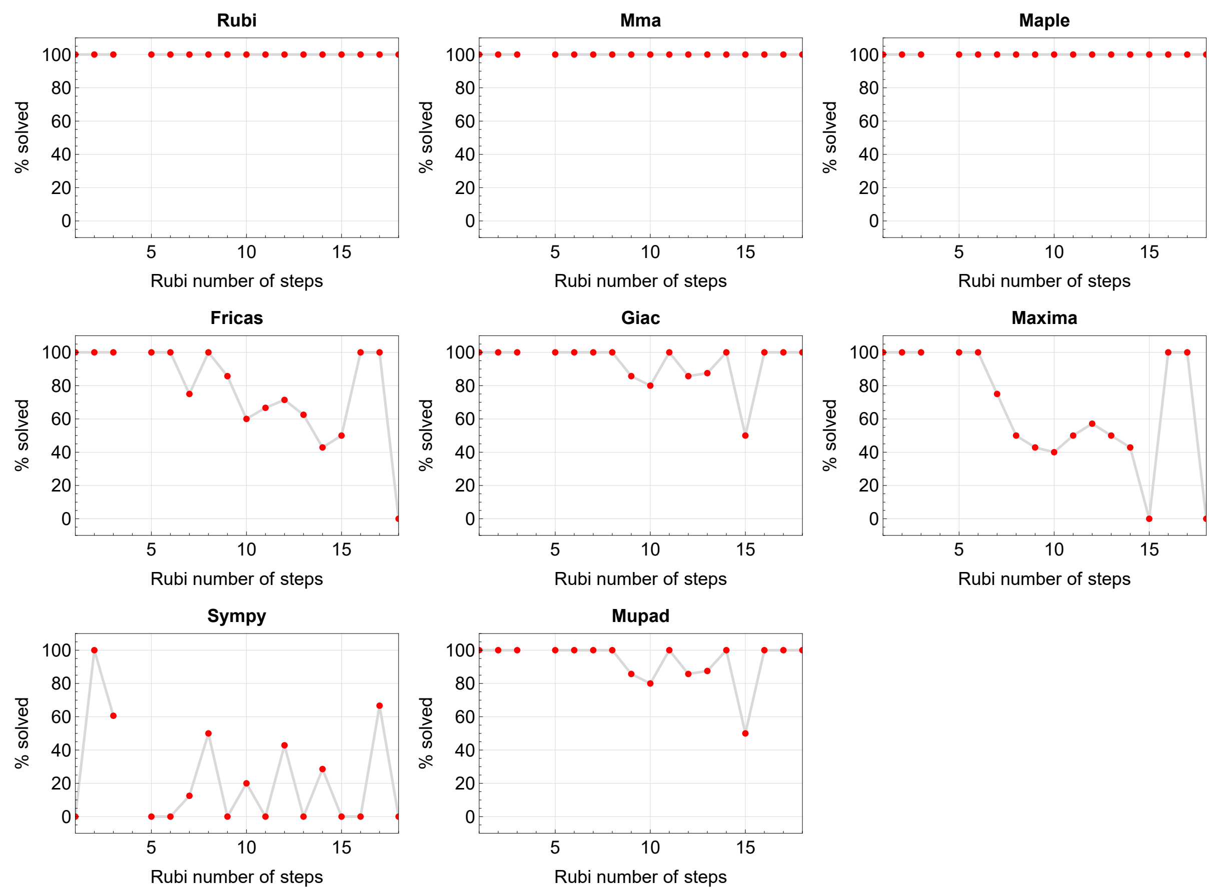


Figure 1.2: Solving statistics per number of Rubi steps used

The above diagram show that the precentage of solved intergals decreases for most CAS systems as the number of steps increases. As expected, for integrals that required less steps by Rubi, CAS systems had more success which indicates the integral was not as hard to solve. As Rubi needed more steps to solve the integral, the solved percentage decreased for most CAS systems which indicates the integral is becoming harder to solve.

1.6 Solved integrals histogram based on leaf size of result

The following shows the distribution of solved integrals for each CAS system based on leaf size of the antiderivatives produced by each CAS. It shows that most integrals solved produced leaf size less than about 100 to 150. The bin size used is 40.

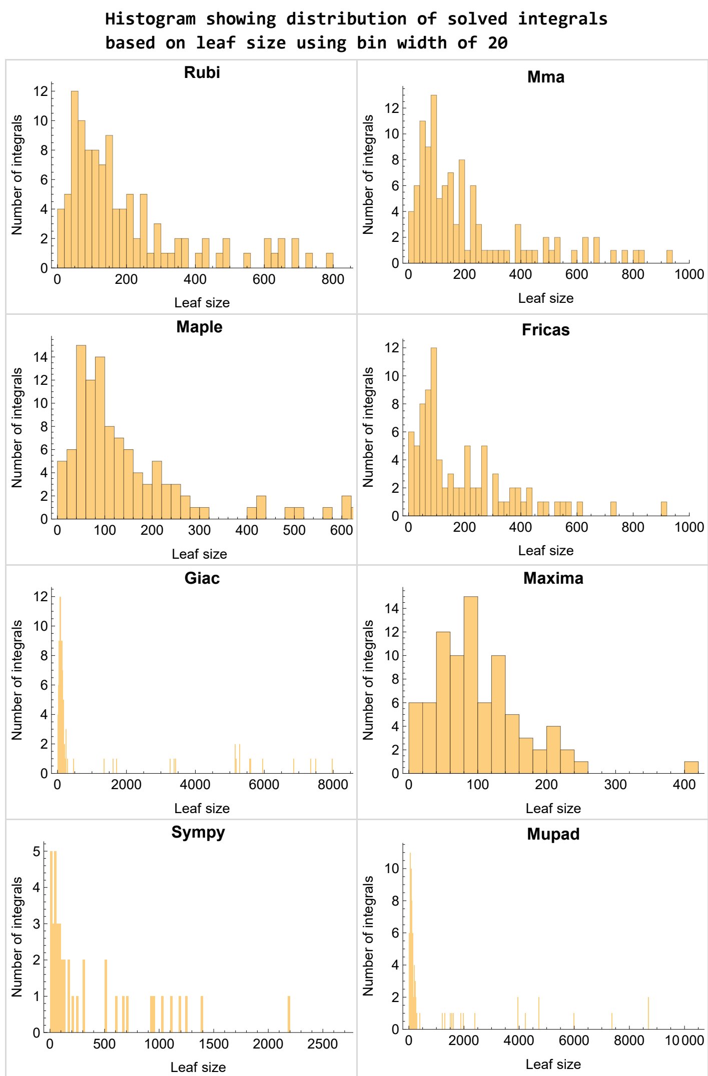


Figure 1.3: Solved integrals based on leaf size distribution

1.7 Solved integrals histogram based on CPU time used

The following shows the distribution of solved integrals for each CAS system based on CPU time used in seconds. The bin size used is 0.1 second.

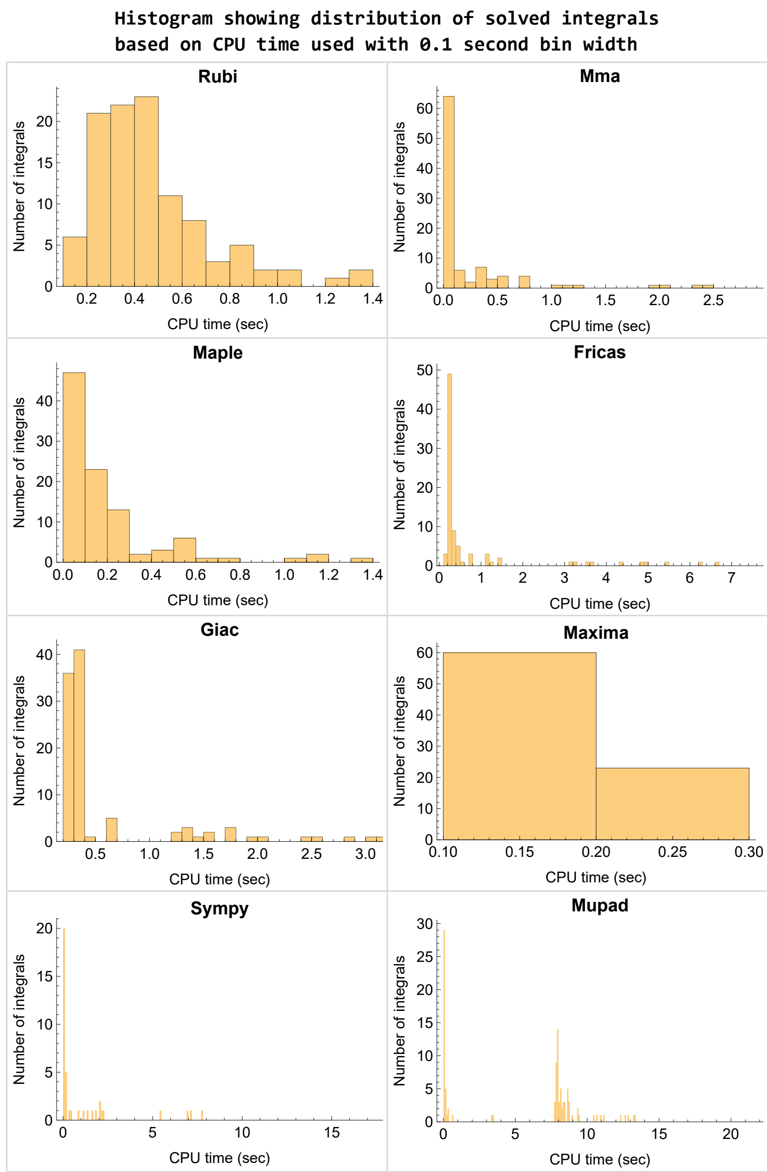


Figure 1.4: Solved integrals histogram based on CPU time used

1.8 Leaf size vs. CPU time used

The following gives the relation between the CPU time used to solve an integral and the leaf size of the antiderivative.

The result for Fricas, Maxima and Giac is shifted more to the right than the other CAS system due to the use of sagemath to call them, which causes an initial slight delay in the timing to start the integration due to overhead of starting a new process each time.

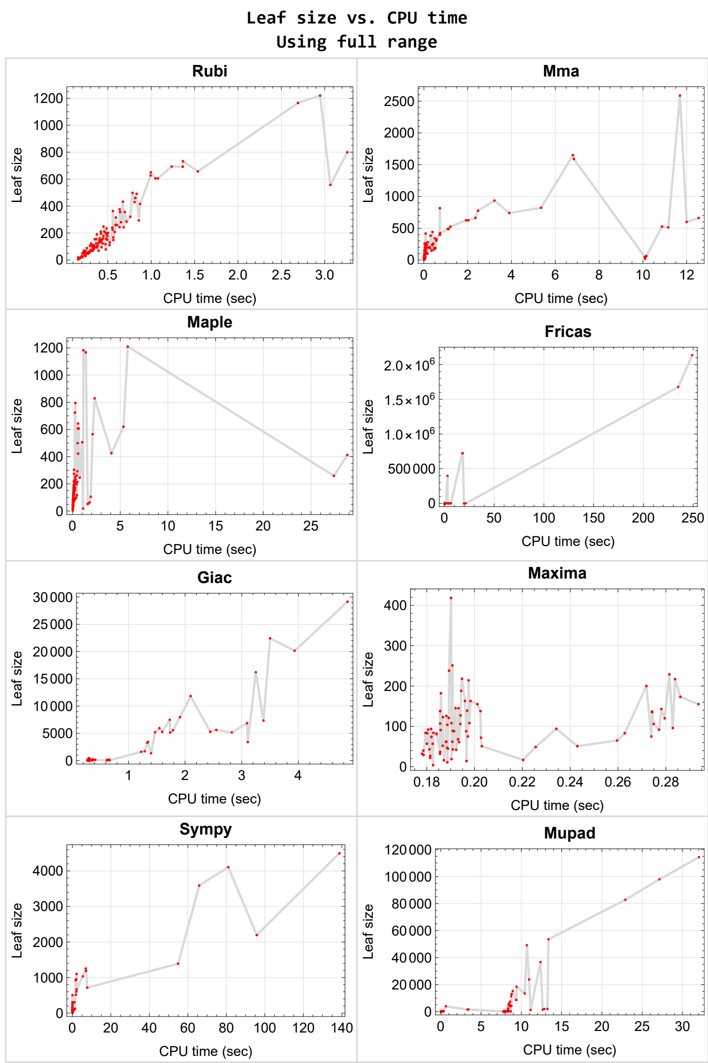


Figure 1.5: Leaf size vs. CPU time. Full range

1.9 list of integrals with no known antiderivative

{}

1.10 List of integrals solved by CAS but has no known antiderivative

Rubi {}

Mathematica {}

Maple {}

Maxima {}

Fricas {}

Sympy {}

Giac {}

Mupad {}

1.11 list of integrals solved by CAS but failed verification

The following are integrals solved by CAS but the verification phase failed to verify the anti-derivative produced is correct. This does not necessarily mean that the anti-derivative is wrong as additional methods of verification might be needed, or more time is needed (3 minutes time limit was used). These integrals are listed here to make it possible to do further investigation to determine why the result could not be verified.

Rubi {}

Mathematica {}

Maple {}

Maxima Verification phase not currently implemented.

Fricas Verification phase not currently implemented.

Sympy Verification phase not currently implemented.

Giac Verification phase not currently implemented.

Mupad Verification phase not currently implemented.

1.12 Timing

The command `AbsoluteTiming[]` was used in Mathematica to obtain the elapsed time for each integrate call. In Maple, the command `Usage` was used as in the following example

```
cpu_time := Usage(assign ('result_of_int',int(expr,x)),output='realtime'
```

For all other CAS systems, the elapsed time to complete each integral was found by taking the difference between the time after the call completed from the time before the call was made. This was done using Python's `time.time()` call.

All elapsed times shown are in seconds. A time limit of 3 CPU minutes was used for each integral. If the integrate command did not complete within this time limit, the integral was aborted and considered to have failed and assigned an F grade. The time used by failed integrals due to time out was not counted in the final statistics.

1.13 Verification

A verification phase was applied on the result of integration for `Rubi` and `Mathematica`.

Future version of this report will implement verification for the other CAS systems. For the integrals whose result was not run through a verification phase, it is assumed that the antiderivative was correct.

Verification phase also had 3 minutes time out. An integral whose result was not verified could still be correct, but further investigation is needed on those integrals. These integrals were marked in the summary table below and also in each integral separate section so they are easy to identify and locate.

1.14 Important notes about some of the results

1.14.1 Important note about Maxima results

Since tests were run in a batch mode, and using an automated script, then any integral where Maxima needed an interactive response from the user to answer a question during the evaluation of the integral will fail.

The exception raised is `ValueError`. Therefore Maxima results is lower than what would result if Maxima was run directly and each question was answered correctly.

The percentage of such failures were not counted for each test file, but for an example, for the `Timofeev` test file, there were about 14 such integrals out of total 705, or about 2 percent. This percentage can be higher or lower depending on the specific input test file.

Such integrals can be identified by looking at the output of the integration in each section for Maxima. The exception message will indicate the cause of error.

Maxima integrate was run using SageMath with the following settings set by default

```
'besselexpand : true'
'display2d : false'
'domain : complex'
'keepfloat : true'
'load(to_poly_solve)'
'load(simplify_sum)'
'load(abs_integrate)' 'load(diag)'
```

SageMath automatic loading of Maxima `abs_integrate` was found to cause some problems. So the following code was added to disable this effect.

```
from sage.interfaces.maxima_lib import maxima_lib
maxima_lib.set('extra_definite_integration_methods', '[]')
maxima_lib.set('extra_integration_methods', '[]')
```

See `https://ask.sagemath.org/question/43088/integrate-results-that-are-different-from-using-maxima/` for reference.

1.14.2 Important note about FriCAS result

There were few integrals which failed due to SageMath interface and not because FriCAS system could not do the integration.

These will fail With error `Exception raised: NotImplementedError`.

The number of such cases seems to be very small. About 1 or 2 percent of all integrals. These can be identified by looking at the exception message given in the result.

1.14.3 Important note about finding leaf size of antiderivative

For Mathematica, Rubi, and Maple, the builtin system function `LeafSize` was used to find the leaf size of each antiderivative.

The other CAS systems (SageMath and Sympy) do not have special builtin function for this purpose at this time. Therefore the leaf size for Fricas and Sympy antiderivative was determined using the following function, thanks to user `slelievre` at `https://ask.sagemath.org/question/57123/could-we-have-a-leaf_count-function-in-base-sagemath/`

```
def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)
```

For Sympy, which was called directly from Python, the following code was used to obtain the leafsize of its result

```
try:
  # 1.7 is a fudge factor since it is low side from actual leaf count
  leafCount = round(1.7*count_ops(anti))

  except Exception as ee:
       leafCount =1
```

1.14.4 Important note about Mupad results

Matlab's symbolic toolbox does not have a leaf count function to measure the size of the antiderivative. Maple was used to determine the leaf size of Mupad output by post processing Mupad result.

Currently no grading of the antiderivative for Mupad is implemented. If it can integrate the problem, it was assigned a B grade automatically as a placeholder. In the future, when grading function is implemented for Mupad, the tests will be rerun again.

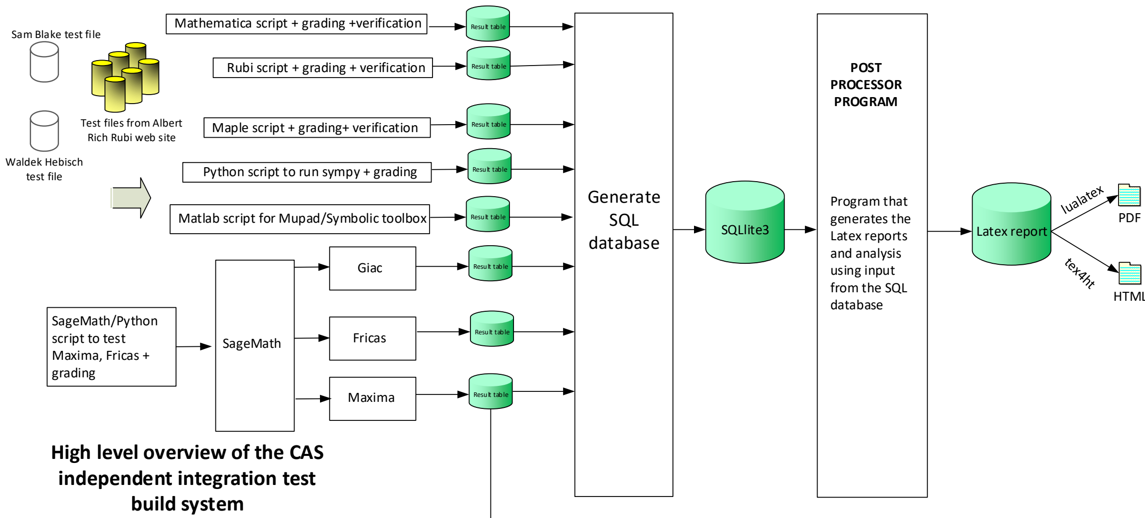
The following is an example of using Matlab's symbolic toolbox (Mupad) to solve an integral

```
integrand = evalin(symengine,'cos(x)*sin(x)')
the_variable = evalin(symengine,'x')
anti = int(integrand,the_variable)
```

Which gives `sin(x)^2/2`

1.15 Design of the test system

The following diagram gives a high level view of the current test build system.



One record (line) per one integral result. The line is CSV comma separated. This is description of each record

1. integer, the problem number.
2. integer. 0 for failed, 1 for passed, -1 for timeout, -2 for CAS specific exception. (this is not the grade field)
3. integer. Leaf size of result.
4. integer. Leaf size of the optimal antiderivative.
5. number. CPU time used to solve this integral. 0 if failed.
6. string. The integral in Latex format
7. string. The input used in CAS own syntax.
8. string. The result (antiderivative) produced by CAS in Latex format
9. string. The optimal antiderivative in Latex format.
10. integer. 0 or 1. Indicates if problem has known antiderivative or not
11. String. The result (antiderivative) in CAS own syntax.
12. String. The grade of the antiderivative. Can be "A", "B", "C", or "F"
13. String. Small string description of why the grade was given.
14. integer. 1 if result was verified or 0 if not verified. (For mma, rubi and maple only)

*The following fields are present only in **Rubi** Table file*

15. integer. Number of steps used.
16. integer. Number of rules used.
17. integer. Integrand leaf size.
18. real number. Ratio. Field 16 over field 17
19. String of form "{n,n,..}" which is list of the rules used by Rubi
20. String. The optimal antiderivative in Mathematica syntx

Nasser M. Abbasi
June 27, 2023
Design.vsdx

CHAPTER 2

DETAILED SUMMARY TABLES OF RESULTS

2.1 List of integrals sorted by grade for each CAS

2.1.1 Rubi

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111 }

B grade { }

C grade { }

F normal fail { }

F(-1) timedout fail { }

F(-2) exception fail { }

2.1.2 Mma

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 108, 109, 110, 111 }

B grade { }

C grade { 15, 16, 17, 18, 19, 31, 32, 33, 34, 35, 47, 48, 49, 50, 51, 103, 104, 105, 106, 107 }

F normal fail { }

F(-1) timedout fail { }

F(-2) exception fail { }

2.1.3 Maple

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 60, 61, 62, 63, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111 }

B grade { }

C grade { 20, 21, 22, 23, 24, 25, 36, 37, 38, 39, 40, 41, 52, 53, 54, 55, 56, 57, 58, 59, 64, 65, 66 }

F normal fail { }

F(-1) timedout fail { }

F(-2) exception fail { }

2.1.4 Fricas

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 31, 32, 33, 34, 35, 47, 48, 49, 60, 61, 62, 63, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 103, 104, 105, 106, 108, 109, 110, 111 }

B grade { 26, 27, 28, 29, 30, 42, 43, 44, 45, 46, 50, 51, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 107 }

C grade { 20, 21, 22, 36, 64 }

F normal fail { }

F(-1) timedout fail { 23, 24, 25, 37, 38, 39, 40, 41, 52, 53, 54, 55, 56, 57, 58, 59, 65, 66 }

F(-2) exception fail { }

2.1.5 Maxima

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 60, 61, 62, 63, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 108, 109, 110, 111 }

B grade { }

C grade { }

F normal fail { 20, 21, 22, 23, 24, 25, 36, 37, 38, 39, 40, 41, 52, 53, 54, 55, 56, 57, 58, 59, 64, 65, 66, 103, 104, 105, 106, 107 }

F(-1) timedout fail { }

F(-2) exception fail { }

2.1.6 Giac

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 60, 61, 62, 63, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102 }

B grade { 20, 21, 22, 23, 24, 25, 36, 37, 38, 39, 40, 41, 52, 53, 54, 55, 56, 57, 58, 59, 64, 65, 66, 108, 109, 110, 111 }

C grade { }

F normal fail { 103, 104, 105, 106, 107 }

F(-1) timedout fail { }

F(-2) exception fail { }

2.1.7 Mupad

A grade { }

B grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 108, 109, 110, 111 }

C grade { }

F normal fail { }

F(-1) timedout fail { 103, 104, 105, 106, 107 }

F(-2) exception fail { }

2.1.8 Sympy

A grade { 1, 2, 3, 4, 5, 6, 7, 8, 9, 60, 61, 62, 63, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 85, 91, 97 }

B grade { 10, 11, 26, 42, 80, 81, 82, 86, 92, 98 }

C grade { 15, 16, 31, 32, 47, 48 }

F normal fail { 103, 104, 105, 106, 108, 109, 110, 111 }

F(-1) timedout fail { 12, 13, 14, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 29, 30, 33, 34, 35, 36, 37, 38, 39, 40, 41, 43, 44, 45, 46, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 64, 65, 66, 83, 84, 87, 88, 89, 90, 93, 94, 95, 96, 99, 100, 101, 102, 107 }

F(-2) exception fail { }

2.2 Detailed conclusion table per each integral for all CAS systems

Detailed conclusion table per each integral is given by the table below. The elapsed time is in seconds. For failed result it is given as **F(-1)** if the failure was due to timeout. It is given as **F(-2)** if the failure was due to an exception being raised, which could indicate a bug in the system. If the failure was due to integral not being evaluated within the time limit, then it is given as **F**.

In this table, the column `N.S.` means **normalized size** and is defined as $\frac{\text{antiderivative leaf size}}{\text{optimal antiderivative leaf size}}$. To make the table fit the page, the name `Mathematica` was abbreviated to `MMA`.

Problem 1	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	50	50	50	41	40	40	46	40	40
N.S.	1	1.00	1.00	0.82	0.80	0.80	0.92	0.80	0.80
time (sec)	N/A	0.209	0.016	0.021	0.179	0.233	0.017	0.297	0.016

Problem 2	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	69	69	69	58	57	57	65	61	59
N.S.	1	1.00	1.00	0.84	0.83	0.83	0.94	0.88	0.86
time (sec)	N/A	0.225	0.022	0.067	0.180	0.239	0.018	0.361	0.019

Problem 3	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	88	88	88	75	74	74	83	82	78
N.S.	1	1.00	1.00	0.85	0.84	0.84	0.94	0.93	0.89
time (sec)	N/A	0.253	0.031	0.141	0.182	0.232	0.018	0.290	7.853

Problem 4	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	105	105	105	90	89	89	102	103	95
N.S.	1	1.00	1.00	0.86	0.85	0.85	0.97	0.98	0.90
time (sec)	N/A	0.295	0.031	0.234	0.191	0.237	0.019	0.366	7.905

Problem 5	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	122	122	122	105	104	104	121	124	112
N.S.	1	1.00	1.00	0.86	0.85	0.85	0.99	1.02	0.92
time (sec)	N/A	0.319	0.032	1.908	0.189	0.246	0.020	0.294	0.048

Problem 6	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	112	112	97	95	94	94	116	100	94
N.S.	1	1.00	0.87	0.85	0.84	0.84	1.04	0.89	0.84
time (sec)	N/A	0.327	0.039	0.099	0.182	0.243	0.024	0.369	0.047

Problem 7	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	154	154	154	139	138	138	165	151	138
N.S.	1	1.00	1.00	0.90	0.90	0.90	1.07	0.98	0.90
time (sec)	N/A	0.352	0.038	0.133	0.202	0.237	0.033	0.306	7.852

Problem 8	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	196	196	196	183	182	182	209	202	182
N.S.	1	1.00	1.00	0.93	0.93	0.93	1.07	1.03	0.93
time (sec)	N/A	0.421	0.044	0.151	0.186	0.250	0.031	0.378	7.879

Problem 9	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	234	234	234	219	218	218	258	253	220
N.S.	1	1.00	1.00	0.94	0.93	0.93	1.10	1.08	0.94
time (sec)	N/A	0.491	0.061	0.232	0.195	0.242	0.033	0.319	0.083

Problem 10	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	B	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	45	50	50	50	43	43	515	47	51
N.S.	1	1.11	1.11	1.11	0.96	0.96	11.44	1.04	1.13
time (sec)	N/A	0.239	0.017	0.050	0.181	0.262	1.809	0.287	7.813

Problem 11	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	B	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	51	55	58	62	51	51	2195	55	63
N.S.	1	1.08	1.14	1.22	1.00	1.00	43.04	1.08	1.24
time (sec)	N/A	0.265	0.022	0.058	0.203	0.289	95.810	0.364	7.763

Problem 12	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	57	62	68	74	61	61	0	65	75
N.S.	1	1.09	1.19	1.30	1.07	1.07	0.00	1.14	1.32
time (sec)	N/A	0.292	0.027	0.069	0.194	0.416	0.000	0.301	7.797

Problem 13	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	64	69	81	89	72	72	0	76	90
N.S.	1	1.08	1.27	1.39	1.12	1.12	0.00	1.19	1.41
time (sec)	N/A	0.395	0.035	0.084	0.203	1.139	0.000	0.311	7.904

Problem 14	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	76	78	98	107	88	88	0	92	108
N.S.	1	1.03	1.29	1.41	1.16	1.16	0.00	1.21	1.42
time (sec)	N/A	0.488	0.045	0.091	0.196	4.998	0.000	0.299	8.146

Problem 15	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	A	A	C	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	92	95	98	68	65	65	923	65	118
N.S.	1	1.03	1.07	0.74	0.71	0.71	10.03	0.71	1.28
time (sec)	N/A	0.330	0.169	0.122	0.260	0.267	1.662	0.304	0.128

Problem 16	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	A	A	C	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	104	108	121	82	75	75	3589	75	159
N.S.	1	1.04	1.16	0.79	0.72	0.72	34.51	0.72	1.53
time (sec)	N/A	0.344	0.110	0.226	0.274	0.289	65.868	0.297	7.978

Problem 17	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	A	A	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	127	133	150	90	83	83	0	83	199
N.S.	1	1.05	1.18	0.71	0.65	0.65	0.00	0.65	1.57
time (sec)	N/A	0.436	0.363	0.267	0.263	0.409	0.000	0.300	8.105

Problem 18	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	A	A	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	136	138	165	99	92	92	0	92	1209
N.S.	1	1.01	1.21	0.73	0.68	0.68	0.00	0.68	8.89
time (sec)	N/A	0.424	0.471	0.386	0.277	1.119	0.000	0.318	11.144

Problem 19	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	A	A	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	151	150	187	117	106	106	0	106	1509
N.S.	1	0.99	1.24	0.77	0.70	0.70	0.00	0.70	9.99
time (sec)	N/A	0.487	0.472	0.457	0.275	4.389	0.000	0.304	12.662

Problem 20	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	C	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	189	190	194	43	0	398481	0	1342	1308
N.S.	1	1.01	1.03	0.23	0.00	2108.37	0.00	7.10	6.92
time (sec)	N/A	0.386	0.172	0.084	0.000	3.158	0.000	1.399	8.363

Problem 21	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	C	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	211	211	234	48	0	723401	0	1714	3942
N.S.	1	1.00	1.11	0.23	0.00	3428.44	0.00	8.12	18.68
time (sec)	N/A	0.428	0.136	0.070	0.000	18.016	0.000	1.287	8.641

Problem 22	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	C	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	245	248	280	53	0	2136355	0	3270	15179
N.S.	1	1.01	1.14	0.22	0.00	8719.82	0.00	13.35	61.96
time (sec)	N/A	0.481	0.181	0.067	0.000	249.076	0.000	1.331	8.942

Problem 23	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	290	293	383	78	0	0	0	5199	5981
N.S.	1	1.01	1.32	0.27	0.00	0.00	0.00	17.93	20.62
time (sec)	N/A	0.897	0.308	0.102	0.000	0.000	0.000	1.470	8.465

Problem 24	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	321	321	441	99	0	0	0	5941	11383
N.S.	1	1.00	1.37	0.31	0.00	0.00	0.00	18.51	35.46
time (sec)	N/A	0.789	0.392	0.096	0.000	0.000	0.000	1.547	8.741

Problem 25	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	545	545	816	246	0	0	0	11830	49150
N.S.	1	1.00	1.50	0.45	0.00	0.00	0.00	21.71	90.18
time (sec)	N/A	2.995	0.729	0.762	0.000	0.000	0.000	2.097	10.688

Problem 26	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	B	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	94	105	90	83	83	169	604	87	84
N.S.	1	1.12	0.96	0.88	0.88	1.80	6.43	0.93	0.89
time (sec)	N/A	0.317	0.038	0.081	0.180	0.278	2.041	0.298	0.053

Problem 27	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	115	126	112	105	106	217	0	109	107
N.S.	1	1.10	0.97	0.91	0.92	1.89	0.00	0.95	0.93
time (sec)	N/A	0.374	0.054	0.088	0.194	0.315	0.000	0.315	0.060

Problem 28	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	138	147	134	125	127	262	0	130	128
N.S.	1	1.07	0.97	0.91	0.92	1.90	0.00	0.94	0.93
time (sec)	N/A	0.424	0.039	0.102	0.194	0.486	0.000	0.295	7.876

Problem 29	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	150	159	159	143	145	304	0	152	146
N.S.	1	1.06	1.06	0.95	0.97	2.03	0.00	1.01	0.97
time (sec)	N/A	0.488	0.049	0.103	0.193	1.405	0.000	0.302	7.980

Problem 30	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	162	167	185	161	163	346	0	173	164
N.S.	1	1.03	1.14	0.99	1.01	2.14	0.00	1.07	1.01
time (sec)	N/A	0.559	0.059	0.123	0.198	6.239	0.000	0.306	0.347

Problem 31	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	A	A	C	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	140	149	146	124	96	154	952	96	149
N.S.	1	1.06	1.04	0.89	0.69	1.10	6.80	0.69	1.06
time (sec)	N/A	0.389	0.351	0.153	0.283	0.277	2.014	0.303	0.145

Problem 32	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	A	A	C	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	165	178	186	154	120	212	4106	124	201
N.S.	1	1.08	1.13	0.93	0.73	1.28	24.88	0.75	1.22
time (sec)	N/A	0.447	0.282	0.268	0.280	0.301	81.025	0.324	0.180

Problem 33	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	A	A	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	179	193	200	172	135	239	0	138	237
N.S.	1	1.08	1.12	0.96	0.75	1.34	0.00	0.77	1.32
time (sec)	N/A	0.489	0.267	0.276	0.274	0.464	0.000	0.331	8.138

Problem 34	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	A	A	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	187	201	234	196	143	255	0	151	1547
N.S.	1	1.07	1.25	1.05	0.76	1.36	0.00	0.81	8.27
time (sec)	N/A	0.510	0.387	0.412	0.278	1.101	0.000	0.415	3.309

Problem 35	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	A	A	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	194	208	243	214	155	279	0	165	1894
N.S.	1	1.07	1.25	1.10	0.80	1.44	0.00	0.85	9.76
time (sec)	N/A	0.577	0.388	0.506	0.294	4.866	0.000	0.298	13.239

Problem 36	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	C	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	330	316	341	209	0	1678440	0	3426	2382
N.S.	1	0.96	1.03	0.63	0.00	5086.18	0.00	10.38	7.22
time (sec)	N/A	0.617	0.503	0.247	0.000	235.031	0.000	1.345	8.365

Problem 37	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	368	356	398	232	0	0	0	5156	4707
N.S.	1	0.97	1.08	0.63	0.00	0.00	0.00	14.01	12.79
time (sec)	N/A	0.668	0.720	0.260	0.000	0.000	0.000	1.737	8.403

Problem 38	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	386	376	421	252	0	0	0	5573	7373
N.S.	1	0.97	1.09	0.65	0.00	0.00	0.00	14.44	19.10
time (sec)	N/A	0.662	0.741	0.279	0.000	0.000	0.000	1.787	8.609

Problem 39	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	439	429	489	274	0	0	0	7495	13024
N.S.	1	0.98	1.11	0.62	0.00	0.00	0.00	17.07	29.67
time (sec)	N/A	0.824	1.093	0.154	0.000	0.000	0.000	1.730	8.795

Problem 40	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	468	460	524	304	0	0	0	7962	18449
N.S.	1	0.98	1.12	0.65	0.00	0.00	0.00	17.01	39.42
time (sec)	N/A	0.825	1.211	0.154	0.000	0.000	0.000	1.908	9.411

Problem 41	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	770	788	935	506	0	0	0	20159	82785
N.S.	1	1.02	1.21	0.66	0.00	0.00	0.00	26.18	107.51
time (sec)	N/A	4.258	3.220	1.013	0.000	0.000	0.000	3.934	22.902

Problem 42	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	B	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	143	160	128	107	121	307	668	115	118
N.S.	1	1.12	0.90	0.75	0.85	2.15	4.67	0.80	0.83
time (sec)	N/A	0.398	0.068	0.090	0.189	0.291	2.230	0.313	0.053

Problem 43	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	175	193	161	139	155	389	0	149	151
N.S.	1	1.10	0.92	0.79	0.89	2.22	0.00	0.85	0.86
time (sec)	N/A	0.495	0.088	0.117	0.201	0.316	0.000	0.320	0.063

Problem 44	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	204	224	193	169	188	470	0	182	182
N.S.	1	1.10	0.95	0.83	0.92	2.30	0.00	0.89	0.89
time (sec)	N/A	0.571	0.062	0.109	0.194	0.505	0.000	0.310	7.862

Problem 45	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	224	244	231	193	214	544	0	216	209
N.S.	1	1.09	1.03	0.86	0.96	2.43	0.00	0.96	0.93
time (sec)	N/A	0.630	0.087	0.124	0.197	1.450	0.000	0.364	0.151

Problem 46	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	239	242	261	217	238	616	0	249	233
N.S.	1	1.01	1.09	0.91	1.00	2.58	0.00	1.04	0.97
time (sec)	N/A	0.679	0.085	0.131	0.189	6.601	0.000	0.282	0.372

Problem 47	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	A	A	C	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	185	201	186	158	137	278	1103	125	185
N.S.	1	1.09	1.01	0.85	0.74	1.50	5.96	0.68	1.00
time (sec)	N/A	0.479	0.530	0.183	0.274	0.294	2.187	0.287	0.153

Problem 48	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	A	A	C	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	223	240	235	202	173	384	4496	165	249
N.S.	1	1.08	1.05	0.91	0.78	1.72	20.16	0.74	1.12
time (sec)	N/A	0.556	0.390	0.277	0.286	0.309	138.761	0.325	7.974

Problem 49	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	A	A	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	243	260	259	232	200	435	0	192	295
N.S.	1	1.07	1.07	0.95	0.82	1.79	0.00	0.79	1.21
time (sec)	N/A	0.614	0.435	0.304	0.272	0.487	0.000	0.307	8.066

Problem 50	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	263	280	303	262	217	485	0	222	1611
N.S.	1	1.06	1.15	1.00	0.83	1.84	0.00	0.84	6.13
time (sec)	N/A	0.659	0.568	0.432	0.284	1.221	0.000	0.307	3.443

Problem 51	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	269	285	325	292	229	521	0	249	1963
N.S.	1	1.06	1.21	1.09	0.85	1.94	0.00	0.93	7.30
time (sec)	N/A	0.717	0.580	0.505	0.281	5.488	0.000	0.295	12.838

Problem 52	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	474	491	488	499	0	0	0	3389	4225
N.S.	1	1.04	1.03	1.05	0.00	0.00	0.00	7.15	8.91
time (sec)	N/A	0.854	1.137	0.528	0.000	0.000	0.000	3.107	8.771

Problem 53	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	621	605	625	607	0	0	0	5284	8689
N.S.	1	0.97	1.01	0.98	0.00	0.00	0.00	8.51	13.99
time (sec)	N/A	1.071	1.929	0.648	0.000	0.000	0.000	2.444	9.384

Problem 54	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	646	627	661	644	0	0	0	5619	13431
N.S.	1	0.97	1.02	1.00	0.00	0.00	0.00	8.70	20.79
time (sec)	N/A	1.020	2.357	0.579	0.000	0.000	0.000	2.552	10.415

Problem 55	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	679	692	739	724	0	0	0	6854	23811
N.S.	1	1.02	1.09	1.07	0.00	0.00	0.00	10.09	35.07
time (sec)	N/A	1.403	3.899	0.240	0.000	0.000	0.000	3.094	10.961

Problem 56	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	728	732	821	795	0	0	0	7340	36653
N.S.	1	1.01	1.13	1.09	0.00	0.00	0.00	10.08	50.35
time (sec)	N/A	1.373	5.350	0.296	0.000	0.000	0.000	3.383	12.374

Problem 57	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	1150	1172	1590	1167	0	0	0	22429	114377
N.S.	1	1.02	1.38	1.01	0.00	0.00	0.00	19.50	99.46
time (sec)	N/A	2.874	6.849	1.392	0.000	0.000	0.000	3.501	32.044

Problem 58	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	645	653	775	422	0	0	0	16214	53538
N.S.	1	1.01	1.20	0.65	0.00	0.00	0.00	25.14	83.00
time (sec)	N/A	1.516	2.474	0.583	0.000	0.000	0.000	3.249	13.368

Problem 59	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	1177	1236	1649	1182	0	0	0	29142	97905
N.S.	1	1.05	1.40	1.00	0.00	0.00	0.00	24.76	83.18
time (sec)	N/A	3.032	6.801	1.140	0.000	0.000	0.000	4.868	27.137

Problem 60	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	416	416	416	412	418	418	503	463	398
N.S.	1	1.00	1.00	0.99	1.00	1.00	1.21	1.11	0.96
time (sec)	N/A	0.912	0.078	28.809	0.190	0.258	0.053	0.304	0.298

Problem 61	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	259	259	259	259	251	251	309	285	246
N.S.	1	1.00	1.00	1.00	0.97	0.97	1.19	1.10	0.95
time (sec)	N/A	0.604	0.035	27.405	0.191	0.270	0.041	0.298	8.113

Problem 62	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	154	154	154	141	138	138	165	151	138
N.S.	1	1.00	1.00	0.92	0.90	0.90	1.07	0.98	0.90
time (sec)	N/A	0.364	0.027	0.134	0.186	0.276	0.027	0.387	0.069

Problem 63	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	20	20	20	17	16	16	15	16	16
N.S.	1	1.00	1.00	0.85	0.80	0.80	0.75	0.80	0.80
time (sec)	N/A	0.198	0.004	0.025	0.187	0.258	0.028	0.619	0.016

Problem 64	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	C	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	211	211	234	48	0	723401	0	1616	3942
N.S.	1	1.00	1.11	0.23	0.00	3428.44	0.00	7.66	18.68
time (sec)	N/A	0.480	0.143	0.063	0.000	18.527	0.000	1.225	0.637

Problem 65	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	368	356	398	232	0	0	0	5159	4707
N.S.	1	0.97	1.08	0.63	0.00	0.00	0.00	14.02	12.79
time (sec)	N/A	0.711	0.738	0.243	0.000	0.000	0.000	2.825	8.620

Problem 66	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	C	F	F(-1)	F(-1)	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	621	605	625	607	0	0	0	5280	8689
N.S.	1	0.97	1.01	0.98	0.00	0.00	0.00	8.50	13.99
time (sec)	N/A	1.108	2.034	0.558	0.000	0.000	0.000	1.596	9.343

Problem 67	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	4	4	4	5	4	4	3	5	4
N.S.	1	1.00	1.00	1.25	1.00	1.00	0.75	1.25	1.00
time (sec)	N/A	0.150	0.002	0.025	0.183	0.269	0.024	0.279	0.012

Problem 68	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	14	14	16	15	14	14	12	15	14
N.S.	1	1.00	1.14	1.07	1.00	1.00	0.86	1.07	1.00
time (sec)	N/A	0.184	0.017	0.033	0.197	0.262	0.057	0.315	7.856

Problem 69	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	31	31	30	28	27	27	26	28	27
N.S.	1	1.00	0.97	0.90	0.87	0.87	0.84	0.90	0.87
time (sec)	N/A	0.221	0.023	0.034	0.181	0.262	0.077	0.306	0.022

Problem 70	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	51	51	45	45	43	43	41	47	44
N.S.	1	1.00	0.88	0.88	0.84	0.84	0.80	0.92	0.86
time (sec)	N/A	0.281	0.022	0.039	0.191	0.261	0.082	0.299	0.022

Problem 71	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	68	68	68	65	62	62	63	72	64
N.S.	1	1.00	1.00	0.96	0.91	0.91	0.93	1.06	0.94
time (sec)	N/A	0.318	0.023	0.042	0.188	0.244	0.099	0.298	0.020

Problem 72	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	92	92	92	88	84	84	88	103	87
N.S.	1	1.00	1.00	0.96	0.91	0.91	0.96	1.12	0.95
time (sec)	N/A	0.347	0.042	0.045	0.183	0.251	0.126	0.295	0.023

Problem 73	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	11	11	11	12	11	11	8	13	8
N.S.	1	1.00	1.00	1.09	1.00	1.00	0.73	1.18	0.73
time (sec)	N/A	0.176	0.045	0.032	0.189	0.254	0.044	0.309	0.051

Problem 74	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	22	22	23	24	22	22	29	24	22
N.S.	1	1.00	1.05	1.09	1.00	1.00	1.32	1.09	1.00
time (sec)	N/A	0.199	0.016	0.038	0.181	0.247	0.165	0.290	7.893

Problem 75	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	29	29	30	31	29	29	44	31	29
N.S.	1	1.00	1.03	1.07	1.00	1.00	1.52	1.07	1.00
time (sec)	N/A	0.240	0.042	0.044	0.179	0.239	0.301	0.304	0.043

Problem 76	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	47	47	44	47	45	45	66	47	45
N.S.	1	1.00	0.94	1.00	0.96	0.96	1.40	1.00	0.96
time (sec)	N/A	0.271	0.070	0.052	0.189	0.253	0.480	0.302	0.039

Problem 77	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	66	66	67	65	62	62	94	67	63
N.S.	1	1.00	1.02	0.98	0.94	0.94	1.42	1.02	0.95
time (sec)	N/A	0.298	0.039	0.056	0.191	0.256	0.850	0.308	0.043

Problem 78	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	90	90	91	88	84	84	122	95	86
N.S.	1	1.00	1.01	0.98	0.93	0.93	1.36	1.06	0.96
time (sec)	N/A	0.330	0.029	0.061	0.179	0.249	1.394	0.299	0.051

Problem 79	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	29	29	29	20	19	19	19	22	19
N.S.	1	1.00	1.00	0.69	0.66	0.66	0.66	0.76	0.66
time (sec)	N/A	0.196	0.007	0.038	0.190	0.251	0.063	0.285	0.041

Problem 80	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	B	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	42	42	39	38	32	32	304	35	38
N.S.	1	1.00	0.93	0.90	0.76	0.76	7.24	0.83	0.90
time (sec)	N/A	0.228	0.014	0.046	0.178	0.261	1.109	0.274	7.785

Problem 81	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	B	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	47	47	44	47	37	37	716	40	47
N.S.	1	1.00	0.94	1.00	0.79	0.79	15.23	0.85	1.00
time (sec)	N/A	0.255	0.017	0.056	0.186	0.250	7.753	0.306	0.066

Problem 82	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	B	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	57	57	55	59	47	47	1389	50	59
N.S.	1	1.00	0.96	1.04	0.82	0.82	24.37	0.88	1.04
time (sec)	N/A	0.271	0.031	0.059	0.189	0.268	54.944	0.301	7.872

Problem 83	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	74	74	71	78	62	62	0	65	78
N.S.	1	1.00	0.96	1.05	0.84	0.84	0.00	0.88	1.05
time (sec)	N/A	0.304	0.028	0.073	0.188	0.284	0.000	0.286	8.034

Problem 84	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	96	96	91	99	82	82	0	87	99
N.S.	1	1.00	0.95	1.03	0.85	0.85	0.00	0.91	1.03
time (sec)	N/A	0.348	0.034	0.076	0.184	0.315	0.000	0.302	7.928

Problem 85	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	46	46	42	33	32	45	34	36	32
N.S.	1	1.00	0.91	0.72	0.70	0.98	0.74	0.78	0.70
time (sec)	N/A	0.232	0.017	0.060	0.186	0.266	0.138	0.293	0.030

Problem 86	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	B	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	71	71	66	64	57	93	1188	61	64
N.S.	1	1.00	0.93	0.90	0.80	1.31	16.73	0.86	0.90
time (sec)	N/A	0.308	0.034	0.069	0.183	0.279	7.105	0.299	7.948

Problem 87	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	82	82	77	79	68	116	0	72	79
N.S.	1	1.00	0.94	0.96	0.83	1.41	0.00	0.88	0.96
time (sec)	N/A	0.341	0.043	0.076	0.193	0.340	0.000	0.301	7.902

Problem 88	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	95	95	90	94	81	141	0	85	94
N.S.	1	1.00	0.95	0.99	0.85	1.48	0.00	0.89	0.99
time (sec)	N/A	0.368	0.041	0.083	0.184	0.722	0.000	0.286	7.943

Problem 89	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	106	106	102	109	92	164	0	96	108
N.S.	1	1.00	0.96	1.03	0.87	1.55	0.00	0.91	1.02
time (sec)	N/A	0.394	0.045	0.098	0.181	3.260	0.000	0.291	8.113

Problem 90	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	122	122	118	127	108	200	0	112	127
N.S.	1	1.00	0.97	1.04	0.89	1.64	0.00	0.92	1.04
time (sec)	N/A	0.445	0.050	0.116	0.190	19.688	0.000	0.297	8.369

Problem 91	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	56	55	48	40	42	72	46	46	42
N.S.	1	0.98	0.86	0.71	0.75	1.29	0.82	0.82	0.75
time (sec)	N/A	0.222	0.019	0.064	0.192	0.254	0.149	0.313	0.027

Problem 92	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	B	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	89	94	80	78	75	153	1255	79	79
N.S.	1	1.06	0.90	0.88	0.84	1.72	14.10	0.89	0.89
time (sec)	N/A	0.438	0.035	0.073	0.197	0.270	6.979	0.290	0.057

Problem 93	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	105	110	97	96	91	191	0	95	97
N.S.	1	1.05	0.92	0.91	0.87	1.82	0.00	0.90	0.92
time (sec)	N/A	0.526	0.055	0.081	0.186	0.340	0.000	0.304	0.077

Problem 94	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	117	117	114	114	107	229	0	111	115
N.S.	1	1.00	0.97	0.97	0.91	1.96	0.00	0.95	0.98
time (sec)	N/A	0.485	0.043	0.100	0.186	0.725	0.000	0.300	7.978

Problem 95	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	131	131	136	132	123	267	0	127	133
N.S.	1	1.00	1.04	1.01	0.94	2.04	0.00	0.97	1.02
time (sec)	N/A	0.518	0.060	0.124	0.187	3.518	0.000	0.290	8.236

Problem 96	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	147	147	153	150	139	305	0	143	151
N.S.	1	1.00	1.04	1.02	0.95	2.07	0.00	0.97	1.03
time (sec)	N/A	0.554	0.063	0.125	0.197	21.463	0.000	0.297	8.666

Problem 97	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	A	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	68	68	60	47	52	103	53	56	52
N.S.	1	1.00	0.88	0.69	0.76	1.51	0.78	0.82	0.76
time (sec)	N/A	0.251	0.021	0.067	0.187	0.251	0.157	0.295	0.026

Problem 98	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	B	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	105	105	97	92	88	211	1034	92	90
N.S.	1	1.00	0.92	0.88	0.84	2.01	9.85	0.88	0.86
time (sec)	N/A	0.339	0.060	0.077	0.191	0.279	5.457	0.300	0.058

Problem 99	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	122	122	121	113	108	267	0	112	113
N.S.	1	1.00	0.99	0.93	0.89	2.19	0.00	0.92	0.93
time (sec)	N/A	0.370	0.042	0.080	0.198	0.361	0.000	0.285	7.901

Problem 100	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	141	141	144	134	126	321	0	130	131
N.S.	1	1.00	1.02	0.95	0.89	2.28	0.00	0.92	0.93
time (sec)	N/A	0.402	0.052	0.088	0.188	0.777	0.000	0.288	8.211

Problem 101	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	158	158	169	155	145	376	0	149	152
N.S.	1	1.00	1.07	0.98	0.92	2.38	0.00	0.94	0.96
time (sec)	N/A	0.447	0.062	0.111	0.192	3.629	0.000	0.293	8.431

Problem 102	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	B	F(-1)	A	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	177	177	195	176	163	430	0	167	170
N.S.	1	1.00	1.10	0.99	0.92	2.43	0.00	0.94	0.96
time (sec)	N/A	0.496	0.071	0.129	0.196	20.839	0.000	0.299	8.687

Problem 103	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	F	A	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	717	693	2588	1209	0	911	0	0	0
N.S.	1	0.97	3.61	1.69	0.00	1.27	0.00	0.00	0.00
time (sec)	N/A	1.249	11.689	5.791	0.000	0.313	0.000	0.000	0.000

Problem 104	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	F	A	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	505	498	661	620	0	574	0	0	0
N.S.	1	0.99	1.31	1.23	0.00	1.14	0.00	0.00	0.00
time (sec)	N/A	0.790	12.543	5.344	0.000	0.255	0.000	0.000	0.000

Problem 105	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	F	A	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	359	363	525	426	0	376	0	0	0
N.S.	1	1.01	1.46	1.19	0.00	1.05	0.00	0.00	0.00
time (sec)	N/A	0.546	10.869	4.082	0.000	0.189	0.000	0.000	0.000

Problem 106	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	F	A	F	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	447	433	513	565	0	723	0	0	0
N.S.	1	0.97	1.15	1.26	0.00	1.62	0.00	0.00	0.00
time (sec)	N/A	0.680	11.157	2.107	0.000	0.115	0.000	0.000	0.000

Problem 107	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	C	A	F	B	F(-1)	F	F(-1)
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	680	650	598	829	0	1948	0	0	0
N.S.	1	0.96	0.88	1.22	0.00	2.86	0.00	0.00	0.00
time (sec)	N/A	1.043	11.995	2.319	0.000	0.145	0.000	0.000	0.000

Problem 108	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	19	19	19	18	17	17	0	60	17
N.S.	1	1.00	1.00	0.95	0.89	0.89	0.00	3.16	0.89
time (sec)	N/A	0.158	10.093	1.102	0.220	0.270	0.000	0.669	8.012

Problem 109	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	57	57	51	52	51	82	0	138	51
N.S.	1	1.00	0.89	0.91	0.89	1.44	0.00	2.42	0.89
time (sec)	N/A	0.273	10.135	1.599	0.243	0.275	0.000	0.662	7.918

Problem 110	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	57	57	48	53	49	80	0	136	51
N.S.	1	1.00	0.84	0.93	0.86	1.40	0.00	2.39	0.89
time (sec)	N/A	0.283	10.083	1.596	0.226	0.276	0.000	0.622	7.940

Problem 111	Optimal	Rubi	MMA	Maple	Maxima	Fricas	Sympy	Giac	Mupad
grade	N/A	A	A	A	A	A	F	B	B
verified	N/A	Yes	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	69	69	61	63	94	92	0	164	62
N.S.	1	1.00	0.88	0.91	1.36	1.33	0.00	2.38	0.90
time (sec)	N/A	0.302	10.155	1.790	0.234	0.286	0.000	0.656	7.995

2.3 Detailed conclusion table specific for Rubi results

The following table is specific to Rubi only. It gives additional statistics for each integral. the column **steps** is the number of steps used by Rubi to obtain the antiderivative. The **rules** column is the number of unique rules used. The **integrand size** column is the leaf size of the integrand. Finally the ratio $\frac{\text{number of rules}}{\text{integrand size}}$ is also given. The larger this ratio is, the harder the integral is to solve. In this test file, problem number [47] had the largest ratio of [1]

Table 2.1: Rubi specific breakdown of results for each integral

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
1	A	2	2	1.00	18	0.111
2	A	2	2	1.00	23	0.087
3	A	2	2	1.00	28	0.071
4	A	2	2	1.00	33	0.061
5	A	2	2	1.00	38	0.053
6	A	2	2	1.00	20	0.100
7	A	2	2	1.00	25	0.080
8	A	2	2	1.00	30	0.067
9	A	2	2	1.00	35	0.057
10	A	8	7	1.11	18	0.389
11	A	8	7	1.08	23	0.304
12	A	7	6	1.09	28	0.214
13	A	7	6	1.08	33	0.182
14	A	7	6	1.03	38	0.158
15	A	12	11	1.03	16	0.688
16	A	12	11	1.04	21	0.524
17	A	13	12	1.05	26	0.462
18	A	9	8	1.01	31	0.258
19	A	7	6	0.99	36	0.167
20	A	8	7	1.01	20	0.350
21	A	8	7	1.00	25	0.280
22	A	9	8	1.01	30	0.267

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Table 2.1 – continued from previous page

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
23	A	9	8	1.01	35	0.229
24	A	7	6	1.00	40	0.150
25	A	7	6	1.00	55	0.109
26	A	10	9	1.12	18	0.500
27	A	10	9	1.10	23	0.391
28	A	9	8	1.07	28	0.286
29	A	9	8	1.06	33	0.242
30	A	11	10	1.03	38	0.263
31	A	14	13	1.06	16	0.812
32	A	14	13	1.08	21	0.619
33	A	13	12	1.08	26	0.462
34	A	13	12	1.07	31	0.387
35	A	14	13	1.07	36	0.361
36	A	11	10	0.96	20	0.500
37	A	11	10	0.97	25	0.400
38	A	10	9	0.97	30	0.300
39	A	10	9	0.98	35	0.257
40	A	11	10	0.98	40	0.250
41	A	12	11	1.02	55	0.200
42	A	12	11	1.12	18	0.611
43	A	12	11	1.10	23	0.478
44	A	11	10	1.10	28	0.357
45	A	11	10	1.09	33	0.303
46	A	13	12	1.01	38	0.316
47	A	17	16	1.09	16	1.000
48	A	17	16	1.08	21	0.762
49	A	16	15	1.07	26	0.577
50	A	16	15	1.06	31	0.484
51	A	17	16	1.06	36	0.444
52	A	14	13	1.04	20	0.650
53	A	14	13	0.97	25	0.520
54	A	13	12	0.97	30	0.400
55	A	13	12	1.02	35	0.343
56	A	14	13	1.01	40	0.325

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Table 2.1 – continued from previous page

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	number of rules / integrand leaf size
57	A	14	13	1.02	55	0.236
58	A	12	11	1.01	50	0.220
59	A	17	16	1.05	50	0.320
60	A	2	2	1.00	63	0.032
61	A	2	2	1.00	63	0.032
62	A	2	2	1.00	61	0.033
63	A	2	2	1.00	63	0.032
64	A	9	8	1.00	63	0.127
65	A	12	11	0.97	63	0.175
66	A	15	14	0.97	63	0.222
67	A	2	2	1.00	26	0.077
68	A	3	3	1.00	31	0.097
69	A	3	3	1.00	36	0.083
70	A	3	3	1.00	41	0.073
71	A	3	3	1.00	46	0.065
72	A	3	3	1.00	51	0.059
73	A	3	3	1.00	21	0.143
74	A	3	3	1.00	26	0.115
75	A	3	3	1.00	31	0.097
76	A	3	3	1.00	36	0.083
77	A	3	3	1.00	41	0.073
78	A	3	3	1.00	46	0.065
79	A	3	3	1.00	16	0.188
80	A	3	3	1.00	21	0.143
81	A	3	3	1.00	26	0.115
82	A	3	3	1.00	31	0.097
83	A	3	3	1.00	36	0.083
84	A	3	3	1.00	41	0.073
85	A	3	3	1.00	26	0.115
86	A	3	3	1.00	31	0.097
87	A	3	3	1.00	36	0.083
88	A	3	3	1.00	41	0.073
89	A	3	3	1.00	46	0.065
90	A	3	3	1.00	51	0.059

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Table 2.1 – continued from previous page

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
91	A	3	3	0.98	21	0.143
92	A	7	7	1.06	26	0.269
93	A	7	7	1.05	31	0.226
94	A	3	3	1.00	36	0.083
95	A	3	3	1.00	41	0.073
96	A	3	3	1.00	46	0.065
97	A	3	3	1.00	16	0.188
98	A	3	3	1.00	21	0.143
99	A	3	3	1.00	26	0.115
100	A	3	3	1.00	31	0.097
101	A	3	3	1.00	36	0.083
102	A	3	3	1.00	41	0.073
103	A	15	14	0.97	32	0.438
104	A	12	11	0.99	32	0.344
105	A	10	9	1.01	32	0.281
106	A	9	8	0.97	32	0.250
107	A	13	12	0.96	32	0.375
108	A	1	1	1.00	28	0.036
109	A	6	5	1.00	31	0.161
110	A	6	5	1.00	33	0.152
111	A	5	4	1.00	36	0.111

CHAPTER 3

Listing of integrals

3.1 $\int (d+ex)(a+bx^2+cx^4)\,dx$

3.1.1 Optimal result

Integrand size = 18, antiderivative size = 50

$$\int (d+ex)\left(a+bx^2+cx^4\right)\,dx=adx+\frac{1}{2}aex^2+\frac{1}{3}bdx^3+\frac{1}{4}bex^4+\frac{1}{5}cdx^5+\frac{1}{6}cex^6$$

output
```
a*d*x+1/2*a*e*x^2+1/3*b*d*x^3+1/4*b*e*x^4+1/5*c*d*x^5+1/6*c*e*x^6
```

3.1.2 Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 50, normalized size of antiderivative = 1.00

$$\int (d+ex)\left(a+bx^2+cx^4\right)\,dx=adx+\frac{1}{2}aex^2+\frac{1}{3}bdx^3+\frac{1}{4}bex^4+\frac{1}{5}cdx^5+\frac{1}{6}cex^6$$

input
```
Integrate[(d + e*x)*(a + b*x^2 + c*x^4),x]
```

output
```
a*d*x + (a*e*x^2)/2 + (b*d*x^3)/3 + (b*e*x^4)/4 + (c*d*x^5)/5 + (c*e*x^6)/
6
```

3.1.3 Rubi [A] (verified)

Time = 0.21 (sec) , antiderivative size = 50, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.111, Rules used = {2200, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int (d+ex)\left(a+bx^2+cx^4\right) \, dx$$

$$\Big\downarrow \text{ 2200}$$

$$\int \left(ad+aex+bdx^2+bex^3+cdx^4+cex^5\right) dx$$

$$\Big\downarrow \text{ 2009}$$

$$adx+\frac{1}{2}aex^2+\frac{1}{3}bdx^3+\frac{1}{4}bex^4+\frac{1}{5}cdx^5+\frac{1}{6}cex^6$$

input
```
Int[(d + e*x)*(a + b*x^2 + c*x^4),x]
```

output
```
a*d*x + (a*e*x^2)/2 + (b*d*x^3)/3 + (b*e*x^4)/4 + (c*d*x^5)/5 + (c*e*x^6)/
6
```

3.1.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2200
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_.), x_Symbol] :> Int[Expa
ndIntegrand[Px*(a + b*x^2 + c*x^4)^p, x], x] /; FreeQ[{a, b, c}, x] && Poly
Q[Px, x] && IGtQ[p, 0]
```

3.1.4 Maple [A] (verified)

Time = 0.02 (sec) , antiderivative size = 41, normalized size of antiderivative = 0.82

method	result	size
gosper	$adx + \frac{1}{2}ae\,x^2 + \frac{1}{3}x^3bd + \frac{1}{4}be\,x^4 + \frac{1}{5}cd\,x^5 + \frac{1}{6}ce\,x^6$	41
default	$adx + \frac{1}{2}ae\,x^2 + \frac{1}{3}x^3bd + \frac{1}{4}be\,x^4 + \frac{1}{5}cd\,x^5 + \frac{1}{6}ce\,x^6$	41
norman	$adx + \frac{1}{2}ae\,x^2 + \frac{1}{3}x^3bd + \frac{1}{4}be\,x^4 + \frac{1}{5}cd\,x^5 + \frac{1}{6}ce\,x^6$	41
risch	$adx + \frac{1}{2}ae\,x^2 + \frac{1}{3}x^3bd + \frac{1}{4}be\,x^4 + \frac{1}{5}cd\,x^5 + \frac{1}{6}ce\,x^6$	41
parallelrisch	$adx + \frac{1}{2}ae\,x^2 + \frac{1}{3}x^3bd + \frac{1}{4}be\,x^4 + \frac{1}{5}cd\,x^5 + \frac{1}{6}ce\,x^6$	41

input
```
int((e*x+d)*(c*x^4+b*x^2+a),x,method=_RETURNVERBOSE)
```

output
```
a*d*x+1/2*a*e*x^2+1/3*x^3*b*d+1/4*b*e*x^4+1/5*c*d*x^5+1/6*c*e*x^6
```

3.1.5 Fricas [A] (verification not implemented)

Time = 0.23 (sec) , antiderivative size = 40, normalized size of antiderivative = 0.80

$$\int (d+ex)\left(a+bx^2+cx^4\right)\,dx = \frac{1}{6}\,cex^6 + \frac{1}{5}\,cdx^5 + \frac{1}{4}\,bex^4 + \frac{1}{3}\,bdx^3 + \frac{1}{2}\,aex^2 + adx$$

input
```
integrate((e*x+d)*(c*x^4+b*x^2+a),x, algorithm="fricas")
```

output
```
1/6*c*e*x^6 + 1/5*c*d*x^5 + 1/4*b*e*x^4 + 1/3*b*d*x^3 + 1/2*a*e*x^2 + a*d*
x
```

3.1.6 Sympy [A] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 46, normalized size of antiderivative = 0.92

$$\int (d+ex)\left(a+bx^2+cx^4\right)\,dx = adx + \frac{aex^2}{2} + \frac{bdx^3}{3} + \frac{bex^4}{4} + \frac{cdx^5}{5} + \frac{cex^6}{6}$$

input
```
integrate((e*x+d)*(c*x**4+b*x**2+a),x)
```

output
```
a*d*x + a*e*x**2/2 + b*d*x**3/3 + b*e*x**4/4 + c*d*x**5/5 + c*e*x**6/6
```

3.1.7 Maxima [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 40, normalized size of antiderivative = 0.80

$$\int (d+ex)\left(a+bx^2+cx^4\right)\,dx=\frac{1}{6}\,cex^6+\frac{1}{5}\,cdx^5+\frac{1}{4}\,bex^4+\frac{1}{3}\,bdx^3+\frac{1}{2}\,aex^2+adx$$

input
```
integrate((e*x+d)*(c*x^4+b*x^2+a),x, algorithm="maxima")
```

output
```
1/6*c*e*x^6 + 1/5*c*d*x^5 + 1/4*b*e*x^4 + 1/3*b*d*x^3 + 1/2*a*e*x^2 + a*d*
x
```

3.1.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 40, normalized size of antiderivative = 0.80

$$\int (d+ex)\left(a+bx^2+cx^4\right)\,dx=\frac{1}{6}\,cex^6+\frac{1}{5}\,cdx^5+\frac{1}{4}\,bex^4+\frac{1}{3}\,bdx^3+\frac{1}{2}\,aex^2+adx$$

input
```
integrate((e*x+d)*(c*x^4+b*x^2+a),x, algorithm="giac")
```

output
```
1/6*c*e*x^6 + 1/5*c*d*x^5 + 1/4*b*e*x^4 + 1/3*b*d*x^3 + 1/2*a*e*x^2 + a*d*
x
```

3.1.9 Mupad [B] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 40, normalized size of antiderivative = 0.80

$$\int (d+ex)\left(a+bx^2+cx^4\right)\,dx=\frac{c\,e\,x^6}{6}+\frac{c\,d\,x^5}{5}+\frac{b\,e\,x^4}{4}+\frac{b\,d\,x^3}{3}+\frac{a\,e\,x^2}{2}+a\,d\,x$$

input
```
int((d + e*x)*(a + b*x^2 + c*x^4),x)
```

output
```
a*d*x + (a*e*x^2)/2 + (b*d*x^3)/3 + (b*e*x^4)/4 + (c*d*x^5)/5 + (c*e*x^6)/
6
```

3.2 $\int (d+ex+fx^2)(a+bx^2+cx^4)\,dx$

3.2.1 Optimal result

Integrand size = 23, antiderivative size = 69

$$\int (d+ex+fx^2)(a+bx^2+cx^4)\,dx = adx+\frac{1}{2}aex^2+\frac{1}{3}(bd+af)x^3+\frac{1}{4}bex^4+\frac{1}{5}(cd+bf)x^5+\frac{1}{6}cex^6+\frac{1}{7}cfx^7$$

output
```
a*d*x+1/2*a*e*x^2+1/3*(a*f+b*d)*x^3+1/4*b*e*x^4+1/5*(b*f+c*d)*x^5+1/6*c*e*
x^6+1/7*c*f*x^7
```

3.2.2 Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 69, normalized size of antiderivative = 1.00

$$\int (d+ex+fx^2)(a+bx^2+cx^4)\,dx = adx+\frac{1}{2}aex^2+\frac{1}{3}(bd+af)x^3+\frac{1}{4}bex^4+\frac{1}{5}(cd+bf)x^5+\frac{1}{6}cex^6+\frac{1}{7}cfx^7$$

input
```
Integrate[(d + e*x + f*x^2)*(a + b*x^2 + c*x^4),x]
```

output
```
a*d*x + (a*e*x^2)/2 + ((b*d + a*f)*x^3)/3 + (b*e*x^4)/4 + ((c*d + b*f)*x^5
)/5 + (c*e*x^6)/6 + (c*f*x^7)/7
```

3.2.3 Rubi [A] (verified)

Time = 0.23 (sec) , antiderivative size = 69, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.087, Rules used = {2188, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \left(a+bx^2+cx^4\right)\left(d+ex+fx^2\right)dx$$

$$\Big\downarrow \text{2188}$$

$$\int \left(x^2(af+bd)+ad+aex+x^4(bf+cd)+bex^3+cex^5+cfx^6\right)dx$$

$$\Big\downarrow \text{2009}$$

$$\frac{1}{3}x^3(af+bd)+adx+\frac{1}{2}aex^2+\frac{1}{5}x^5(bf+cd)+\frac{1}{4}bex^4+\frac{1}{6}cex^6+\frac{1}{7}cfx^7$$

input
```
Int[(d + e*x + f*x^2)*(a + b*x^2 + c*x^4),x]
```

output
```
a*d*x + (a*e*x^2)/2 + ((b*d + a*f)*x^3)/3 + (b*e*x^4)/4 + ((c*d + b*f)*x^5
)/5 + (c*e*x^6)/6 + (c*f*x^7)/7
```

3.2.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2188
```
Int[(Pq_)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_.), x_Symbol] :> Int[Expand
Integrand[Pq*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c}, x] && PolyQ[Pq
, x] && IGtQ[p, -2]
```

3.2.4 Maple [A] (verified)

Time = 0.07 (sec) , antiderivative size = 58, normalized size of antiderivative = 0.84

method	result	size
default	$adx+\frac{ae\,x^{2}}{2}+\frac{(af+bd)x^{3}}{3}+\frac{be\,x^{4}}{4}+\frac{(bf+cd)x^{5}}{5}+\frac{ce\,x^{6}}{6}+\frac{cf\,x^{7}}{7}$	58
norman	$\frac{cf\,x^{7}}{7}+\frac{ce\,x^{6}}{6}+\left(\frac{bf}{5}+\frac{cd}{5}\right)x^{5}+\frac{be\,x^{4}}{4}+\left(\frac{af}{3}+\frac{bd}{3}\right)x^{3}+\frac{ae\,x^{2}}{2}+adx$	60
gosper	$\frac{1}{7}cf\,x^{7}+\frac{1}{6}ce\,x^{6}+\frac{1}{5}x^{5}bf+\frac{1}{5}cd\,x^{5}+\frac{1}{4}be\,x^{4}+\frac{1}{3}x^{3}af+\frac{1}{3}x^{3}bd+\frac{1}{2}ae\,x^{2}+adx$	62
risch	$\frac{1}{7}cf\,x^{7}+\frac{1}{6}ce\,x^{6}+\frac{1}{5}x^{5}bf+\frac{1}{5}cd\,x^{5}+\frac{1}{4}be\,x^{4}+\frac{1}{3}x^{3}af+\frac{1}{3}x^{3}bd+\frac{1}{2}ae\,x^{2}+adx$	62
parallelrisch	$\frac{1}{7}cf\,x^{7}+\frac{1}{6}ce\,x^{6}+\frac{1}{5}x^{5}bf+\frac{1}{5}cd\,x^{5}+\frac{1}{4}be\,x^{4}+\frac{1}{3}x^{3}af+\frac{1}{3}x^{3}bd+\frac{1}{2}ae\,x^{2}+adx$	62

input
```
int((f*x^2+e*x+d)*(c*x^4+b*x^2+a),x,method=_RETURNVERBOSE)
```

output
```
a*d*x+1/2*a*e*x^2+1/3*(a*f+b*d)*x^3+1/4*b*e*x^4+1/5*(b*f+c*d)*x^5+1/6*c*e*
x^6+1/7*c*f*x^7
```

3.2.5 Fricas [A] (verification not implemented)

Time = 0.24 (sec) , antiderivative size = 57, normalized size of antiderivative = 0.83

$$\int \left(d+ex+fx^{2}\right)\left(a+bx^{2}+cx^{4}\right)\,dx=\frac{1}{7}\,cfx^{7}+\frac{1}{6}\,cex^{6}+\frac{1}{4}\,bex^{4}+\frac{1}{5}\,(cd+bf)x^{5}+\frac{1}{2}\,aex^{2}+\frac{1}{3}\,(bd+af)x^{3}+adx$$

input
```
integrate((f*x^2+e*x+d)*(c*x^4+b*x^2+a),x, algorithm="fricas")
```

output
```
1/7*c*f*x^7 + 1/6*c*e*x^6 + 1/4*b*e*x^4 + 1/5*(c*d + b*f)*x^5 + 1/2*a*e*x^
2 + 1/3*(b*d + a*f)*x^3 + a*d*x
```

3.2.6 Sympy [A] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 65, normalized size of antiderivative = 0.94

$$\int (d+ex+fx^2)(a+bx^2+cx^4)\,dx = adx + \frac{aex^2}{2} + \frac{bex^4}{4} + \frac{cex^6}{6} + \frac{cfx^7}{7} + x^5\left(\frac{bf}{5} + \frac{cd}{5}\right) + x^3\left(\frac{af}{3} + \frac{bd}{3}\right)$$

input
```
integrate((f*x**2+e*x+d)*(c*x**4+b*x**2+a),x)
```

output
```
a*d*x + a*e*x**2/2 + b*e*x**4/4 + c*e*x**6/6 + c*f*x**7/7 + x**5*(b*f/5 +
c*d/5) + x**3*(a*f/3 + b*d/3)
```

3.2.7 Maxima [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 57, normalized size of antiderivative = 0.83

$$\int (d+ex+fx^2)(a+bx^2+cx^4)\,dx = \frac{1}{7}cfx^7 + \frac{1}{6}cex^6 + \frac{1}{4}bex^4 + \frac{1}{5}(cd+bf)x^5 + \frac{1}{2}aex^2 + \frac{1}{3}(bd+af)x^3 + adx$$

input
```
integrate((f*x^2+e*x+d)*(c*x^4+b*x^2+a),x, algorithm="maxima")
```

output
```
1/7*c*f*x^7 + 1/6*c*e*x^6 + 1/4*b*e*x^4 + 1/5*(c*d + b*f)*x^5 + 1/2*a*e*x^
2 + 1/3*(b*d + a*f)*x^3 + a*d*x
```

3.2.8 Giac [A] (verification not implemented)

Time = 0.36 (sec) , antiderivative size = 61, normalized size of antiderivative = 0.88

$$\int (d+ex+fx^2)(a+bx^2+cx^4)\,dx = \frac{1}{7}cfx^7 + \frac{1}{6}cex^6 + \frac{1}{5}cdx^5 + \frac{1}{5}bfx^5 + \frac{1}{4}bex^4 + \frac{1}{3}bdx^3 + \frac{1}{3}afx^3 + \frac{1}{2}aex^2 + adx$$

input
```
integrate((f*x^2+e*x+d)*(c*x^4+b*x^2+a),x, algorithm="giac")
```

output
```
1/7*c*f*x^7 + 1/6*c*e*x^6 + 1/5*c*d*x^5 + 1/5*b*f*x^5 + 1/4*b*e*x^4 + 1/3*
b*d*x^3 + 1/3*a*f*x^3 + 1/2*a*e*x^2 + a*d*x
```

3.2.9 Mupad [B] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 59, normalized size of antiderivative = 0.86

$$\int \left(d+ex+fx^2\right)\left(a+bx^2+cx^4\right)\,dx=\frac{c\,f\,x^7}{7}+\frac{c\,e\,x^6}{6}+\left(\frac{c\,d}{5}+\frac{b\,f}{5}\right)\,x^5+\frac{b\,e\,x^4}{4}+\left(\frac{b\,d}{3}+\frac{a\,f}{3}\right)\,x^3+\frac{a\,e\,x^2}{2}+a\,d\,x$$

input
```
int((d + e*x + f*x^2)*(a + b*x^2 + c*x^4),x)
```

output
```
x^3*((b*d)/3 + (a*f)/3) + x^5*((c*d)/5 + (b*f)/5) + a*d*x + (a*e*x^2)/2 +
(b*e*x^4)/4 + (c*e*x^6)/6 + (c*f*x^7)/7
```

3.3 $\int (d+ex+fx^2+gx^3)(a+bx^2+cx^4)\,dx$

3.3.1 Optimal result

Integrand size = 28, antiderivative size = 88

$$\int (d+ex+fx^2+gx^3)(a+bx^2+cx^4)\,dx = adx+\frac{1}{2}aex^2+\frac{1}{3}(bd+af)x^3+\frac{1}{4}(be+ag)x^4+\frac{1}{5}(cd+bf)x^5+\frac{1}{6}(ce+bg)x^6+\frac{1}{7}cfx^7+\frac{1}{8}cgx^8$$

output
```
a*d*x+1/2*a*e*x^2+1/3*(a*f+b*d)*x^3+1/4*(a*g+b*e)*x^4+1/5*(b*f+c*d)*x^5+1/
6*(b*g+c*e)*x^6+1/7*c*f*x^7+1/8*c*g*x^8
```

3.3.2 Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 88, normalized size of antiderivative = 1.00

$$\int (d+ex+fx^2+gx^3)(a+bx^2+cx^4)\,dx = adx+\frac{1}{2}aex^2+\frac{1}{3}(bd+af)x^3+\frac{1}{4}(be+ag)x^4+\frac{1}{5}(cd+bf)x^5+\frac{1}{6}(ce+bg)x^6+\frac{1}{7}cfx^7+\frac{1}{8}cgx^8$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3)*(a + b*x^2 + c*x^4),x]
```

output
```
a*d*x + (a*e*x^2)/2 + ((b*d + a*f)*x^3)/3 + ((b*e + a*g)*x^4)/4 + ((c*d +
b*f)*x^5)/5 + ((c*e + b*g)*x^6)/6 + (c*f*x^7)/7 + (c*g*x^8)/8
```

3.3.3 Rubi [A] (verified)

Time = 0.25 (sec) , antiderivative size = 88, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.071, Rules used = {2200, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \left(a+bx^2+cx^4\right)\left(d+ex+fx^2+gx^3\right)\,dx$$

$$\Big\downarrow \ 2200$$

$$\int \left(x^2(af+bd)+x^3(ag+be)+ad+aex+x^4(bf+cd)+x^5(bg+ce)+cfx^6+cgx^7\right)dx$$

$$\Big\downarrow \ 2009$$

$$\frac{1}{3}x^3(af+bd)+\frac{1}{4}x^4(ag+be)+adx+\frac{1}{2}aex^2+\frac{1}{5}x^5(bf+cd)+\frac{1}{6}x^6(bg+ce)+\frac{1}{7}cfx^7+\frac{1}{8}cgx^8$$

input
```
Int[(d + e*x + f*x^2 + g*x^3)*(a + b*x^2 + c*x^4),x]
```

output
```
a*d*x + (a*e*x^2)/2 + ((b*d + a*f)*x^3)/3 + ((b*e + a*g)*x^4)/4 + ((c*d +
b*f)*x^5)/5 + ((c*e + b*g)*x^6)/6 + (c*f*x^7)/7 + (c*g*x^8)/8
```

3.3.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2200
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_.), x_Symbol] :> Int[Expa
ndIntegrand[Px*(a + b*x^2 + c*x^4)^p, x], x] /; FreeQ[{a, b, c}, x] && Poly
Q[Px, x] && IGtQ[p, 0]
```

3.3.4 Maple [A] (verified)

Time = 0.14 (sec) , antiderivative size = 75, normalized size of antiderivative = 0.85

method	result
default	$adx+\frac{ae\,x^{2}}{2}+\frac{(af+bd)x^{3}}{3}+\frac{(ag+be)x^{4}}{4}+\frac{(bf+cd)x^{5}}{5}+\frac{(bg+ec)x^{6}}{6}+\frac{cf\,x^{7}}{7}+\frac{cg\,x^{8}}{8}$
norman	$\frac{cg\,x^{8}}{8}+\frac{cf\,x^{7}}{7}+\left(\frac{bg}{6}+\frac{ec}{6}\right)x^{6}+\left(\frac{bf}{5}+\frac{cd}{5}\right)x^{5}+\left(\frac{ag}{4}+\frac{be}{4}\right)x^{4}+\left(\frac{af}{3}+\frac{bd}{3}\right)x^{3}+\frac{ae\,x^{2}}{2}+adx$
gosper	$\frac{1}{8}cg\,x^{8}+\frac{1}{7}cf\,x^{7}+\frac{1}{6}x^{6}bg+\frac{1}{6}ce\,x^{6}+\frac{1}{5}x^{5}bf+\frac{1}{5}cd\,x^{5}+\frac{1}{4}x^{4}ag+\frac{1}{4}be\,x^{4}+\frac{1}{3}x^{3}af+\frac{1}{3}x^{3}bd+$
risch	$\frac{1}{8}cg\,x^{8}+\frac{1}{7}cf\,x^{7}+\frac{1}{6}x^{6}bg+\frac{1}{6}ce\,x^{6}+\frac{1}{5}x^{5}bf+\frac{1}{5}cd\,x^{5}+\frac{1}{4}x^{4}ag+\frac{1}{4}be\,x^{4}+\frac{1}{3}x^{3}af+\frac{1}{3}x^{3}bd+$
parallelrisch	$\frac{1}{8}cg\,x^{8}+\frac{1}{7}cf\,x^{7}+\frac{1}{6}x^{6}bg+\frac{1}{6}ce\,x^{6}+\frac{1}{5}x^{5}bf+\frac{1}{5}cd\,x^{5}+\frac{1}{4}x^{4}ag+\frac{1}{4}be\,x^{4}+\frac{1}{3}x^{3}af+\frac{1}{3}x^{3}bd+$

input
```
int((g*x^3+f*x^2+e*x+d)*(c*x^4+b*x^2+a),x,method=_RETURNVERBOSE)
```

output
```
a*d*x+1/2*a*e*x^2+1/3*(a*f+b*d)*x^3+1/4*(a*g+b*e)*x^4+1/5*(b*f+c*d)*x^5+1/
6*(b*g+c*e)*x^6+1/7*c*f*x^7+1/8*c*g*x^8
```

3.3.5 Fricas [A] (verification not implemented)

Time = 0.23 (sec) , antiderivative size = 74, normalized size of antiderivative = 0.84

$$\int (d+ex+fx^2+gx^3)\left(a+bx^2+cx^4\right)\,dx=\frac{1}{8}\,cgx^8+\frac{1}{7}\,cfx^7+\frac{1}{6}\,(ce+bg)x^6+\frac{1}{5}\,(cd+bf)x^5+\frac{1}{4}\,(be+ag)x^4+\frac{1}{2}\,aex^2+\frac{1}{3}\,(bd+af)x^3+adx$$

input
```
integrate((g*x^3+f*x^2+e*x+d)*(c*x^4+b*x^2+a),x, algorithm="fricas")
```

output
```
1/8*c*g*x^8 + 1/7*c*f*x^7 + 1/6*(c*e + b*g)*x^6 + 1/5*(c*d + b*f)*x^5 + 1/
4*(b*e + a*g)*x^4 + 1/2*a*e*x^2 + 1/3*(b*d + a*f)*x^3 + a*d*x
```

3.3.6 Sympy [A] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 83, normalized size of antiderivative = 0.94

$$\int \left(d+ex+fx^2+gx^3\right)\left(a+bx^2+cx^4\right)\,dx = adx+\frac{aex^2}{2}+\frac{cfx^7}{7}+\frac{cgx^8}{8}+x^6\left(\frac{bg}{6}+\frac{ce}{6}\right)+x^5\left(\frac{bf}{5}+\frac{cd}{5}\right)+x^4\left(\frac{ag}{4}+\frac{be}{4}\right)+x^3\left(\frac{af}{3}+\frac{bd}{3}\right)$$

input
```
integrate((g*x**3+f*x**2+e*x+d)*(c*x**4+b*x**2+a),x)
```

output
```
a*d*x + a*e*x**2/2 + c*f*x**7/7 + c*g*x**8/8 + x**6*(b*g/6 + c*e/6) + x**5
*(b*f/5 + c*d/5) + x**4*(a*g/4 + b*e/4) + x**3*(a*f/3 + b*d/3)
```

3.3.7 Maxima [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 74, normalized size of antiderivative = 0.84

$$\int \left(d+ex+fx^2+gx^3\right)\left(a+bx^2+cx^4\right)\,dx = \frac{1}{8}\,cgx^8+\frac{1}{7}\,cfx^7+\frac{1}{6}\,(ce+bg)x^6+\frac{1}{5}\,(cd+bf)x^5+\frac{1}{4}\,(be+ag)x^4+\frac{1}{2}\,aex^2+\frac{1}{3}\,(bd+af)x^3+adx$$

input
```
integrate((g*x^3+f*x^2+e*x+d)*(c*x^4+b*x^2+a),x, algorithm="maxima")
```

output
```
1/8*c*g*x^8 + 1/7*c*f*x^7 + 1/6*(c*e + b*g)*x^6 + 1/5*(c*d + b*f)*x^5 + 1/
4*(b*e + a*g)*x^4 + 1/2*a*e*x^2 + 1/3*(b*d + a*f)*x^3 + a*d*x
```

3.3.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 82, normalized size of antiderivative = 0.93

$$\int (d+ex+fx^2+gx^3)\left(a+bx^2+cx^4\right)\,dx = \frac{1}{8}cgx^8+\frac{1}{7}cfx^7+\frac{1}{6}cex^6+\frac{1}{6}bgx^6+\frac{1}{5}cdx^5+\frac{1}{5}bfx^5+\frac{1}{4}bex^4+\frac{1}{4}agx^4+\frac{1}{3}bdx^3+\frac{1}{3}afx^3+\frac{1}{2}aex^2+adx$$

input
```
integrate((g*x^3+f*x^2+e*x+d)*(c*x^4+b*x^2+a),x, algorithm="giac")
```

output
```
1/8*c*g*x^8 + 1/7*c*f*x^7 + 1/6*c*e*x^6 + 1/6*b*g*x^6 + 1/5*c*d*x^5 + 1/5*
b*f*x^5 + 1/4*b*e*x^4 + 1/4*a*g*x^4 + 1/3*b*d*x^3 + 1/3*a*f*x^3 + 1/2*a*e*
x^2 + a*d*x
```

3.3.9 Mupad [B] (verification not implemented)

Time = 7.85 (sec) , antiderivative size = 78, normalized size of antiderivative = 0.89

$$\int (d+ex+fx^2+gx^3)\left(a+bx^2+cx^4\right)\,dx = \frac{c\,g\,x^8}{8}+\frac{c\,f\,x^7}{7}+\left(\frac{c\,e}{6}+\frac{b\,g}{6}\right)x^6+\left(\frac{c\,d}{5}+\frac{b\,f}{5}\right)x^5+\left(\frac{b\,e}{4}+\frac{a\,g}{4}\right)x^4+\left(\frac{b\,d}{3}+\frac{a\,f}{3}\right)x^3+\frac{a\,e\,x^2}{2}+a\,d\,x$$

input
```
int((a + b*x^2 + c*x^4)*(d + e*x + f*x^2 + g*x^3),x)
```

output
```
x^3*((b*d)/3 + (a*f)/3) + x^4*((b*e)/4 + (a*g)/4) + x^5*((c*d)/5 + (b*f)/5
) + x^6*((c*e)/6 + (b*g)/6) + (c*g*x^8)/8 + a*d*x + (a*e*x^2)/2 + (c*f*x^7
)/7
```

3.4 $\int (a+bx^2+cx^4)(d+ex+fx^2+gx^3+hx^4)\,dx$

3.4.1 Optimal result

Integrand size = 33, antiderivative size = 105

$$\int (a+bx^2+cx^4)(d+ex+fx^2+gx^3+hx^4)\,dx = adx+\frac{1}{2}aex^2+\frac{1}{3}(bd+af)x^3+\frac{1}{4}(be+ag)x^4+\frac{1}{5}(cd+bf+ah)x^5 + \frac{1}{6}(ce+bg)x^6+\frac{1}{7}(cf+bh)x^7+\frac{1}{8}cgx^8+\frac{1}{9}chx^9$$

output
```
a*d*x+1/2*a*e*x^2+1/3*(a*f+b*d)*x^3+1/4*(a*g+b*e)*x^4+1/5*(a*h+b*f+c*d)*x^
5+1/6*(b*g+c*e)*x^6+1/7*(b*h+c*f)*x^7+1/8*c*g*x^8+1/9*c*h*x^9
```

3.4.2 Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.00

$$\int (a+bx^2+cx^4)(d+ex+fx^2+gx^3+hx^4)\,dx = adx+\frac{1}{2}aex^2+\frac{1}{3}(bd+af)x^3+\frac{1}{4}(be+ag)x^4+\frac{1}{5}(cd+bf+ah)x^5 + \frac{1}{6}(ce+bg)x^6+\frac{1}{7}(cf+bh)x^7+\frac{1}{8}cgx^8+\frac{1}{9}chx^9$$

input
```
Integrate[(a + b*x^2 + c*x^4)*(d + e*x + f*x^2 + g*x^3 + h*x^4),x]
```

output
```
a*d*x + (a*e*x^2)/2 + ((b*d + a*f)*x^3)/3 + ((b*e + a*g)*x^4)/4 + ((c*d +
b*f + a*h)*x^5)/5 + ((c*e + b*g)*x^6)/6 + ((c*f + b*h)*x^7)/7 + (c*g*x^8)/
8 + (c*h*x^9)/9
```

3.4.3 Rubi [A] (verified)

Time = 0.29 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.061$, Rules used = {2200, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \left(a+bx^2+cx^4\right)\left(d+ex+fx^2+gx^3+hx^4\right)\,dx$$

$$\Big\downarrow \text{ 2200}$$

$$\int \left(x^4(ah+bf+cd)+x^2(af+bd)+x^3(ag+be)+ad+aex+x^5(bg+ce)+x^6(bh+cf)+cgx^7+chx^8\right)dx$$

$$\Big\downarrow \text{ 2009}$$

$$\frac{1}{5}x^5(ah+bf+cd)+\frac{1}{3}x^3(af+bd)+\frac{1}{4}x^4(ag+be)+adx+\frac{1}{2}aex^2+\frac{1}{6}x^6(bg+ce)+\frac{1}{7}x^7(bh+cf)+\frac{1}{8}cgx^8+\frac{1}{9}chx^9$$

input
```
Int[(a + b*x^2 + c*x^4)*(d + e*x + f*x^2 + g*x^3 + h*x^4),x]
```

output
```
a*d*x + (a*e*x^2)/2 + ((b*d + a*f)*x^3)/3 + ((b*e + a*g)*x^4)/4 + ((c*d +
b*f + a*h)*x^5)/5 + ((c*e + b*g)*x^6)/6 + ((c*f + b*h)*x^7)/7 + (c*g*x^8)/
8 + (c*h*x^9)/9
```

3.4.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2200
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_.), x_Symbol] :> Int[Expa
ndIntegrand[Px*(a + b*x^2 + c*x^4)^p, x], x] /; FreeQ[{a, b, c}, x] && Poly
Q[Px, x] && IGtQ[p, 0]
```

3.4.4 Maple [A] (verified)

Time = 0.23 (sec) , antiderivative size = 90, normalized size of antiderivative = 0.86

method	result
default	$adx+\frac{ae\,x^2}{2}+\frac{(af+bd)x^3}{3}+\frac{(ag+be)x^4}{4}+\frac{(ah+bf+cd)x^5}{5}+\frac{(bg+ec)x^6}{6}+\frac{(bh+cf)x^7}{7}+\frac{cg\,x^8}{8}+\frac{ch\,x^9}{9}$
norman	$\frac{ch\,x^9}{9}+\frac{cg\,x^8}{8}+\left(\frac{bh}{7}+\frac{cf}{7}\right)x^7+\left(\frac{bg}{6}+\frac{ec}{6}\right)x^6+\left(\frac{ah}{5}+\frac{bf}{5}+\frac{cd}{5}\right)x^5+\left(\frac{ag}{4}+\frac{be}{4}\right)x^4+\left(\frac{af}{3}+\frac{bd}{3}\right)x^3$
gosper	$\frac{1}{9}ch\,x^9+\frac{1}{8}cg\,x^8+\frac{1}{7}x^7bh+\frac{1}{7}cf\,x^7+\frac{1}{6}x^6bg+\frac{1}{6}ce\,x^6+\frac{1}{5}x^5ah+\frac{1}{5}x^5bf+\frac{1}{5}cd\,x^5+\frac{1}{4}x^4ag+$
risch	$\frac{1}{9}ch\,x^9+\frac{1}{8}cg\,x^8+\frac{1}{7}x^7bh+\frac{1}{7}cf\,x^7+\frac{1}{6}x^6bg+\frac{1}{6}ce\,x^6+\frac{1}{5}x^5ah+\frac{1}{5}x^5bf+\frac{1}{5}cd\,x^5+\frac{1}{4}x^4ag+$
parallelrisch	$\frac{1}{9}ch\,x^9+\frac{1}{8}cg\,x^8+\frac{1}{7}x^7bh+\frac{1}{7}cf\,x^7+\frac{1}{6}x^6bg+\frac{1}{6}ce\,x^6+\frac{1}{5}x^5ah+\frac{1}{5}x^5bf+\frac{1}{5}cd\,x^5+\frac{1}{4}x^4ag+$

input
```
int((c*x^4+b*x^2+a)*(h*x^4+g*x^3+f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

output
```
a*d*x+1/2*a*e*x^2+1/3*(a*f+b*d)*x^3+1/4*(a*g+b*e)*x^4+1/5*(a*h+b*f+c*d)*x^
5+1/6*(b*g+c*e)*x^6+1/7*(b*h+c*f)*x^7+1/8*c*g*x^8+1/9*c*h*x^9
```

3.4.5 Fricas [A] (verification not implemented)

Time = 0.24 (sec) , antiderivative size = 89, normalized size of antiderivative = 0.85

$$\begin{aligned}\int (a+bx^2+cx^4)\,(d+ex+fx^2+gx^3+hx^4)\,dx &= \frac{1}{9}\,chx^9+\frac{1}{8}\,cgx^8+\frac{1}{7}\,(cf+bh)x^7+\frac{1}{6}\,(ce+bg)x^6+\frac{1}{5}\,(cd+bf+ah)x^5\\ &\quad+\frac{1}{4}\,(be+ag)x^4+\frac{1}{2}\,aex^2+\frac{1}{3}\,(bd+af)x^3+adx\end{aligned}$$

input
```
integrate((c*x^4+b*x^2+a)*(h*x^4+g*x^3+f*x^2+e*x+d),x, algorithm="fricas")
```

output
```
1/9*c*h*x^9 + 1/8*c*g*x^8 + 1/7*(c*f + b*h)*x^7 + 1/6*(c*e + b*g)*x^6 + 1/
5*(c*d + b*f + a*h)*x^5 + 1/4*(b*e + a*g)*x^4 + 1/2*a*e*x^2 + 1/3*(b*d + a
*f)*x^3 + a*d*x
```

3.4.6 Sympy [A] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 102, normalized size of antiderivative = 0.97

$$\int (a+bx^2+cx^4)(d+ex+fx^2+gx^3+hx^4)\,dx$$
$$= adx + \frac{aex^2}{2} + \frac{cgx^8}{8} + \frac{chx^9}{9} + x^7\left(\frac{bh}{7}+\frac{cf}{7}\right) + x^6\left(\frac{bg}{6}+\frac{ce}{6}\right) + x^5\left(\frac{ah}{5}+\frac{bf}{5}+\frac{cd}{5}\right) + x^4\left(\frac{ag}{4}+\frac{be}{4}\right) + x^3\left(\frac{af}{3}+\frac{bd}{3}\right)$$

input
```
integrate((c*x**4+b*x**2+a)*(h*x**4+g*x**3+f*x**2+e*x+d),x)
```

output
```
a*d*x + a*e*x**2/2 + c*g*x**8/8 + c*h*x**9/9 + x**7*(b*h/7 + c*f/7) + x**6
*(b*g/6 + c*e/6) + x**5*(a*h/5 + b*f/5 + c*d/5) + x**4*(a*g/4 + b*e/4) + x
**3*(a*f/3 + b*d/3)
```

3.4.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 89, normalized size of antiderivative = 0.85

$$\int (a+bx^2+cx^4)(d+ex+fx^2+gx^3+hx^4)\,dx$$
$$= \frac{1}{9}chx^9 + \frac{1}{8}cgx^8 + \frac{1}{7}(cf+bh)x^7 + \frac{1}{6}(ce+bg)x^6 + \frac{1}{5}(cd+bf+ah)x^5 + \frac{1}{4}(be+ag)x^4 + \frac{1}{2}aex^2 + \frac{1}{3}(bd+af)x^3 + adx$$

input
```
integrate((c*x^4+b*x^2+a)*(h*x^4+g*x^3+f*x^2+e*x+d),x, algorithm="maxima")
```

output
```
1/9*c*h*x^9 + 1/8*c*g*x^8 + 1/7*(c*f + b*h)*x^7 + 1/6*(c*e + b*g)*x^6 + 1/
5*(c*d + b*f + a*h)*x^5 + 1/4*(b*e + a*g)*x^4 + 1/2*a*e*x^2 + 1/3*(b*d + a
*f)*x^3 + a*d*x
```

3.4.8 Giac [A] (verification not implemented)

Time = 0.37 (sec) , antiderivative size = 103, normalized size of antiderivative = 0.98

$$\int (a+bx^2+cx^4)(d+ex+fx^2+gx^3+hx^4)\,dx = \frac{1}{9}chx^9+\frac{1}{8}cgx^8+\frac{1}{7}cfx^7+\frac{1}{7}bhx^7+\frac{1}{6}cex^6+\frac{1}{6}bgx^6+\frac{1}{5}cdx^5+\frac{1}{5}bfx^5+\frac{1}{5}ahx^5+\frac{1}{4}bex^4+\frac{1}{4}agx^4+\frac{1}{3}bdx^3+\frac{1}{3}afx^3+\frac{1}{2}aex^2+adx$$

input
```
integrate((c*x^4+b*x^2+a)*(h*x^4+g*x^3+f*x^2+e*x+d),x, algorithm="giac")
```

output
```
1/9*c*h*x^9 + 1/8*c*g*x^8 + 1/7*c*f*x^7 + 1/7*b*h*x^7 + 1/6*c*e*x^6 + 1/6*
b*g*x^6 + 1/5*c*d*x^5 + 1/5*b*f*x^5 + 1/5*a*h*x^5 + 1/4*b*e*x^4 + 1/4*a*g*
x^4 + 1/3*b*d*x^3 + 1/3*a*f*x^3 + 1/2*a*e*x^2 + a*d*x
```

3.4.9 Mupad [B] (verification not implemented)

Time = 7.90 (sec) , antiderivative size = 95, normalized size of antiderivative = 0.90

$$\int (a+bx^2+cx^4)(d+ex+fx^2+gx^3+hx^4)\,dx = \frac{c\,h\,x^9}{9}+\frac{c\,g\,x^8}{8}+\left(\frac{c\,f}{7}+\frac{b\,h}{7}\right)x^7+\left(\frac{c\,e}{6}+\frac{b\,g}{6}\right)x^6+\left(\frac{c\,d}{5}+\frac{b\,f}{5}+\frac{a\,h}{5}\right)x^5+\left(\frac{b\,e}{4}+\frac{a\,g}{4}\right)x^4+\left(\frac{b\,d}{3}+\frac{a\,f}{3}\right)x^3+\frac{a\,e\,x^2}{2}+a\,d\,x$$

input
```
int((a + b*x^2 + c*x^4)*(d + e*x + f*x^2 + g*x^3 + h*x^4),x)
```

output
```
x^5*((c*d)/5 + (b*f)/5 + (a*h)/5) + x^3*((b*d)/3 + (a*f)/3) + x^4*((b*e)/4
 + (a*g)/4) + x^6*((c*e)/6 + (b*g)/6) + x^7*((c*f)/7 + (b*h)/7) + (c*g*x^8
)/8 + (c*h*x^9)/9 + a*d*x + (a*e*x^2)/2
```

3.5 $\int (a+bx^2+cx^4) (d+ex+fx^2+gx^3+hx^4+ix^5) \, dx$

3.5.1 Optimal result

Integrand size = 38, antiderivative size = 122

$$\begin{aligned}\int (a+bx^2+cx^4) (d+ex+fx^2+gx^3+hx^4+ix^5) \, dx \\ = adx+\frac{1}{2}aex^2+\frac{1}{3}(bd+af)x^3+\frac{1}{4}(be+ag)x^4+\frac{1}{5}(cd+bf+ah)x^5 \\ +\frac{1}{6}(ce+bg+ai)x^6+\frac{1}{7}(cf+bh)x^7+\frac{1}{8}(cg+bi)x^8+\frac{1}{9}chx^9+\frac{1}{10}cix^{10}\end{aligned}$$

output
```
a*d*x+1/2*a*e*x^2+1/3*(a*f+b*d)*x^3+1/4*(a*g+b*e)*x^4+1/5*(a*h+b*f+c*d)*x^
5+1/6*(a*i+b*g+c*e)*x^6+1/7*(b*h+c*f)*x^7+1/8*(b*i+c*g)*x^8+1/9*c*h*x^9+1/
10*c*i*x^10
```

3.5.2 Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 122, normalized size of antiderivative = 1.00

$$\begin{aligned}\int (a+bx^2+cx^4) (d+ex+fx^2+gx^3+hx^4+ix^5) \, dx \\ = adx+\frac{1}{2}aex^2+\frac{1}{3}(bd+af)x^3+\frac{1}{4}(be+ag)x^4+\frac{1}{5}(cd+bf+ah)x^5 \\ +\frac{1}{6}(ce+bg+ai)x^6+\frac{1}{7}(cf+bh)x^7+\frac{1}{8}(cg+bi)x^8+\frac{1}{9}chx^9+\frac{1}{10}cix^{10}\end{aligned}$$

input
```
Integrate[(a + b*x^2 + c*x^4)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5),x]
```

output
```
a*d*x + (a*e*x^2)/2 + ((b*d + a*f)*x^3)/3 + ((b*e + a*g)*x^4)/4 + ((c*d +
b*f + a*h)*x^5)/5 + ((c*e + b*g + a*i)*x^6)/6 + ((c*f + b*h)*x^7)/7 + ((c*
g + b*i)*x^8)/8 + (c*h*x^9)/9 + (c*i*x^10)/10
```

3.5.3 Rubi [A] (verified)

Time = 0.32 (sec) , antiderivative size = 122, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.053$, Rules used = {2200, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \left(a+bx^2+cx^4\right)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)\,dx$$

$$\Big\downarrow \text{2200}$$

$$\int \left(x^4(ah+bf+cd)+x^5(ai+bg+ce)+x^2(af+bd)+x^3(ag+be)+ad+aex+x^6(bh+cf)+x^7(bi+cg)+c\right.$$

$$\Big\downarrow \text{2009}$$

$$\frac{1}{5}x^5(ah+bf+cd)+\frac{1}{6}x^6(ai+bg+ce)+\frac{1}{3}x^3(af+bd)+\frac{1}{4}x^4(ag+be)+adx+\frac{1}{2}aex^2+\frac{1}{7}x^7(bh+cf)+\frac{1}{8}x^8(bi+cg)+\frac{1}{9}chx^9+\frac{1}{10}cix^{10}$$

input
```
Int[(a + b*x^2 + c*x^4)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5),x]
```

output
```
a*d*x + (a*e*x^2)/2 + ((b*d + a*f)*x^3)/3 + ((b*e + a*g)*x^4)/4 + ((c*d +
b*f + a*h)*x^5)/5 + ((c*e + b*g + a*i)*x^6)/6 + ((c*f + b*h)*x^7)/7 + ((c*
g + b*i)*x^8)/8 + (c*h*x^9)/9 + (c*i*x^10)/10
```

3.5.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2200
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_.), x_Symbol] :> Int[Expa
ndIntegrand[Px*(a + b*x^2 + c*x^4)^p, x], x] /; FreeQ[{a, b, c}, x] && Poly
Q[Px, x] && IGtQ[p, 0]
```

3.5.4 Maple [A] (verified)

Time = 1.91 (sec) , antiderivative size = 105, normalized size of antiderivative = 0.86

method	result
default	$adx+\frac{ae\,x^2}{2}+\frac{(af+bd)x^3}{3}+\frac{(ag+be)x^4}{4}+\frac{(ah+bf+cd)x^5}{5}+\frac{(ai+bg+ec)x^6}{6}+\frac{(bh+cf)x^7}{7}+\frac{(bi+gc)x^8}{8}+\frac{ch\,x^9}{9}$
norman	$\frac{ci\,x^{10}}{10}+\frac{ch\,x^9}{9}+\left(\frac{bi}{8}+\frac{gc}{8}\right)x^8+\left(\frac{bh}{7}+\frac{cf}{7}\right)x^7+\left(\frac{ai}{6}+\frac{bg}{6}+\frac{ec}{6}\right)x^6+\left(\frac{ah}{5}+\frac{bf}{5}+\frac{cd}{5}\right)x^5+\left(\frac{ag}{4}+\right.$
gosper	$\frac{1}{10}ci\,x^{10}+\frac{1}{9}ch\,x^9+\frac{1}{8}x^8bi+\frac{1}{8}cg\,x^8+\frac{1}{7}x^7bh+\frac{1}{7}cf\,x^7+\frac{1}{6}x^6ai+\frac{1}{6}x^6bg+\frac{1}{6}ce\,x^6+\frac{1}{5}x^5ah+$
risch	$\frac{1}{10}ci\,x^{10}+\frac{1}{9}ch\,x^9+\frac{1}{8}x^8bi+\frac{1}{8}cg\,x^8+\frac{1}{7}x^7bh+\frac{1}{7}cf\,x^7+\frac{1}{6}x^6ai+\frac{1}{6}x^6bg+\frac{1}{6}ce\,x^6+\frac{1}{5}x^5ah+$
parallelrisch	$\frac{1}{10}ci\,x^{10}+\frac{1}{9}ch\,x^9+\frac{1}{8}x^8bi+\frac{1}{8}cg\,x^8+\frac{1}{7}x^7bh+\frac{1}{7}cf\,x^7+\frac{1}{6}x^6ai+\frac{1}{6}x^6bg+\frac{1}{6}ce\,x^6+\frac{1}{5}x^5ah+$

input
```
int((c*x^4+b*x^2+a)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d),x,method=_RETURNVERBOS
E)
```

output
```
a*d*x+1/2*a*e*x^2+1/3*(a*f+b*d)*x^3+1/4*(a*g+b*e)*x^4+1/5*(a*h+b*f+c*d)*x^
5+1/6*(a*i+b*g+c*e)*x^6+1/7*(b*h+c*f)*x^7+1/8*(b*i+c*g)*x^8+1/9*c*h*x^9+1/
10*c*i*x^10
```

3.5.5 Fricas [A] (verification not implemented)

Time = 0.25 (sec) , antiderivative size = 104, normalized size of antiderivative = 0.85

$$\begin{aligned}&\int (a+bx^2+cx^4)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)\,dx\\&=\frac{1}{10}cix^{10}+\frac{1}{9}chx^9+\frac{1}{8}(cg+bi)x^8+\frac{1}{7}(cf+bh)x^7+\frac{1}{6}(ce+bg+ai)x^6\\&\quad+\frac{1}{5}(cd+bf+ah)x^5+\frac{1}{4}(be+ag)x^4+\frac{1}{2}aex^2+\frac{1}{3}(bd+af)x^3+adx\end{aligned}$$

input
```
integrate((c*x^4+b*x^2+a)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d),x, algorithm="fr
icas")
```

output
```
1/10*c*i*x^10 + 1/9*c*h*x^9 + 1/8*(c*g + b*i)*x^8 + 1/7*(c*f + b*h)*x^7 +
1/6*(c*e + b*g + a*i)*x^6 + 1/5*(c*d + b*f + a*h)*x^5 + 1/4*(b*e + a*g)*x^
4 + 1/2*a*e*x^2 + 1/3*(b*d + a*f)*x^3 + a*d*x
```

3.5.6 Sympy [A] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 121, normalized size of antiderivative = 0.99

$$\int (a+bx^2+cx^4)(d+ex+fx^2+gx^3+hx^4+ix^5)\,dx = adx+\frac{aex^2}{2}+\frac{chx^9}{9}+\frac{cix^{10}}{10}+x^8\left(\frac{bi}{8}+\frac{cg}{8}\right)+x^7\left(\frac{bh}{7}+\frac{cf}{7}\right)+x^6\left(\frac{ai}{6}+\frac{bg}{6}+\frac{ce}{6}\right)+x^5\left(\frac{ah}{5}+\frac{bf}{5}+\frac{cd}{5}\right)+x^4\left(\frac{ag}{4}+\frac{be}{4}\right)+x^3\left(\frac{af}{3}+\frac{bd}{3}\right)$$

input
```
integrate((c*x**4+b*x**2+a)*(i*x**5+h*x**4+g*x**3+f*x**2+e*x+d),x)
```

output
```
a*d*x + a*e*x**2/2 + c*h*x**9/9 + c*i*x**10/10 + x**8*(b*i/8 + c*g/8) + x*
*7*(b*h/7 + c*f/7) + x**6*(a*i/6 + b*g/6 + c*e/6) + x**5*(a*h/5 + b*f/5 +
c*d/5) + x**4*(a*g/4 + b*e/4) + x**3*(a*f/3 + b*d/3)
```

3.5.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 104, normalized size of antiderivative = 0.85

$$\int (a+bx^2+cx^4)(d+ex+fx^2+gx^3+hx^4+ix^5)\,dx = \frac{1}{10}cix^{10}+\frac{1}{9}chx^9+\frac{1}{8}(cg+bi)x^8+\frac{1}{7}(cf+bh)x^7+\frac{1}{6}(ce+bg+ai)x^6+\frac{1}{5}(cd+bf+ah)x^5+\frac{1}{4}(be+ag)x^4+\frac{1}{2}aex^2+\frac{1}{3}(bd+af)x^3+adx$$

input
```
integrate((c*x^4+b*x^2+a)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d),x, algorithm="ma
xima")
```

output
```
1/10*c*i*x^10 + 1/9*c*h*x^9 + 1/8*(c*g + b*i)*x^8 + 1/7*(c*f + b*h)*x^7 +
1/6*(c*e + b*g + a*i)*x^6 + 1/5*(c*d + b*f + a*h)*x^5 + 1/4*(b*e + a*g)*x^
4 + 1/2*a*e*x^2 + 1/3*(b*d + a*f)*x^3 + a*d*x
```

3.5.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 124, normalized size of antiderivative = 1.02

$$\int (a + bx^2 + cx^4) (d + ex + fx^2 + gx^3 + hx^4 + ix^5) dx$$
$$= \frac{1}{10} cix^{10} + \frac{1}{9} chx^9 + \frac{1}{8} cgx^8 + \frac{1}{8} bix^8 + \frac{1}{7} cfx^7 + \frac{1}{7} bhx^7 + \frac{1}{6} cex^6 + \frac{1}{6} bgx^6 + \frac{1}{6} aix^6 + \frac{1}{5} cdx^5 + \frac{1}{5} bfx^5 + \frac{1}{5} ahx^5 + \frac{1}{4} bex^4 + \frac{1}{4} agx^4 + \frac{1}{3} bdx^3 + \frac{1}{3} afx^3 + \frac{1}{2} aex^2 + adx$$

input
```
integrate((c*x^4+b*x^2+a)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d),x, algorithm="gi
ac")
```

output
```
1/10*c*i*x^10 + 1/9*c*h*x^9 + 1/8*c*g*x^8 + 1/8*b*i*x^8 + 1/7*c*f*x^7 + 1/
7*b*h*x^7 + 1/6*c*e*x^6 + 1/6*b*g*x^6 + 1/6*a*i*x^6 + 1/5*c*d*x^5 + 1/5*b*
f*x^5 + 1/5*a*h*x^5 + 1/4*b*e*x^4 + 1/4*a*g*x^4 + 1/3*b*d*x^3 + 1/3*a*f*x^
3 + 1/2*a*e*x^2 + a*d*x
```

3.5.9 Mupad [B] (verification not implemented)

Time = 0.05 (sec) , antiderivative size = 112, normalized size of antiderivative = 0.92

$$\int (a + bx^2 + cx^4) (d + ex + fx^2 + gx^3 + hx^4 + ix^5) dx$$
$$= \frac{c\,i\,x^{10}}{10} + \frac{c\,h\,x^9}{9} + \left(\frac{c\,g}{8} + \frac{b\,i}{8}\right) x^8 + \left(\frac{c\,f}{7} + \frac{b\,h}{7}\right) x^7 + \left(\frac{c\,e}{6} + \frac{b\,g}{6} + \frac{a\,i}{6}\right) x^6 + \left(\frac{c\,d}{5} + \frac{b\,f}{5} + \frac{a\,h}{5}\right) x^5 + \left(\frac{b\,e}{4} + \frac{a\,g}{4}\right) x^4 + \left(\frac{b\,d}{3} + \frac{a\,f}{3}\right) x^3 + \frac{a\,e\,x^2}{2} + a\,d\,x$$

input
```
int((a + b*x^2 + c*x^4)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5),x)
```

output
```
x^5*((c*d)/5 + (b*f)/5 + (a*h)/5) + x^6*((c*e)/6 + (b*g)/6 + (a*i)/6) + x^
3*((b*d)/3 + (a*f)/3) + x^4*((b*e)/4 + (a*g)/4) + x^7*((c*f)/7 + (b*h)/7)
+ x^8*((c*g)/8 + (b*i)/8) + (c*h*x^9)/9 + (c*i*x^10)/10 + a*d*x + (a*e*x^2
)/2
```

3.6 $\int (d+ex)\left(a+bx^2+cx^4\right)^2 \, dx$

3.6.1 Optimal result

Integrand size = 20, antiderivative size = 112

$$\int (d+ex)\left(a+bx^2+cx^4\right)^2 \, dx = a^2dx+\frac{1}{2}a^2ex^2+\frac{2}{3}abdx^3+\frac{1}{2}abex^4+\frac{1}{5}\left(b^2+2ac\right)dx^5 + \frac{1}{6}\left(b^2+2ac\right)ex^6+\frac{2}{7}bcdx^7+\frac{1}{4}bcex^8+\frac{1}{9}c^2dx^9+\frac{1}{10}c^2ex^{10}$$

output
```
a^2*d*x+1/2*a^2*e*x^2+2/3*a*b*d*x^3+1/2*a*b*e*x^4+1/5*(2*a*c+b^2)*d*x^5+1/
6*(2*a*c+b^2)*e*x^6+2/7*b*c*d*x^7+1/4*b*c*e*x^8+1/9*c^2*d*x^9+1/10*c^2*e*x
^10
```

3.6.2 Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 97, normalized size of antiderivative = 0.87

$$\int (d+ex)\left(a+bx^2+cx^4\right)^2 \, dx = \frac{630a^2x(2d+ex)+42b^2x^5(6d+5ex)+45bcx^7(8d+7ex)+14c^2x^9(10d+9ex)+42a(5bx^3(4d+3ex)+2}{1260}$$

input
```
Integrate[(d + e*x)*(a + b*x^2 + c*x^4)^2,x]
```

output
```
(630*a^2*x*(2*d + e*x) + 42*b^2*x^5*(6*d + 5*e*x) + 45*b*c*x^7*(8*d + 7*e*
x) + 14*c^2*x^9*(10*d + 9*e*x) + 42*a*(5*b*x^3*(4*d + 3*e*x) + 2*c*x^5*(6*
d + 5*e*x)))/1260
```

3.6.3 Rubi [A] (verified)

Time = 0.33 (sec) , antiderivative size = 112, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.100, Rules used = {2200, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int (d+ex)\left(a+bx^2+cx^4\right)^2 dx$$

$$\Big\downarrow \text{2200}$$

$$\int \left(a^2d+a^2ex+dx^4\left(2ac+b^2\right)+ex^5\left(2ac+b^2\right)+2abdx^2+2abex^3+2bcdx^6+2bcex^7+c^2dx^8+c^2ex^9\right)dx$$

$$\Big\downarrow \text{2009}$$

$$a^2dx+\frac{1}{2}a^2ex^2+\frac{1}{5}dx^5\left(2ac+b^2\right)+\frac{1}{6}ex^6\left(2ac+b^2\right)+\frac{2}{3}abdx^3+\frac{1}{2}abex^4+\frac{2}{7}bcdx^7+\frac{1}{4}bcex^8+\frac{1}{9}c^2dx^9+\frac{1}{10}c^2ex^{10}$$

input
```
Int[(d + e*x)*(a + b*x^2 + c*x^4)^2,x]
```

output
```
a^2*d*x + (a^2*e*x^2)/2 + (2*a*b*d*x^3)/3 + (a*b*e*x^4)/2 + ((b^2 + 2*a*c)
*d*x^5)/5 + ((b^2 + 2*a*c)*e*x^6)/6 + (2*b*c*d*x^7)/7 + (b*c*e*x^8)/4 + (c
^2*d*x^9)/9 + (c^2*e*x^10)/10
```

3.6.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2200
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_.), x_Symbol] :> Int[Expa
ndIntegrand[Px*(a + b*x^2 + c*x^4)^p, x], x] /; FreeQ[{a, b, c}, x] && Poly
Q[Px, x] && IGtQ[p, 0]
```

3.6.4 Maple [A] (verified)

Time = 0.10 (sec) , antiderivative size = 95, normalized size of antiderivative = 0.85

method	result
default	$a^2dx+\frac{a^2e\,x^2}{2}+\frac{2x^3dab}{3}+\frac{abe\,x^4}{2}+\frac{(2ac+b^2)\,d\,x^5}{5}+\frac{(2ac+b^2)\,e\,x^6}{6}+\frac{2x^7bcd}{7}+\frac{bce\,x^8}{4}+\frac{c^2d\,x^9}{9}+\frac{c^2e\,x^{10}}{10}$
norman	$\frac{c^2e\,x^{10}}{10}+\frac{c^2d\,x^9}{9}+\frac{bce\,x^8}{4}+\frac{2x^7bcd}{7}+\left(\frac{1}{3}ace+\frac{1}{6}b^2e\right)x^6+\left(\frac{2}{5}acd+\frac{1}{5}b^2d\right)x^5+\frac{abe\,x^4}{2}+\frac{2x^3dab}{3}+$
gosper	$\frac{1}{10}c^2e\,x^{10}+\frac{1}{9}c^2d\,x^9+\frac{1}{4}bce\,x^8+\frac{2}{7}x^7bcd+\frac{1}{3}x^6ace+\frac{1}{6}x^6b^2e+\frac{2}{5}acd\,x^5+\frac{1}{5}x^5b^2d+\frac{1}{2}abe\,x^4+$
risch	$\frac{1}{10}c^2e\,x^{10}+\frac{1}{9}c^2d\,x^9+\frac{1}{4}bce\,x^8+\frac{2}{7}x^7bcd+\frac{1}{3}x^6ace+\frac{1}{6}x^6b^2e+\frac{2}{5}acd\,x^5+\frac{1}{5}x^5b^2d+\frac{1}{2}abe\,x^4+$
parallelrisch	$\frac{1}{10}c^2e\,x^{10}+\frac{1}{9}c^2d\,x^9+\frac{1}{4}bce\,x^8+\frac{2}{7}x^7bcd+\frac{1}{3}x^6ace+\frac{1}{6}x^6b^2e+\frac{2}{5}acd\,x^5+\frac{1}{5}x^5b^2d+\frac{1}{2}abe\,x^4+$

input
```
int((e*x+d)*(c*x^4+b*x^2+a)^2,x,method=_RETURNVERBOSE)
```

output
```
a^2*d*x+1/2*a^2*e*x^2+2/3*x^3*d*a*b+1/2*a*b*e*x^4+1/5*(2*a*c+b^2)*d*x^5+1/
6*(2*a*c+b^2)*e*x^6+2/7*x^7*b*c*d+1/4*b*c*e*x^8+1/9*c^2*d*x^9+1/10*c^2*e*x
^10
```

3.6.5 Fricas [A] (verification not implemented)

Time = 0.24 (sec) , antiderivative size = 94, normalized size of antiderivative = 0.84

$$\int (d+ex)\left(a+bx^2+cx^4\right)^2\,dx=\frac{1}{10}\,c^2ex^{10}+\frac{1}{9}\,c^2dx^9+\frac{1}{4}\,bcex^8+\frac{2}{7}\,bcdx^7+\frac{1}{6}\,(b^2+2\,ac)ex^6+\frac{1}{2}\,abex^4+\frac{1}{5}\,(b^2+2\,ac)dx^5+\frac{2}{3}\,abdx^3+\frac{1}{2}\,a^2ex^2+a^2dx$$

input
```
integrate((e*x+d)*(c*x^4+b*x^2+a)^2,x, algorithm="fricas")
```

output
```
1/10*c^2*e*x^10 + 1/9*c^2*d*x^9 + 1/4*b*c*e*x^8 + 2/7*b*c*d*x^7 + 1/6*(b^2
 + 2*a*c)*e*x^6 + 1/2*a*b*e*x^4 + 1/5*(b^2 + 2*a*c)*d*x^5 + 2/3*a*b*d*x^3
+ 1/2*a^2*e*x^2 + a^2*d*x
```

3.6.6 Sympy [A] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 116, normalized size of antiderivative = 1.04

$$\int (d+ex)\left(a+bx^2+cx^4\right)^2\,dx = a^2dx + \frac{a^2ex^2}{2} + \frac{2abdx^3}{3} + \frac{abex^4}{2} + \frac{2bcdx^7}{7} + \frac{bcex^8}{4} + \frac{c^2dx^9}{9} + \frac{c^2ex^{10}}{10} + x^6\left(\frac{ace}{3} + \frac{b^2e}{6}\right) + x^5\cdot\left(\frac{2acd}{5} + \frac{b^2d}{5}\right)$$

input

```
integrate((e*x+d)*(c*x**4+b*x**2+a)**2,x)
```

output

```
a**2*d*x + a**2*e*x**2/2 + 2*a*b*d*x**3/3 + a*b*e*x**4/2 + 2*b*c*d*x**7/7
+ b*c*e*x**8/4 + c**2*d*x**9/9 + c**2*e*x**10/10 + x**6*(a*c*e/3 + b**2*e/
6) + x**5*(2*a*c*d/5 + b**2*d/5)
```

3.6.7 Maxima [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 94, normalized size of antiderivative = 0.84

$$\int (d+ex)\left(a+bx^2+cx^4\right)^2\,dx = \frac{1}{10}\,c^2ex^{10} + \frac{1}{9}\,c^2dx^9 + \frac{1}{4}\,bcex^8 + \frac{2}{7}\,bcdx^7 + \frac{1}{6}\,(b^2+2\,ac)ex^6 + \frac{1}{2}\,abex^4 + \frac{1}{5}\,(b^2+2\,ac)dx^5 + \frac{2}{3}\,abdx^3 + \frac{1}{2}\,a^2ex^2 + a^2dx$$

input

```
integrate((e*x+d)*(c*x^4+b*x^2+a)^2,x, algorithm="maxima")
```

output

```
1/10*c^2*e*x^10 + 1/9*c^2*d*x^9 + 1/4*b*c*e*x^8 + 2/7*b*c*d*x^7 + 1/6*(b^2
 + 2*a*c)*e*x^6 + 1/2*a*b*e*x^4 + 1/5*(b^2 + 2*a*c)*d*x^5 + 2/3*a*b*d*x^3
+ 1/2*a^2*e*x^2 + a^2*d*x
```

3.6.8 Giac [A] (verification not implemented)

Time = 0.37 (sec) , antiderivative size = 100, normalized size of antiderivative = 0.89

$$\int (d+ex)\left(a+bx^2+cx^4\right)^2\,dx = \frac{1}{10}\,c^2ex^{10} + \frac{1}{9}\,c^2dx^9 + \frac{1}{4}\,bcex^8 + \frac{2}{7}\,bcdx^7 + \frac{1}{6}\,b^2ex^6 + \frac{1}{3}\,acex^6 + \frac{1}{5}\,b^2dx^5 + \frac{2}{5}\,acdx^5 + \frac{1}{2}\,abex^4 + \frac{2}{3}\,abdx^3 + \frac{1}{2}\,a^2ex^2 + a^2dx$$

input
```
integrate((e*x+d)*(c*x^4+b*x^2+a)^2,x, algorithm="giac")
```

output
```
1/10*c^2*e*x^10 + 1/9*c^2*d*x^9 + 1/4*b*c*e*x^8 + 2/7*b*c*d*x^7 + 1/6*b^2*
e*x^6 + 1/3*a*c*e*x^6 + 1/5*b^2*d*x^5 + 2/5*a*c*d*x^5 + 1/2*a*b*e*x^4 + 2/
3*a*b*d*x^3 + 1/2*a^2*e*x^2 + a^2*d*x
```

3.6.9 Mupad [B] (verification not implemented)

Time = 0.05 (sec) , antiderivative size = 94, normalized size of antiderivative = 0.84

$$\int (d+ex)\left(a+bx^2+cx^4\right)^2\,dx = \frac{a^2\,e\,x^2}{2}+\frac{c^2\,d\,x^9}{9}+\frac{c^2\,e\,x^{10}}{10}+\frac{d\,x^5\,\left(b^2+2\,a\,c\right)}{5}+\frac{e\,x^6\,\left(b^2+2\,a\,c\right)}{6}+a^2\,d\,x+\frac{2\,a\,b\,d\,x^3}{3}+\frac{a\,b\,e\,x^4}{2}+\frac{2\,b\,c\,d\,x^7}{7}+\frac{b\,c\,e\,x^8}{4}$$

input
```
int((d + e*x)*(a + b*x^2 + c*x^4)^2,x)
```

output
```
(a^2*e*x^2)/2 + (c^2*d*x^9)/9 + (c^2*e*x^10)/10 + (d*x^5*(2*a*c + b^2))/5
+ (e*x^6*(2*a*c + b^2))/6 + a^2*d*x + (2*a*b*d*x^3)/3 + (a*b*e*x^4)/2 + (2
*b*c*d*x^7)/7 + (b*c*e*x^8)/4
```

3.7 $\int (d+ex+fx^2)(a+bx^2+cx^4)^2\,dx$

3.7.1 Optimal result

Integrand size = 25, antiderivative size = 154

$$\begin{aligned}\int (d+ex+fx^2)\left(a+bx^2+cx^4\right)^2\,dx &= a^2dx+\frac{1}{2}a^2ex^2+\frac{1}{3}a(2bd+af)x^3+\frac{1}{2}abex^4\\&+\frac{1}{5}\left(b^2d+2acd+2abf\right)x^5+\frac{1}{6}\left(b^2+2ac\right)ex^6\\&+\frac{1}{7}\left(2bcd+b^2f+2acf\right)x^7+\frac{1}{4}bcex^8\\&+\frac{1}{9}c(cd+2bf)x^9+\frac{1}{10}c^2ex^{10}+\frac{1}{11}c^2fx^{11}\end{aligned}$$

output
```
a^2*d*x+1/2*a^2*e*x^2+1/3*a*(a*f+2*b*d)*x^3+1/2*a*b*e*x^4+1/5*(2*a*b*f+2*a
*c*d+b^2*d)*x^5+1/6*(2*a*c+b^2)*e*x^6+1/7*(2*a*c*f+b^2*f+2*b*c*d)*x^7+1/4*
b*c*e*x^8+1/9*c*(2*b*f+c*d)*x^9+1/10*c^2*e*x^10+1/11*c^2*f*x^11
```

3.7.2 Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 154, normalized size of antiderivative = 1.00

$$\begin{aligned}\int (d+ex+fx^2)\left(a+bx^2+cx^4\right)^2\,dx &= a^2dx+\frac{1}{2}a^2ex^2+\frac{1}{3}a(2bd+af)x^3+\frac{1}{2}abex^4\\&+\frac{1}{5}\left(b^2d+2acd+2abf\right)x^5+\frac{1}{6}\left(b^2+2ac\right)ex^6\\&+\frac{1}{7}\left(2bcd+b^2f+2acf\right)x^7+\frac{1}{4}bcex^8\\&+\frac{1}{9}c(cd+2bf)x^9+\frac{1}{10}c^2ex^{10}+\frac{1}{11}c^2fx^{11}\end{aligned}$$

input
```
Integrate[(d + e*x + f*x^2)*(a + b*x^2 + c*x^4)^2,x]
```

output
```
a^2*d*x + (a^2*e*x^2)/2 + (a*(2*b*d + a*f)*x^3)/3 + (a*b*e*x^4)/2 + ((b^2*
d + 2*a*c*d + 2*a*b*f)*x^5)/5 + ((b^2 + 2*a*c)*e*x^6)/6 + ((2*b*c*d + b^2*
f + 2*a*c*f)*x^7)/7 + (b*c*e*x^8)/4 + (c*(c*d + 2*b*f)*x^9)/9 + (c^2*e*x^1
0)/10 + (c^2*f*x^11)/11
```

3.7.3 Rubi [A] (verified)

Time = 0.35 (sec) , antiderivative size = 154, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.080$, Rules used = {2188, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \left(a+bx^2+cx^4\right)^2 \left(d+ex+fx^2\right) dx$$

$$\Big\downarrow \text{2188}$$

$$\int \left(a^2d+a^2ex+x^6\left(2acf+b^2f+2bcd\right)+x^4\left(2abf+2acd+b^2d\right)+ex^5\left(2ac+b^2\right)+ax^2(af+2bd)+2abex^3+$$

$$\Big\downarrow \text{2009}$$

$$a^2dx+\frac{1}{2}a^2ex^2+\frac{1}{7}x^7\left(2acf+b^2f+2bcd\right)+\frac{1}{5}x^5\left(2abf+2acd+b^2d\right)+\frac{1}{6}ex^6\left(2ac+b^2\right)+\frac{1}{3}ax^3(af+2bd)+\frac{1}{2}abex^4+\frac{1}{9}cx^9(2bf+cd)+\frac{1}{4}bcex^8+\frac{1}{10}c^2ex^{10}+\frac{1}{11}c^2fx^{11}$$

input
```
Int[(d + e*x + f*x^2)*(a + b*x^2 + c*x^4)^2,x]
```

output
```
a^2*d*x + (a^2*e*x^2)/2 + (a*(2*b*d + a*f)*x^3)/3 + (a*b*e*x^4)/2 + ((b^2*
d + 2*a*c*d + 2*a*b*f)*x^5)/5 + ((b^2 + 2*a*c)*e*x^6)/6 + ((2*b*c*d + b^2*
f + 2*a*c*f)*x^7)/7 + (b*c*e*x^8)/4 + (c*(c*d + 2*b*f)*x^9)/9 + (c^2*e*x^1
0)/10 + (c^2*f*x^11)/11
```

3.7.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2188
```
Int[(Pq_)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_.), x_Symbol] :> Int[Expand
Integrand[Pq*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c}, x] && PolyQ[Pq
, x] && IGtQ[p, -2]
```

3.7.4 Maple [A] (verified)

Time = 0.13 (sec) , antiderivative size = 139, normalized size of antiderivative = 0.90

method	result
default	$\frac{c^2 f x^{11}}{11}+\frac{c^2 e x^{10}}{10}+\frac{(2fbc+c^2d)x^9}{9}+\frac{bce x^8}{4}+\frac{(2bcd+f(2ac+b^2))x^7}{7}+\frac{(2ac+b^2)e x^6}{6}+\frac{(d(2ac+b^2)+2abf)x^5}{5}+$
norman	$\frac{c^2 f x^{11}}{11}+\frac{c^2 e x^{10}}{10}+\left(\frac{2}{9}fbc+\frac{1}{9}c^2d\right)x^9+\frac{bce x^8}{4}+\left(\frac{2}{7}acf+\frac{1}{7}b^2f+\frac{2}{7}bcd\right)x^7+\left(\frac{1}{3}ace+\frac{1}{6}b^2e\right)x^6$
gosper	$\frac{1}{11}c^2 f x^{11}+\frac{1}{10}c^2 e x^{10}+\frac{2}{9}x^9 fbc+\frac{1}{9}c^2 d x^9+\frac{1}{4}bce x^8+\frac{2}{7}x^7 acf+\frac{1}{7}x^7 b^2 f+\frac{2}{7}x^7 bcd+\frac{1}{3}x^6 ace$
risch	$\frac{1}{11}c^2 f x^{11}+\frac{1}{10}c^2 e x^{10}+\frac{2}{9}x^9 fbc+\frac{1}{9}c^2 d x^9+\frac{1}{4}bce x^8+\frac{2}{7}x^7 acf+\frac{1}{7}x^7 b^2 f+\frac{2}{7}x^7 bcd+\frac{1}{3}x^6 ace$
parallelrisch	$\frac{1}{11}c^2 f x^{11}+\frac{1}{10}c^2 e x^{10}+\frac{2}{9}x^9 fbc+\frac{1}{9}c^2 d x^9+\frac{1}{4}bce x^8+\frac{2}{7}x^7 acf+\frac{1}{7}x^7 b^2 f+\frac{2}{7}x^7 bcd+\frac{1}{3}x^6 ace$

input
```
int((f*x^2+e*x+d)*(c*x^4+b*x^2+a)^2,x,method=_RETURNVERBOSE)
```

output
```
1/11*c^2*f*x^11+1/10*c^2*e*x^10+1/9*(2*b*c*f+c^2*d)*x^9+1/4*b*c*e*x^8+1/7*
(2*b*c*d+f*(2*a*c+b^2))*x^7+1/6*(2*a*c+b^2)*e*x^6+1/5*(d*(2*a*c+b^2)+2*a*b
*f)*x^5+1/2*a*b*e*x^4+1/3*(a^2*f+2*a*b*d)*x^3+1/2*a^2*e*x^2+a^2*d*x
```

3.7.5 Fricas [A] (verification not implemented)

Time = 0.24 (sec) , antiderivative size = 138, normalized size of antiderivative = 0.90

$$\begin{aligned}\int (d+ex+fx^2)\left(a+bx^2+cx^4\right)^2\,dx &= \frac{1}{11}c^2fx^{11}+\frac{1}{10}c^2ex^{10}+\frac{1}{4}bcex^8+\frac{1}{9}\left(c^2d+2\,bcf\right)x^9\\ &\quad+\frac{1}{6}\left(b^2+2\,ac\right)ex^6+\frac{1}{7}\left(2\,bcd+\left(b^2+2\,ac\right)f\right)x^7\\ &\quad+\frac{1}{2}\,abex^4+\frac{1}{5}\left(2\,abf+\left(b^2+2\,ac\right)d\right)x^5\\ &\quad+\frac{1}{2}\,a^2ex^2+a^2dx+\frac{1}{3}\left(2\,abd+a^2f\right)x^3\end{aligned}$$

input
```
integrate((f*x^2+e*x+d)*(c*x^4+b*x^2+a)^2,x, algorithm="fricas")
```

output
```
1/11*c^2*f*x^11 + 1/10*c^2*e*x^10 + 1/4*b*c*e*x^8 + 1/9*(c^2*d + 2*b*c*f)*
x^9 + 1/6*(b^2 + 2*a*c)*e*x^6 + 1/7*(2*b*c*d + (b^2 + 2*a*c)*f)*x^7 + 1/2*
a*b*e*x^4 + 1/5*(2*a*b*f + (b^2 + 2*a*c)*d)*x^5 + 1/2*a^2*e*x^2 + a^2*d*x
+ 1/3*(2*a*b*d + a^2*f)*x^3
```

3.7.6 Sympy [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 165, normalized size of antiderivative = 1.07

$$\int (d+ex+fx^2)(a+bx^2+cx^4)^2\,dx = a^2dx + \frac{a^2ex^2}{2} + \frac{abex^4}{2} + \frac{bcex^8}{4} + \frac{c^2ex^{10}}{10} + \frac{c^2fx^{11}}{11} + x^9\cdot\left(\frac{2bcf}{9} + \frac{c^2d}{9}\right) + x^7\cdot\left(\frac{2acf}{7} + \frac{b^2f}{7} + \frac{2bcd}{7}\right) + x^6\left(\frac{ace}{3} + \frac{b^2e}{6}\right) + x^5\cdot\left(\frac{2abf}{5} + \frac{2acd}{5} + \frac{b^2d}{5}\right) + x^3\left(\frac{a^2f}{3} + \frac{2abd}{3}\right)$$

input
```
integrate((f*x**2+e*x+d)*(c*x**4+b*x**2+a)**2,x)
```

output
```
a**2*d*x + a**2*e*x**2/2 + a*b*e*x**4/2 + b*c*e*x**8/4 + c**2*e*x**10/10 +
 c**2*f*x**11/11 + x**9*(2*b*c*f/9 + c**2*d/9) + x**7*(2*a*c*f/7 + b**2*f/
7 + 2*b*c*d/7) + x**6*(a*c*e/3 + b**2*e/6) + x**5*(2*a*b*f/5 + 2*a*c*d/5 +
 b**2*d/5) + x**3*(a**2*f/3 + 2*a*b*d/3)
```

3.7.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 138, normalized size of antiderivative = 0.90

$$\int (d+ex+fx^2)(a+bx^2+cx^4)^2\,dx = \frac{1}{11}c^2fx^{11} + \frac{1}{10}c^2ex^{10} + \frac{1}{4}bcex^8 + \frac{1}{9}(c^2d+2bcf)x^9 + \frac{1}{6}(b^2+2ac)ex^6 + \frac{1}{7}(2bcd+(b^2+2ac)f)x^7 + \frac{1}{2}abex^4 + \frac{1}{5}(2abf+(b^2+2ac)d)x^5 + \frac{1}{2}a^2ex^2 + a^2dx + \frac{1}{3}(2abd+a^2f)x^3$$

input
```
integrate((f*x^2+e*x+d)*(c*x^4+b*x^2+a)^2,x, algorithm="maxima")
```

output
```
1/11*c^2*f*x^11 + 1/10*c^2*e*x^10 + 1/4*b*c*e*x^8 + 1/9*(c^2*d + 2*b*c*f)*
x^9 + 1/6*(b^2 + 2*a*c)*e*x^6 + 1/7*(2*b*c*d + (b^2 + 2*a*c)*f)*x^7 + 1/2*
a*b*e*x^4 + 1/5*(2*a*b*f + (b^2 + 2*a*c)*d)*x^5 + 1/2*a^2*e*x^2 + a^2*d*x
+ 1/3*(2*a*b*d + a^2*f)*x^3
```

3.7.8 Giac [A] (verification not implemented)

Time = 0.31 (sec) , antiderivative size = 151, normalized size of antiderivative = 0.98

$$\int (d+ex+fx^2)(a+bx^2+cx^4)^2 dx = \frac{1}{11}c^2fx^{11} + \frac{1}{10}c^2ex^{10} + \frac{1}{9}c^2dx^9 + \frac{2}{9}bcfx^9 + \frac{1}{4}bcex^8 + \frac{2}{7}bcdx^7 + \frac{1}{7}b^2fx^7 + \frac{2}{7}acfx^7 + \frac{1}{6}b^2ex^6 + \frac{1}{3}acex^6 + \frac{1}{5}b^2dx^5 + \frac{2}{5}acdx^5 + \frac{2}{5}abfx^5 + \frac{1}{2}abex^4 + \frac{2}{3}abdx^3 + \frac{1}{3}a^2fx^3 + \frac{1}{2}a^2ex^2 + a^2dx$$

input
```
integrate((f*x^2+e*x+d)*(c*x^4+b*x^2+a)^2,x, algorithm="giac")
```

output
```
1/11*c^2*f*x^11 + 1/10*c^2*e*x^10 + 1/9*c^2*d*x^9 + 2/9*b*c*f*x^9 + 1/4*b*
c*e*x^8 + 2/7*b*c*d*x^7 + 1/7*b^2*f*x^7 + 2/7*a*c*f*x^7 + 1/6*b^2*e*x^6 +
1/3*a*c*e*x^6 + 1/5*b^2*d*x^5 + 2/5*a*c*d*x^5 + 2/5*a*b*f*x^5 + 1/2*a*b*e*
x^4 + 2/3*a*b*d*x^3 + 1/3*a^2*f*x^3 + 1/2*a^2*e*x^2 + a^2*d*x
```

3.7.9 Mupad [B] (verification not implemented)

Time = 7.85 (sec) , antiderivative size = 138, normalized size of antiderivative = 0.90

$$\int (d+ex+fx^2)(a+bx^2+cx^4)^2 dx = x^5\left(\frac{d\,b^2}{5} + \frac{2\,a\,f\,b}{5} + \frac{2\,a\,c\,d}{5}\right) + x^7\left(\frac{f\,b^2}{7} + \frac{2\,c\,d\,b}{7} + \frac{2\,a\,c\,f}{7}\right) + x^3\left(\frac{f\,a^2}{3} + \frac{2\,b\,d\,a}{3}\right) + x^9\left(\frac{d\,c^2}{9} + \frac{2\,b\,f\,c}{9}\right) + \frac{a^2\,e\,x^2}{2} + \frac{c^2\,e\,x^{10}}{10} + \frac{c^2\,f\,x^{11}}{11} + \frac{e\,x^6\,(b^2+2\,a\,c)}{6} + a^2\,d\,x + \frac{a\,b\,e\,x^4}{2} + \frac{b\,c\,e\,x^8}{4}$$

input
```
int((d + e*x + f*x^2)*(a + b*x^2 + c*x^4)^2,x)
```

output
```
x^5*((b^2*d)/5 + (2*a*c*d)/5 + (2*a*b*f)/5) + x^7*((b^2*f)/7 + (2*b*c*d)/7
 + (2*a*c*f)/7) + x^3*((a^2*f)/3 + (2*a*b*d)/3) + x^9*((c^2*d)/9 + (2*b*c*
f)/9) + (a^2*e*x^2)/2 + (c^2*e*x^10)/10 + (c^2*f*x^11)/11 + (e*x^6*(2*a*c
+ b^2))/6 + a^2*d*x + (a*b*e*x^4)/2 + (b*c*e*x^8)/4
```

3.7. $\int (d + ex + fx^2) (a + bx^2 + cx^4)^2 \, dx$

3.8 $\int (d+ex+fx^2+gx^3)(a+bx^2+cx^4)^2 \, dx$

3.8.1 Optimal result

Integrand size = 30, antiderivative size = 196

$$\begin{aligned}\int (d+ex+fx^2+gx^3)(a+bx^2+cx^4)^2 \, dx = {} & a^2dx+\frac{1}{2}a^2ex^2+\frac{1}{3}a(2bd+af)x^3 \\ & +\frac{1}{4}a(2be+ag)x^4+\frac{1}{5}(b^2d+2acd+2abf)x^5 \\ & +\frac{1}{6}(b^2e+2ace+2abg)x^6 \\ & +\frac{1}{7}(2bcd+b^2f+2acf)x^7 \\ & +\frac{1}{8}(2bce+b^2g+2acg)x^8+\frac{1}{9}c(cd+2bf)x^9 \\ & +\frac{1}{10}c(ce+2bg)x^{10}+\frac{1}{11}c^2fx^{11}+\frac{1}{12}c^2gx^{12}\end{aligned}$$

output
```
a^2*d*x+1/2*a^2*e*x^2+1/3*a*(a*f+2*b*d)*x^3+1/4*a*(a*g+2*b*e)*x^4+1/5*(2*a
*b*f+2*a*c*d+b^2*d)*x^5+1/6*(2*a*b*g+2*a*c*e+b^2*e)*x^6+1/7*(2*a*c*f+b^2*f
+2*b*c*d)*x^7+1/8*(2*a*c*g+b^2*g+2*b*c*e)*x^8+1/9*c*(2*b*f+c*d)*x^9+1/10*c
*(2*b*g+c*e)*x^10+1/11*c^2*f*x^11+1/12*c^2*g*x^12
```

3.8.2 Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 196, normalized size of antiderivative = 1.00

$$\int \left(d+ex+fx^2+gx^3\right) \left(a+bx^2+cx^4\right)^2 \, dx = a^2dx+\frac{1}{2}a^2ex^2+\frac{1}{3}a(2bd+af)x^3 +\frac{1}{4}a(2be+ag)x^4+\frac{1}{5}\left(b^2d+2acd+2abf\right)x^5 +\frac{1}{6}\left(b^2e+2ace+2abg\right)x^6 +\frac{1}{7}\left(2bcd+b^2f+2acf\right)x^7 +\frac{1}{8}\left(2bce+b^2g+2acg\right)x^8+\frac{1}{9}c(cd+2bf)x^9 +\frac{1}{10}c(ce+2bg)x^{10}+\frac{1}{11}c^2fx^{11}+\frac{1}{12}c^2gx^{12}$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3)*(a + b*x^2 + c*x^4)^2,x]
```

output
```
a^2*d*x + (a^2*e*x^2)/2 + (a*(2*b*d + a*f)*x^3)/3 + (a*(2*b*e + a*g)*x^4)/
4 + ((b^2*d + 2*a*c*d + 2*a*b*f)*x^5)/5 + ((b^2*e + 2*a*c*e + 2*a*b*g)*x^6
)/6 + ((2*b*c*d + b^2*f + 2*a*c*f)*x^7)/7 + ((2*b*c*e + b^2*g + 2*a*c*g)*x
^8)/8 + (c*(c*d + 2*b*f)*x^9)/9 + (c*(c*e + 2*b*g)*x^10)/10 + (c^2*f*x^11)
/11 + (c^2*g*x^12)/12
```

3.8.3 Rubi [A] (verified)

Time = 0.42 (sec) , antiderivative size = 196, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.067, Rules used = {2200, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \left(a+bx^2+cx^4\right)^2 \left(d+ex+fx^2+gx^3\right) \, dx$$

$$\Big\downarrow \text{2200}$$

$$\int \left(a^2d+a^2ex+x^6\left(2acf+b^2f+2bcd\right)+x^4\left(2abf+2acd+b^2d\right)+x^7\left(2acg+b^2g+2bce\right)+x^5\left(2abg+2ace+b\right.\right.$$

$$\Big\downarrow \text{2009}$$

$$a^2dx + \frac{1}{2}a^2ex^2 + \frac{1}{7}x^7(2acf + b^2f + 2bcd) + \frac{1}{5}x^5(2abf + 2acd + b^2d) + \frac{1}{8}x^8(2acg + b^2g + 2bce) + \frac{1}{6}x^6(2abg + 2ace + b^2e) + \frac{1}{3}ax^3(af + 2bd) + \frac{1}{4}ax^4(ag + 2be) + \frac{1}{9}cx^9(2bf + cd) + \frac{1}{10}cx^{10}(2bg + ce) + \frac{1}{11}c^2fx^{11} + \frac{1}{12}c^2gx^{12}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3)*(a + b*x^2 + c*x^4)^2,x]
```

output
```
a^2*d*x + (a^2*e*x^2)/2 + (a*(2*b*d + a*f)*x^3)/3 + (a*(2*b*e + a*g)*x^4)/
4 + ((b^2*d + 2*a*c*d + 2*a*b*f)*x^5)/5 + ((b^2*e + 2*a*c*e + 2*a*b*g)*x^6
)/6 + ((2*b*c*d + b^2*f + 2*a*c*f)*x^7)/7 + ((2*b*c*e + b^2*g + 2*a*c*g)*x
^8)/8 + (c*(c*d + 2*b*f)*x^9)/9 + (c*(c*e + 2*b*g)*x^10)/10 + (c^2*f*x^11)
/11 + (c^2*g*x^12)/12
```

3.8.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2200
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_.), x_Symbol] :> Int[Expa
ndIntegrand[Px*(a + b*x^2 + c*x^4)^p, x], x] /; FreeQ[{a, b, c}, x] && Poly
Q[Px, x] && IGtQ[p, 0]
```

3.8.4 Maple [A] (verified)

Time = 0.15 (sec) , antiderivative size = 183, normalized size of antiderivative = 0.93

method	result
default	$\frac{c^2 g\,x^{12}}{12}+\frac{c^2 f\,x^{11}}{11}+\frac{(2gbc+e\,c^2)x^{10}}{10}+\frac{(2fbc+c^2 d)x^9}{9}+\frac{(2ebc+g(2ac+b^2))x^8}{8}+\frac{(2bcd+f(2ac+b^2))x^7}{7}+\frac{(e(2ac+b$
norman	$\frac{c^2 g\,x^{12}}{12}+\frac{c^2 f\,x^{11}}{11}+\left(\frac{1}{5}gbc+\frac{1}{10}e\,c^2\right)x^{10}+\left(\frac{2}{9}fbc+\frac{1}{9}c^2 d\right)x^9+\left(\frac{1}{4}acg+\frac{1}{8}b^2 g+\frac{1}{4}ebc\right)x^8+\left(\frac{2}{7}ac$
gosper	$\frac{1}{12}c^2 g\,x^{12}+\frac{1}{11}c^2 f\,x^{11}+\frac{1}{5}x^{10}gbc+\frac{1}{10}c^2 e\,x^{10}+\frac{2}{9}x^9 fbc+\frac{1}{9}c^2 d\,x^9+\frac{1}{4}x^8 acg+\frac{1}{8}x^8 b^2 g+\frac{1}{4}bce$
risch	$\frac{1}{12}c^2 g\,x^{12}+\frac{1}{11}c^2 f\,x^{11}+\frac{1}{5}x^{10}gbc+\frac{1}{10}c^2 e\,x^{10}+\frac{2}{9}x^9 fbc+\frac{1}{9}c^2 d\,x^9+\frac{1}{4}x^8 acg+\frac{1}{8}x^8 b^2 g+\frac{1}{4}bce$
parallelrisch	$\frac{1}{12}c^2 g\,x^{12}+\frac{1}{11}c^2 f\,x^{11}+\frac{1}{5}x^{10}gbc+\frac{1}{10}c^2 e\,x^{10}+\frac{2}{9}x^9 fbc+\frac{1}{9}c^2 d\,x^9+\frac{1}{4}x^8 acg+\frac{1}{8}x^8 b^2 g+\frac{1}{4}bce$

input
```
int((g*x^3+f*x^2+e*x+d)*(c*x^4+b*x^2+a)^2,x,method=_RETURNVERBOSE)
```

output
```
1/12*c^2*g*x^12+1/11*c^2*f*x^11+1/10*(2*b*c*g+c^2*e)*x^10+1/9*(2*b*c*f+c^2
*d)*x^9+1/8*(2*e*b*c+g*(2*a*c+b^2))*x^8+1/7*(2*b*c*d+f*(2*a*c+b^2))*x^7+1/
6*(e*(2*a*c+b^2)+2*a*b*g)*x^6+1/5*(d*(2*a*c+b^2)+2*a*b*f)*x^5+1/4*(a^2*g+2
*a*b*e)*x^4+1/3*(a^2*f+2*a*b*d)*x^3+1/2*a^2*e*x^2+a^2*d*x
```

3.8.5 Fricas [A] (verification not implemented)

Time = 0.25 (sec) , antiderivative size = 182, normalized size of antiderivative = 0.93

$$\int \left(d+ex+fx^2+gx^3\right)\left(a+bx^2+cx^4\right)^2\,dx$$
$$=\frac{1}{12}c^2gx^{12}+\frac{1}{11}c^2fx^{11}+\frac{1}{10}\left(c^2e+2\,bcg\right)x^{10}+\frac{1}{9}\left(c^2d+2\,bcf\right)x^9+\frac{1}{8}\left(2\,bce+\left(b^2+2\,ac\right)g\right)x^8+\frac{1}{7}\left(2\,bcd+\left(b^2+2\,ac\right)f\right)x^7+\frac{1}{6}\left(2\,abg+\left(b^2+2\,ac\right)e\right)x^6+\frac{1}{5}\left(2\,abf+\left(b^2+2\,ac\right)d\right)x^5+\frac{1}{2}a^2ex^2+\frac{1}{4}\left(2\,abe+a^2g\right)x^4+a^2dx+\frac{1}{3}\left(2\,abd+a^2f\right)x^3$$

input
```
integrate((g*x^3+f*x^2+e*x+d)*(c*x^4+b*x^2+a)^2,x, algorithm="fricas")
```

output
```
1/12*c^2*g*x^12 + 1/11*c^2*f*x^11 + 1/10*(c^2*e + 2*b*c*g)*x^10 + 1/9*(c^2
*d + 2*b*c*f)*x^9 + 1/8*(2*b*c*e + (b^2 + 2*a*c)*g)*x^8 + 1/7*(2*b*c*d + (
b^2 + 2*a*c)*f)*x^7 + 1/6*(2*a*b*g + (b^2 + 2*a*c)*e)*x^6 + 1/5*(2*a*b*f +
 (b^2 + 2*a*c)*d)*x^5 + 1/2*a^2*e*x^2 + 1/4*(2*a*b*e + a^2*g)*x^4 + a^2*d*
x + 1/3*(2*a*b*d + a^2*f)*x^3
```

3.8.6 Sympy [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 209, normalized size of antiderivative = 1.07

$$\int (d+ex+fx^2+gx^3)(a+bx^2+cx^4)^2\,dx = a^2dx + \frac{a^2ex^2}{2} + \frac{c^2fx^{11}}{11} + \frac{c^2gx^{12}}{12} + x^{10}\left(\frac{bcg}{5}+\frac{c^2e}{10}\right) + x^9\cdot\left(\frac{2bcf}{9}+\frac{c^2d}{9}\right) + x^8\left(\frac{acg}{4}+\frac{b^2g}{8}+\frac{bce}{4}\right) + x^7\cdot\left(\frac{2acf}{7}+\frac{b^2f}{7}+\frac{2bcd}{7}\right) + x^6\left(\frac{abg}{3}+\frac{ace}{3}+\frac{b^2e}{6}\right) + x^5\cdot\left(\frac{2abf}{5}+\frac{2acd}{5}+\frac{b^2d}{5}\right) + x^4\left(\frac{a^2g}{4}+\frac{abe}{2}\right) + x^3\left(\frac{a^2f}{3}+\frac{2abd}{3}\right)$$

input
```
integrate((g*x**3+f*x**2+e*x+d)*(c*x**4+b*x**2+a)**2,x)
```

output
```
a**2*d*x + a**2*e*x**2/2 + c**2*f*x**11/11 + c**2*g*x**12/12 + x**10*(b*c*
g/5 + c**2*e/10) + x**9*(2*b*c*f/9 + c**2*d/9) + x**8*(a*c*g/4 + b**2*g/8
+ b*c*e/4) + x**7*(2*a*c*f/7 + b**2*f/7 + 2*b*c*d/7) + x**6*(a*b*g/3 + a*c
*e/3 + b**2*e/6) + x**5*(2*a*b*f/5 + 2*a*c*d/5 + b**2*d/5) + x**4*(a**2*g/
4 + a*b*e/2) + x**3*(a**2*f/3 + 2*a*b*d/3)
```

3.8.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 182, normalized size of antiderivative = 0.93

$$\int (d+ex+fx^2+gx^3)(a+bx^2+cx^4)^2\,dx = \frac{1}{12}c^2gx^{12} + \frac{1}{11}c^2fx^{11} + \frac{1}{10}(c^2e+2\,bcg)x^{10} + \frac{1}{9}(c^2d+2\,bcf)x^9 + \frac{1}{8}(2\,bce+(b^2+2\,ac)g)x^8 + \frac{1}{7}(2\,bcd+(b^2+2\,ac)f)x^7 + \frac{1}{6}(2\,abg+(b^2+2\,ac)e)x^6 + \frac{1}{5}(2\,abf+(b^2+2\,ac)d)x^5 + \frac{1}{2}a^2ex^2 + \frac{1}{4}(2\,abe+a^2g)x^4 + a^2dx + \frac{1}{3}(2\,abd+a^2f)x^3$$

input
```
integrate((g*x^3+f*x^2+e*x+d)*(c*x^4+b*x^2+a)^2,x, algorithm="maxima")
```

output
```
1/12*c^2*g*x^12 + 1/11*c^2*f*x^11 + 1/10*(c^2*e + 2*b*c*g)*x^10 + 1/9*(c^2
*d + 2*b*c*f)*x^9 + 1/8*(2*b*c*e + (b^2 + 2*a*c)*g)*x^8 + 1/7*(2*b*c*d + (
b^2 + 2*a*c)*f)*x^7 + 1/6*(2*a*b*g + (b^2 + 2*a*c)*e)*x^6 + 1/5*(2*a*b*f +
 (b^2 + 2*a*c)*d)*x^5 + 1/2*a^2*e*x^2 + 1/4*(2*a*b*e + a^2*g)*x^4 + a^2*d*
x + 1/3*(2*a*b*d + a^2*f)*x^3
```

3.8.8 Giac [A] (verification not implemented)

Time = 0.38 (sec) , antiderivative size = 202, normalized size of antiderivative = 1.03

$$\int (d+ex+fx^2+gx^3)(a+bx^2+cx^4)^2 \, dx = \frac{1}{12}c^2gx^{12} + \frac{1}{11}c^2fx^{11} + \frac{1}{10}c^2ex^{10} + \frac{1}{5}bcgx^{10} + \frac{1}{9}c^2dx^9 + \frac{2}{9}bcfx^9 + \frac{1}{4}bcex^8 + \frac{1}{8}b^2gx^8 + \frac{1}{4}acgx^8 + \frac{2}{7}bcdx^7 + \frac{1}{7}b^2fx^7 + \frac{2}{7}acfx^7 + \frac{1}{6}b^2ex^6 + \frac{1}{3}acex^6 + \frac{1}{3}abgx^6 + \frac{1}{5}b^2dx^5 + \frac{2}{5}acdx^5 + \frac{2}{5}abfx^5 + \frac{1}{2}abex^4 + \frac{1}{4}a^2gx^4 + \frac{2}{3}abdx^3 + \frac{1}{3}a^2fx^3 + \frac{1}{2}a^2ex^2 + a^2dx$$

input
```
integrate((g*x^3+f*x^2+e*x+d)*(c*x^4+b*x^2+a)^2,x, algorithm="giac")
```

output
```
1/12*c^2*g*x^12 + 1/11*c^2*f*x^11 + 1/10*c^2*e*x^10 + 1/5*b*c*g*x^10 + 1/9
*c^2*d*x^9 + 2/9*b*c*f*x^9 + 1/4*b*c*e*x^8 + 1/8*b^2*g*x^8 + 1/4*a*c*g*x^8
 + 2/7*b*c*d*x^7 + 1/7*b^2*f*x^7 + 2/7*a*c*f*x^7 + 1/6*b^2*e*x^6 + 1/3*a*c
*e*x^6 + 1/3*a*b*g*x^6 + 1/5*b^2*d*x^5 + 2/5*a*c*d*x^5 + 2/5*a*b*f*x^5 + 1
/2*a*b*e*x^4 + 1/4*a^2*g*x^4 + 2/3*a*b*d*x^3 + 1/3*a^2*f*x^3 + 1/2*a^2*e*x
^2 + a^2*d*x
```

3.8.9 Mupad [B] (verification not implemented)

Time = 7.88 (sec) , antiderivative size = 182, normalized size of antiderivative = 0.93

$$\int (d+ex+fx^2+gx^3)\left(a+bx^2+cx^4\right)^2\,dx$$
$$= x^5\left(\frac{d\,b^2}{5}+\frac{2\,a\,f\,b}{5}+\frac{2\,a\,c\,d}{5}\right)+x^6\left(\frac{e\,b^2}{6}+\frac{a\,g\,b}{3}+\frac{a\,c\,e}{3}\right)+x^7\left(\frac{f\,b^2}{7}+\frac{2\,c\,d\,b}{7}+\frac{2\,a\,c\,f}{7}\right)$$
$$+x^8\left(\frac{g\,b^2}{8}+\frac{c\,e\,b}{4}+\frac{a\,c\,g}{4}\right)+x^3\left(\frac{f\,a^2}{3}+\frac{2\,b\,d\,a}{3}\right)+x^4\left(\frac{g\,a^2}{4}+\frac{b\,e\,a}{2}\right)$$
$$+x^9\left(\frac{d\,c^2}{9}+\frac{2\,b\,f\,c}{9}\right)+x^{10}\left(\frac{e\,c^2}{10}+\frac{b\,g\,c}{5}\right)+\frac{a^2\,e\,x^2}{2}+\frac{c^2\,f\,x^{11}}{11}+\frac{c^2\,g\,x^{12}}{12}+a^2\,d\,x$$

input
```
int((a + b*x^2 + c*x^4)^2*(d + e*x + f*x^2 + g*x^3),x)
```

output
```
x^5*((b^2*d)/5 + (2*a*c*d)/5 + (2*a*b*f)/5) + x^6*((b^2*e)/6 + (a*c*e)/3 +
 (a*b*g)/3) + x^7*((b^2*f)/7 + (2*b*c*d)/7 + (2*a*c*f)/7) + x^8*((b^2*g)/8
 + (b*c*e)/4 + (a*c*g)/4) + x^3*((a^2*f)/3 + (2*a*b*d)/3) + x^4*((a^2*g)/4
 + (a*b*e)/2) + x^9*((c^2*d)/9 + (2*b*c*f)/9) + x^10*((c^2*e)/10 + (b*c*g)
/5) + (a^2*e*x^2)/2 + (c^2*f*x^11)/11 + (c^2*g*x^12)/12 + a^2*d*x
```

3.9 $\int (a+bx^2+cx^4)^2 (d+ex+fx^2+gx^3+hx^4) \, dx$

3.9.1 Optimal result

Integrand size = 35, antiderivative size = 234

$$\begin{aligned}
&\int (a+bx^2+cx^4)^2 (d+ex+fx^2+gx^3+hx^4) \, dx \\
&= a^2dx+\frac{1}{2}a^2ex^2+\frac{1}{3}a(2bd+af)x^3+\frac{1}{4}a(2be+ag)x^4+\frac{1}{5}\left(b^2d+2abf+a(2cd+ah)\right)x^5 \\
&\quad+\frac{1}{6}\left(b^2e+2ace+2abg\right)x^6+\frac{1}{7}\left(b^2f+2acf+2b(cd+ah)\right)x^7 \\
&\quad+\frac{1}{8}\left(2bce+b^2g+2acg\right)x^8+\frac{1}{9}\left(c^2d+b^2h+2c(bf+ah)\right)x^9 \\
&\quad+\frac{1}{10}c(ce+2bg)x^{10}+\frac{1}{11}c(cf+2bh)x^{11}+\frac{1}{12}c^2gx^{12}+\frac{1}{13}c^2hx^{13}
\end{aligned}$$

output
```
a^2*d*x+1/2*a^2*e*x^2+1/3*a*(a*f+2*b*d)*x^3+1/4*a*(a*g+2*b*e)*x^4+1/5*(b^2
*d+2*a*b*f+a*(a*h+2*c*d))*x^5+1/6*(2*a*b*g+2*a*c*e+b^2*e)*x^6+1/7*(b^2*f+2
*a*c*f+2*b*(a*h+c*d))*x^7+1/8*(2*a*c*g+b^2*g+2*b*c*e)*x^8+1/9*(c^2*d+b^2*h
+2*c*(a*h+b*f))*x^9+1/10*c*(2*b*g+c*e)*x^10+1/11*c*(2*b*h+c*f)*x^11+1/12*c
^2*g*x^12+1/13*c^2*h*x^13
```

3.9.2 Mathematica [A] (verified)

Time = 0.06 (sec) , antiderivative size = 234, normalized size of antiderivative = 1.00

$$\int \left(a+bx^2+cx^4\right)^2 \left(d+ex+fx^2+gx^3+hx^4\right) \, dx$$
$$= a^2dx+\frac{1}{2}a^2ex^2+\frac{1}{3}a(2bd+af)x^3+\frac{1}{4}a(2be+ag)x^4+\frac{1}{5}\left(b^2d+2acd+2abf+a^2h\right)x^5+\frac{1}{6}\left(b^2e+2ace+2abg\right)x^6+\frac{1}{7}\left(2bcd+b^2f+2acf+2abh\right)x^7+\frac{1}{8}\left(2bce+b^2g+2acg\right)x^8+\frac{1}{9}\left(c^2d+2bcf+b^2h+2ach\right)x^9+\frac{1}{10}c(ce+2bg)x^{10}+\frac{1}{11}c(cf+2bh)x^{11}+\frac{1}{12}c^2gx^{12}+\frac{1}{13}c^2hx^{13}$$

input
```
Integrate[(a + b*x^2 + c*x^4)^2*(d + e*x + f*x^2 + g*x^3 + h*x^4),x]
```

output
```
a^2*d*x + (a^2*e*x^2)/2 + (a*(2*b*d + a*f)*x^3)/3 + (a*(2*b*e + a*g)*x^4)/
4 + ((b^2*d + 2*a*c*d + 2*a*b*f + a^2*h)*x^5)/5 + ((b^2*e + 2*a*c*e + 2*a*
b*g)*x^6)/6 + ((2*b*c*d + b^2*f + 2*a*c*f + 2*a*b*h)*x^7)/7 + ((2*b*c*e +
b^2*g + 2*a*c*g)*x^8)/8 + ((c^2*d + 2*b*c*f + b^2*h + 2*a*c*h)*x^9)/9 + (c
*(c*e + 2*b*g)*x^10)/10 + (c*(c*f + 2*b*h)*x^11)/11 + (c^2*g*x^12)/12 + (c
^2*h*x^13)/13
```

3.9.3 Rubi [A] (verified)

Time = 0.49 (sec) , antiderivative size = 234, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.057$, Rules used = {2200, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \left(a+bx^2+cx^4\right)^2 \left(d+ex+fx^2+gx^3+hx^4\right) \, dx$$

$$\Big\downarrow \ 2200$$

$$\int \left(a^2d+a^2ex+x^8\left(2c(ah+bf)+b^2h+c^2d\right)+x^6\left(2b(ah+cd)+2acf+b^2f\right)+x^4\left(2abf+a(ah+2cd)+b^2d\right)\right.$$

$$\Big\downarrow \ 2009$$

$$a^2dx+\frac{1}{2}a^2ex^2+\frac{1}{9}x^9(2c(ah+bf)+b^2h+c^2d)+\frac{1}{7}x^7(2b(ah+cd)+2acf+b^2f)+\frac{1}{5}x^5(2abf+a(ah+2cd)+b^2d)+\frac{1}{8}x^8(2acg+b^2g+2bce)+\frac{1}{6}x^6(2abg+2ace+b^2e)+\frac{1}{3}ax^3(af+2bd)+\frac{1}{4}ax^4(ag+2be)+\frac{1}{10}cx^{10}(2bg+ce)+\frac{1}{11}cx^{11}(2bh+cf)+\frac{1}{12}c^2gx^{12}+\frac{1}{13}c^2hx^{13}$$

input
```
Int[(a + b*x^2 + c*x^4)^2*(d + e*x + f*x^2 + g*x^3 + h*x^4),x]
```

output
```
a^2*d*x + (a^2*e*x^2)/2 + (a*(2*b*d + a*f)*x^3)/3 + (a*(2*b*e + a*g)*x^4)/
4 + ((b^2*d + 2*a*b*f + a*(2*c*d + a*h))*x^5)/5 + ((b^2*e + 2*a*c*e + 2*a*
b*g)*x^6)/6 + ((b^2*f + 2*a*c*f + 2*b*(c*d + a*h))*x^7)/7 + ((2*b*c*e + b^
2*g + 2*a*c*g)*x^8)/8 + ((c^2*d + b^2*h + 2*c*(b*f + a*h))*x^9)/9 + (c*(c*
e + 2*b*g)*x^10)/10 + (c*(c*f + 2*b*h)*x^11)/11 + (c^2*g*x^12)/12 + (c^2*h
*x^13)/13
```

3.9.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2200
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_.), x_Symbol] :> Int[Expa
ndIntegrand[Px*(a + b*x^2 + c*x^4)^p, x], x] /; FreeQ[{a, b, c}, x] && Poly
Q[Px, x] && IGtQ[p, 0]
```

3.9.4 Maple [A] (verified)

Time = 0.23 (sec) , antiderivative size = 219, normalized size of antiderivative = 0.94

method	result
default	$\frac{c^2hx^{13}}{13}+\frac{c^2gx^{12}}{12}+\frac{(2bch+fc^2)x^{11}}{11}+\frac{(2gbc+ec^2)x^{10}}{10}+\frac{((2ac+b^2)h+2fbc+c^2d)x^9}{9}+\frac{(2ebc+g(2ac+b^2))x^8}{8}+$
norman	$\frac{c^2hx^{13}}{13}+\frac{c^2gx^{12}}{12}+\left(\frac{2}{11}bch+\frac{1}{11}fc^2\right)x^{11}+\left(\frac{1}{5}gbc+\frac{1}{10}ec^2\right)x^{10}+\left(\frac{2}{9}ach+\frac{1}{9}b^2h+\frac{2}{9}fbc+\frac{1}{9}c^2d\right.$
gosper	$\frac{1}{9}x^9b^2h+\frac{1}{5}x^5a^2h+\frac{1}{8}x^8b^2g+\frac{1}{4}x^4ga^2+\frac{2}{9}x^9fbc+\frac{2}{7}x^7acf+\frac{2}{5}x^5abf+\frac{1}{3}x^3fa^2+\frac{1}{7}x^7b^2f+$
risch	$\frac{1}{9}x^9b^2h+\frac{1}{5}x^5a^2h+\frac{1}{8}x^8b^2g+\frac{1}{4}x^4ga^2+\frac{2}{9}x^9fbc+\frac{2}{7}x^7acf+\frac{2}{5}x^5abf+\frac{1}{3}x^3fa^2+\frac{1}{7}x^7b^2f+$
parallelrisch	$\frac{1}{9}x^9b^2h+\frac{1}{5}x^5a^2h+\frac{1}{8}x^8b^2g+\frac{1}{4}x^4ga^2+\frac{2}{9}x^9fbc+\frac{2}{7}x^7acf+\frac{2}{5}x^5abf+\frac{1}{3}x^3fa^2+\frac{1}{7}x^7b^2f+$

input
```
int((c*x^4+b*x^2+a)^2*(h*x^4+g*x^3+f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

output
```
1/13*c^2*h*x^13+1/12*c^2*g*x^12+1/11*(2*b*c*h+c^2*f)*x^11+1/10*(2*b*c*g+c^
2*e)*x^10+1/9*((2*a*c+b^2)*h+2*f*b*c+c^2*d)*x^9+1/8*(2*e*b*c+g*(2*a*c+b^2)
)*x^8+1/7*(2*a*b*h+f*(2*a*c+b^2)+2*b*c*d)*x^7+1/6*(e*(2*a*c+b^2)+2*a*b*g)*
x^6+1/5*(a^2*h+2*a*b*f+d*(2*a*c+b^2))*x^5+1/4*(a^2*g+2*a*b*e)*x^4+1/3*(a^2
*f+2*a*b*d)*x^3+1/2*a^2*e*x^2+a^2*d*x
```

3.9.5 Fricas [A] (verification not implemented)

Time = 0.24 (sec) , antiderivative size = 218, normalized size of antiderivative = 0.93

$$\begin{aligned}\int \left(a+bx^2+cx^4\right)^2\left(d+ex+fx^2+gx^3+hx^4\right)\,dx &= \frac{1}{13}\,c^2hx^{13}+\frac{1}{12}\,c^2gx^{12}+\frac{1}{11}\left(c^2f+2\,bch\right)x^{11}+\frac{1}{10}\left(c^2e+2\,bcg\right)x^{10}\\ &\quad+\frac{1}{9}\left(c^2d+2\,bcf+\left(b^2+2\,ac\right)h\right)x^9+\frac{1}{8}\left(2\,bce+\left(b^2+2\,ac\right)g\right)x^8\\ &\quad+\frac{1}{7}\left(2\,bcd+2\,abh+\left(b^2+2\,ac\right)f\right)x^7+\frac{1}{6}\left(2\,abg+\left(b^2+2\,ac\right)e\right)x^6\\ &\quad+\frac{1}{5}\left(2\,abf+a^2h+\left(b^2+2\,ac\right)d\right)x^5+\frac{1}{2}\,a^2ex^2\\ &\quad+\frac{1}{4}\left(2\,abe+a^2g\right)x^4+a^2dx+\frac{1}{3}\left(2\,abd+a^2f\right)x^3\end{aligned}$$

input
```
integrate((c*x^4+b*x^2+a)^2*(h*x^4+g*x^3+f*x^2+e*x+d),x, algorithm="fricas
")
```

output
```
1/13*c^2*h*x^13 + 1/12*c^2*g*x^12 + 1/11*(c^2*f + 2*b*c*h)*x^11 + 1/10*(c^
2*e + 2*b*c*g)*x^10 + 1/9*(c^2*d + 2*b*c*f + (b^2 + 2*a*c)*h)*x^9 + 1/8*(2
*b*c*e + (b^2 + 2*a*c)*g)*x^8 + 1/7*(2*b*c*d + 2*a*b*h + (b^2 + 2*a*c)*f)*
x^7 + 1/6*(2*a*b*g + (b^2 + 2*a*c)*e)*x^6 + 1/5*(2*a*b*f + a^2*h + (b^2 +
2*a*c)*d)*x^5 + 1/2*a^2*e*x^2 + 1/4*(2*a*b*e + a^2*g)*x^4 + a^2*d*x + 1/3*
(2*a*b*d + a^2*f)*x^3
```

3.9.6 Sympy [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 258, normalized size of antiderivative = 1.10

$$\int (a+bx^2+cx^4)^2 (d+ex+fx^2+gx^3+hx^4)\, dx$$
$$= a^2dx + \frac{a^2ex^2}{2} + \frac{c^2gx^{12}}{12} + \frac{c^2hx^{13}}{13} + x^{11}\cdot\left(\frac{2bch}{11}+\frac{c^2f}{11}\right) + x^{10}\left(\frac{bcg}{5}+\frac{c^2e}{10}\right)$$
$$+ x^9\cdot\left(\frac{2ach}{9}+\frac{b^2h}{9}+\frac{2bcf}{9}+\frac{c^2d}{9}\right) + x^8\left(\frac{acg}{4}+\frac{b^2g}{8}+\frac{bce}{4}\right)$$
$$+ x^7\cdot\left(\frac{2abh}{7}+\frac{2acf}{7}+\frac{b^2f}{7}+\frac{2bcd}{7}\right) + x^6\left(\frac{abg}{3}+\frac{ace}{3}+\frac{b^2e}{6}\right)$$
$$+ x^5\left(\frac{a^2h}{5}+\frac{2abf}{5}+\frac{2acd}{5}+\frac{b^2d}{5}\right) + x^4\left(\frac{a^2g}{4}+\frac{abe}{2}\right) + x^3\left(\frac{a^2f}{3}+\frac{2abd}{3}\right)$$

input
```
integrate((c*x**4+b*x**2+a)**2*(h*x**4+g*x**3+f*x**2+e*x+d),x)
```

output
```
a**2*d*x + a**2*e*x**2/2 + c**2*g*x**12/12 + c**2*h*x**13/13 + x**11*(2*b*
c*h/11 + c**2*f/11) + x**10*(b*c*g/5 + c**2*e/10) + x**9*(2*a*c*h/9 + b**2
*h/9 + 2*b*c*f/9 + c**2*d/9) + x**8*(a*c*g/4 + b**2*g/8 + b*c*e/4) + x**7*
(2*a*b*h/7 + 2*a*c*f/7 + b**2*f/7 + 2*b*c*d/7) + x**6*(a*b*g/3 + a*c*e/3 +
 b**2*e/6) + x**5*(a**2*h/5 + 2*a*b*f/5 + 2*a*c*d/5 + b**2*d/5) + x**4*(a*
*2*g/4 + a*b*e/2) + x**3*(a**2*f/3 + 2*a*b*d/3)
```

3.9.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 218, normalized size of antiderivative = 0.93

$$\int (a+bx^2+cx^4)^2 (d+ex+fx^2+gx^3+hx^4)\, dx$$
$$= \frac{1}{13}c^2hx^{13} + \frac{1}{12}c^2gx^{12} + \frac{1}{11}(c^2f+2\,bch)x^{11} + \frac{1}{10}(c^2e+2\,bcg)x^{10}$$
$$+ \frac{1}{9}(c^2d+2\,bcf+(b^2+2\,ac)h)x^9 + \frac{1}{8}(2\,bce+(b^2+2\,ac)g)x^8$$
$$+ \frac{1}{7}(2\,bcd+2\,abh+(b^2+2\,ac)f)x^7 + \frac{1}{6}(2\,abg+(b^2+2\,ac)e)x^6$$
$$+ \frac{1}{5}(2\,abf+a^2h+(b^2+2\,ac)d)x^5 + \frac{1}{2}a^2ex^2$$
$$+ \frac{1}{4}(2\,abe+a^2g)x^4 + a^2dx + \frac{1}{3}(2\,abd+a^2f)x^3$$

input
```
integrate((c*x^4+b*x^2+a)^2*(h*x^4+g*x^3+f*x^2+e*x+d),x, algorithm="maxima
")
```

output
```
1/13*c^2*h*x^13 + 1/12*c^2*g*x^12 + 1/11*(c^2*f + 2*b*c*h)*x^11 + 1/10*(c^
2*e + 2*b*c*g)*x^10 + 1/9*(c^2*d + 2*b*c*f + (b^2 + 2*a*c)*h)*x^9 + 1/8*(2
*b*c*e + (b^2 + 2*a*c)*g)*x^8 + 1/7*(2*b*c*d + 2*a*b*h + (b^2 + 2*a*c)*f)*
x^7 + 1/6*(2*a*b*g + (b^2 + 2*a*c)*e)*x^6 + 1/5*(2*a*b*f + a^2*h + (b^2 +
2*a*c)*d)*x^5 + 1/2*a^2*e*x^2 + 1/4*(2*a*b*e + a^2*g)*x^4 + a^2*d*x + 1/3*
(2*a*b*d + a^2*f)*x^3
```

3.9.8 Giac [A] (verification not implemented)

Time = 0.32 (sec) , antiderivative size = 253, normalized size of antiderivative = 1.08

$$\begin{aligned}\int (a+bx^2+cx^4)^2 \left(d+ex+fx^2+gx^3+hx^4\right) dx &= \frac{1}{13}c^2hx^{13}+\frac{1}{12}c^2gx^{12}+\frac{1}{11}c^2fx^{11}+\frac{2}{11}bchx^{11}+\frac{1}{10}c^2ex^{10}+\frac{1}{5}bcgx^{10}+\frac{1}{9}c^2dx^9\\ &+\frac{2}{9}bcfx^9+\frac{1}{9}b^2hx^9+\frac{2}{9}achx^9+\frac{1}{4}bcex^8+\frac{1}{8}b^2gx^8+\frac{1}{4}acgx^8+\frac{2}{7}bcdx^7+\frac{1}{7}b^2fx^7\\ &+\frac{2}{7}acfx^7+\frac{2}{7}abhx^7+\frac{1}{6}b^2ex^6+\frac{1}{3}acex^6+\frac{1}{3}abgx^6+\frac{1}{5}b^2dx^5+\frac{2}{5}acdx^5\\ &+\frac{2}{5}abfx^5+\frac{1}{5}a^2hx^5+\frac{1}{2}abex^4+\frac{1}{4}a^2gx^4+\frac{2}{3}abdx^3+\frac{1}{3}a^2fx^3+\frac{1}{2}a^2ex^2+a^2dx\end{aligned}$$

input
```
integrate((c*x^4+b*x^2+a)^2*(h*x^4+g*x^3+f*x^2+e*x+d),x, algorithm="giac")
```

output
```
1/13*c^2*h*x^13 + 1/12*c^2*g*x^12 + 1/11*c^2*f*x^11 + 2/11*b*c*h*x^11 + 1/
10*c^2*e*x^10 + 1/5*b*c*g*x^10 + 1/9*c^2*d*x^9 + 2/9*b*c*f*x^9 + 1/9*b^2*h
*x^9 + 2/9*a*c*h*x^9 + 1/4*b*c*e*x^8 + 1/8*b^2*g*x^8 + 1/4*a*c*g*x^8 + 2/7
*b*c*d*x^7 + 1/7*b^2*f*x^7 + 2/7*a*c*f*x^7 + 2/7*a*b*h*x^7 + 1/6*b^2*e*x^6
 + 1/3*a*c*e*x^6 + 1/3*a*b*g*x^6 + 1/5*b^2*d*x^5 + 2/5*a*c*d*x^5 + 2/5*a*b
*f*x^5 + 1/5*a^2*h*x^5 + 1/2*a*b*e*x^4 + 1/4*a^2*g*x^4 + 2/3*a*b*d*x^3 + 1
/3*a^2*f*x^3 + 1/2*a^2*e*x^2 + a^2*d*x
```

3.9.9 Mupad [B] (verification not implemented)

Time = 0.08 (sec) , antiderivative size = 220, normalized size of antiderivative = 0.94

$$\int (a+bx^2+cx^4)^2 (d+ex+fx^2+gx^3+hx^4) \, dx$$
$$= x^6 \left(\frac{e\,b^2}{6}+\frac{a\,g\,b}{3}+\frac{a\,c\,e}{3}\right)+x^8 \left(\frac{g\,b^2}{8}+\frac{c\,e\,b}{4}+\frac{a\,c\,g}{4}\right)+x^3 \left(\frac{f\,a^2}{3}+\frac{2\,b\,d\,a}{3}\right)$$
$$+x^4 \left(\frac{g\,a^2}{4}+\frac{b\,e\,a}{2}\right)+x^{10} \left(\frac{e\,c^2}{10}+\frac{b\,g\,c}{5}\right)+x^{11} \left(\frac{f\,c^2}{11}+\frac{2\,b\,h\,c}{11}\right)$$
$$+x^5 \left(\frac{h\,a^2}{5}+\frac{2\,f\,a\,b}{5}+\frac{2\,c\,d\,a}{5}+\frac{d\,b^2}{5}\right)+x^7 \left(\frac{b^2\,f}{7}+\frac{2\,b\,c\,d}{7}+\frac{2\,a\,c\,f}{7}+\frac{2\,a\,b\,h}{7}\right)$$
$$+x^9 \left(\frac{h\,b^2}{9}+\frac{2\,f\,b\,c}{9}+\frac{d\,c^2}{9}+\frac{2\,a\,h\,c}{9}\right)+\frac{a^2\,e\,x^2}{2}+\frac{c^2\,g\,x^{12}}{12}+\frac{c^2\,h\,x^{13}}{13}+a^2\,d\,x$$

input
```
int((a + b*x^2 + c*x^4)^2*(d + e*x + f*x^2 + g*x^3 + h*x^4),x)
```

output
```
x^6*((b^2*e)/6 + (a*c*e)/3 + (a*b*g)/3) + x^8*((b^2*g)/8 + (b*c*e)/4 + (a*
c*g)/4) + x^3*((a^2*f)/3 + (2*a*b*d)/3) + x^4*((a^2*g)/4 + (a*b*e)/2) + x^
10*((c^2*e)/10 + (b*c*g)/5) + x^11*((c^2*f)/11 + (2*b*c*h)/11) + x^5*((b^2
*d)/5 + (a^2*h)/5 + (2*a*c*d)/5 + (2*a*b*f)/5) + x^7*((b^2*f)/7 + (2*b*c*d
)/7 + (2*a*c*f)/7 + (2*a*b*h)/7) + x^9*((c^2*d)/9 + (b^2*h)/9 + (2*b*c*f)/
9 + (2*a*c*h)/9) + (a^2*e*x^2)/2 + (c^2*g*x^12)/12 + (c^2*h*x^13)/13 + a^2
*d*x
```

3.10 $\int \frac{d+ex}{4-5x^2+x^4} \, dx$

3.10.1 Optimal result

Integrand size = 18, antiderivative size = 45

$$\int \frac{d+ex}{4-5x^2+x^4} \, dx = -\frac{1}{6}d\text{arctanh}\left(\frac{x}{2}\right)+\frac{1}{3}d\text{arctanh}(x)-\frac{1}{6}e\log\left(1-x^2\right)+\frac{1}{6}e\log\left(4-x^2\right)$$

output
```
-1/6*d*arctanh(1/2*x)+1/3*d*arctanh(x)-1/6*e*ln(-x^2+1)+1/6*e*ln(-x^2+4)
```

3.10.2 Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 50, normalized size of antiderivative = 1.11

$$\int \frac{d+ex}{4-5x^2+x^4} \, dx = \frac{1}{12}(-2(d+e)\log(1-x)+(d+2e)\log(2-x)+2(d-e)\log(1+x) - (d-2e)\log(2+x))$$

input
```
Integrate[(d + e*x)/(4 - 5*x^2 + x^4),x]
```

output
```
(-2*(d + e)*Log[1 - x] + (d + 2*e)*Log[2 - x] + 2*(d - e)*Log[1 + x] - (d
- 2*e)*Log[2 + x])/12
```

3.10.3 Rubi [A] (verified)

Time = 0.24 (sec) , antiderivative size = 50, normalized size of antiderivative = 1.11, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.389, Rules used = {2202, 27, 1406, 220, 1432, 1081, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex}{x^4-5x^2+4}\,dx$$
$$\Big\downarrow \text{ 2202}$$
$$\int \frac{d}{x^4-5x^2+4}dx+\int \frac{ex}{x^4-5x^2+4}dx$$
$$\Big\downarrow \text{ 27}$$
$$d\int \frac{1}{x^4-5x^2+4}dx+e\int \frac{x}{x^4-5x^2+4}dx$$
$$\Big\downarrow \text{ 1406}$$
$$d\left(\frac{1}{3}\int \frac{1}{x^2-4}dx-\frac{1}{3}\int \frac{1}{x^2-1}dx\right)+e\int \frac{x}{x^4-5x^2+4}dx$$
$$\Big\downarrow \text{ 220}$$
$$e\int \frac{x}{x^4-5x^2+4}dx+d\left(\frac{\text{arctanh}(x)}{3}-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)\right)$$
$$\Big\downarrow \text{ 1432}$$
$$\frac{1}{2}e\int \frac{1}{x^4-5x^2+4}dx^2+d\left(\frac{\text{arctanh}(x)}{3}-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)\right)$$
$$\Big\downarrow \text{ 1081}$$
$$\frac{1}{2}e\int \left(\frac{1}{3(1-x^2)}-\frac{1}{3(4-x^2)}\right)dx^2+d\left(\frac{\text{arctanh}(x)}{3}-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)\right)$$
$$\Big\downarrow \text{ 2009}$$
$$d\left(\frac{\text{arctanh}(x)}{3}-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)\right)+\frac{1}{2}e\left(\frac{1}{3}\log\left(4-x^2\right)-\frac{1}{3}\log\left(1-x^2\right)\right)$$

input
```
Int[(d + e*x)/(4 - 5*x^2 + x^4),x]
```

output
```
d*(-1/6*ArcTanh[x/2] + ArcTanh[x]/3) + (e*(-1/3*Log[1 - x^2] + Log[4 - x^2
]/3))/2
```

3.10.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 220
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[b, 2])^(-
1))*ArcTanh[Rt[b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] &&
 (LtQ[a, 0] || GtQ[b, 0])
```

rule 1081
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> With[{q = Rt[b^2
- 4*a*c, 2]}, Simp[c   Int[ExpandIntegrand[1/((b/2 - q/2 + c*x)*(b/2 + q/2
+ c*x)), x], x], x]] /; FreeQ[{a, b, c}, x] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 1406
```
Int[((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(-1), x_Symbol] :> With[{q = Rt[b^
2 - 4*a*c, 2]}, Simp[c/q   Int[1/(b/2 - q/2 + c*x^2), x], x] - Simp[c/q   I
nt[1/(b/2 + q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c}, x] && NeQ[b^2 - 4*a*c
, 0] && PosQ[b^2 - 4*a*c]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.10.4 Maple [A] (verified)

Time = 0.05 (sec) , antiderivative size = 50, normalized size of antiderivative = 1.11

method	result	size
default	$\left(-\frac{d}{12}+\frac{e}{6}\right)\ln\left(x+2\right)+\left(\frac{d}{6}-\frac{e}{6}\right)\ln\left(x+1\right)+\left(-\frac{d}{6}-\frac{e}{6}\right)\ln\left(x-1\right)+\left(\frac{d}{12}+\frac{e}{6}\right)\ln\left(x-2\right)$	50
norman	$\left(-\frac{d}{12}+\frac{e}{6}\right)\ln\left(x+2\right)+\left(\frac{d}{6}-\frac{e}{6}\right)\ln\left(x+1\right)+\left(-\frac{d}{6}-\frac{e}{6}\right)\ln\left(x-1\right)+\left(\frac{d}{12}+\frac{e}{6}\right)\ln\left(x-2\right)$	50
parallelrisch	$\frac{\ln(x-2)d}{12}+\frac{\ln(x-2)e}{6}-\frac{\ln(x-1)d}{6}-\frac{\ln(x-1)e}{6}+\frac{\ln(x+1)d}{6}-\frac{\ln(x+1)e}{6}-\frac{\ln(x+2)d}{12}+\frac{\ln(x+2)e}{6}$	58
risch	$\frac{\ln(2-x)d}{12}+\frac{\ln(2-x)e}{6}-\frac{\ln(x+2)d}{12}+\frac{\ln(x+2)e}{6}-\frac{\ln(1-x)d}{6}-\frac{\ln(1-x)e}{6}+\frac{\ln(x+1)d}{6}-\frac{\ln(x+1)e}{6}$	66

input
```
int((e*x+d)/(x^4-5*x^2+4),x,method=_RETURNVERBOSE)
```

output
```
(-1/12*d+1/6*e)*ln(x+2)+(1/6*d-1/6*e)*ln(x+1)+(-1/6*d-1/6*e)*ln(x-1)+(1/12
*d+1/6*e)*ln(x-2)
```

3.10.5 Fricas [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 43, normalized size of antiderivative = 0.96

$$\int \frac{d+ex}{4-5x^2+x^4}\,dx = -\frac{1}{12}\left(d-2\,e\right)\log\left(x+2\right)+\frac{1}{6}\left(d-e\right)\log\left(x+1\right) - \frac{1}{6}\left(d+e\right)\log\left(x-1\right)+\frac{1}{12}\left(d+2\,e\right)\log\left(x-2\right)$$

input
```
integrate((e*x+d)/(x^4-5*x^2+4),x, algorithm="fricas")
```

output
```
-1/12*(d - 2*e)*log(x + 2) + 1/6*(d - e)*log(x + 1) - 1/6*(d + e)*log(x -
1) + 1/12*(d + 2*e)*log(x - 2)
```

3.10.6 Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 515 vs. 2(34) = 68.

Time = 1.81 (sec) , antiderivative size = 515, normalized size of antiderivative = 11.44

$$\int \frac{d+ex}{4-5x^2+x^4}\,dx =$$

$$-\frac{(d-2e)\log\left(x+\frac{-35d^4e+\frac{51d^4(d-2e)}{2}-180d^2e^3-90d^2e^2(d-2e)+41d^2e(d-2e)^2-\frac{15d^2(d-2e)^3}{2}+320e^5-96e^4(d-2e)-80e^3(d-2e)^2+24}{9d^5-160d^3e^2+256de^4}\right.}{12}$$

$$+\frac{(d-e)\log\left(x+\frac{-35d^4e-51d^4(d-e)-180d^2e^3+180d^2e^2(d-e)+164d^2e(d-e)^2+60d^2(d-e)^3+320e^5+192e^4(d-e)-320e^3(d-e)^2-192e}{9d^5-160d^3e^2+256de^4}\right.}{6}$$

$$-\frac{(d+e)\log\left(x+\frac{-35d^4e+51d^4(d+e)-180d^2e^3-180d^2e^2(d+e)+164d^2e(d+e)^2-60d^2(d+e)^3+320e^5-192e^4(d+e)-320e^3(d+e)^2+192e}{9d^5-160d^3e^2+256de^4}\right.}{6}$$

$$+\frac{(d+2e)\log\left(x+\frac{-35d^4e-\frac{51d^4(d+2e)}{2}-180d^2e^3+90d^2e^2(d+2e)+41d^2e(d+2e)^2+\frac{15d^2(d+2e)^3}{2}+320e^5+96e^4(d+2e)-80e^3(d+2e)^2-24}{9d^5-160d^3e^2+256de^4}\right.}{12}$$

input
```
integrate((e*x+d)/(x**4-5*x**2+4),x)
```

output
```
-(d - 2*e)*log(x + (-35*d**4*e + 51*d**4*(d - 2*e)/2 - 180*d**2*e**3 - 90*
d**2*e**2*(d - 2*e) + 41*d**2*e*(d - 2*e)**2 - 15*d**2*(d - 2*e)**3/2 + 32
0*e**5 - 96*e**4*(d - 2*e) - 80*e**3*(d - 2*e)**2 + 24*e**2*(d - 2*e)**3)/
(9*d**5 - 160*d**3*e**2 + 256*d*e**4))/12 + (d - e)*log(x + (-35*d**4*e -
51*d**4*(d - e) - 180*d**2*e**3 + 180*d**2*e**2*(d - e) + 164*d**2*e*(d -
e)**2 + 60*d**2*(d - e)**3 + 320*e**5 + 192*e**4*(d - e) - 320*e**3*(d - e
)**2 - 192*e**2*(d - e)**3)/(9*d**5 - 160*d**3*e**2 + 256*d*e**4))/6 - (d
+ e)*log(x + (-35*d**4*e + 51*d**4*(d + e) - 180*d**2*e**3 - 180*d**2*e**2
*(d + e) + 164*d**2*e*(d + e)**2 - 60*d**2*(d + e)**3 + 320*e**5 - 192*e**
4*(d + e) - 320*e**3*(d + e)**2 + 192*e**2*(d + e)**3)/(9*d**5 - 160*d**3*
e**2 + 256*d*e**4))/6 + (d + 2*e)*log(x + (-35*d**4*e - 51*d**4*(d + 2*e)/
2 - 180*d**2*e**3 + 90*d**2*e**2*(d + 2*e) + 41*d**2*e*(d + 2*e)**2 + 15*d
**2*(d + 2*e)**3/2 + 320*e**5 + 96*e**4*(d + 2*e) - 80*e**3*(d + 2*e)**2 -
 24*e**2*(d + 2*e)**3)/(9*d**5 - 160*d**3*e**2 + 256*d*e**4))/12
```

3.10.7 Maxima [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 43, normalized size of antiderivative = 0.96

$$\int \frac{d+ex}{4-5x^2+x^4}\,dx = -\frac{1}{12}\,(d-2\,e)\log{(x+2)} + \frac{1}{6}\,(d-e)\log{(x+1)} - \frac{1}{6}\,(d+e)\log{(x-1)} + \frac{1}{12}\,(d+2\,e)\log{(x-2)}$$

input
```
integrate((e*x+d)/(x^4-5*x^2+4),x, algorithm="maxima")
```

output
```
-1/12*(d - 2*e)*log(x + 2) + 1/6*(d - e)*log(x + 1) - 1/6*(d + e)*log(x -
1) + 1/12*(d + 2*e)*log(x - 2)
```

3.10.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 47, normalized size of antiderivative = 1.04

$$\int \frac{d+ex}{4-5x^2+x^4}\,dx = -\frac{1}{12}\,(d-2\,e)\log{(|x+2|)} + \frac{1}{6}\,(d-e)\log{(|x+1|)} - \frac{1}{6}\,(d+e)\log{(|x-1|)} + \frac{1}{12}\,(d+2\,e)\log{(|x-2|)}$$

input
```
integrate((e*x+d)/(x^4-5*x^2+4),x, algorithm="giac")
```

output
```
-1/12*(d - 2*e)*log(abs(x + 2)) + 1/6*(d - e)*log(abs(x + 1)) - 1/6*(d + e
)*log(abs(x - 1)) + 1/12*(d + 2*e)*log(abs(x - 2))
```

3.10.9 Mupad [B] (verification not implemented)

Time = 7.81 (sec) , antiderivative size = 51, normalized size of antiderivative = 1.13

$$\int \frac{d+ex}{4-5x^2+x^4}\,dx = \ln{(x+1)}\,\left(\frac{d}{6}-\frac{e}{6}\right) - \ln{(x-1)}\,\left(\frac{d}{6}+\frac{e}{6}\right) + \ln{(x-2)}\,\left(\frac{d}{12}+\frac{e}{6}\right) - \ln{(x+2)}\,\left(\frac{d}{12}-\frac{e}{6}\right)$$

input
```
int((d + e*x)/(x^4 - 5*x^2 + 4),x)
```

output
```
log(x + 1)*(d/6 - e/6) - log(x - 1)*(d/6 + e/6) + log(x - 2)*(d/12 + e/6)
- log(x + 2)*(d/12 - e/6)
```

3.10. $\int \frac{d+ex}{4-5x^2+x^4}\,dx$

3.11 $\int \frac{d+ex+fx^2}{4-5x^2+x^4} dx$

3.11.1 Optimal result

Integrand size = 23, antiderivative size = 51

$$\int \frac{d+ex+fx^2}{4-5x^2+x^4} dx = -\frac{1}{6}(d+4f)\text{arctanh}\left(\frac{x}{2}\right)+\frac{1}{3}(d+f)\text{arctanh}(x) -\frac{1}{6}e\log\left(1-x^2\right)+\frac{1}{6}e\log\left(4-x^2\right)$$

output
```
-1/6*(d+4*f)*arctanh(1/2*x)+1/3*(d+f)*arctanh(x)-1/6*e*ln(-x^2+1)+1/6*e*ln
(-x^2+4)
```

3.11.2 Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 58, normalized size of antiderivative = 1.14

$$\int \frac{d+ex+fx^2}{4-5x^2+x^4} dx = \frac{1}{12}(-2(d+e+f)\log(1-x)+(d+2e+4f)\log(2-x) +2(d-e+f)\log(1+x)-(d-2e+4f)\log(2+x))$$

input
```
Integrate[(d + e*x + f*x^2)/(4 - 5*x^2 + x^4),x]
```

output
```
(-2*(d + e + f)*Log[1 - x] + (d + 2*e + 4*f)*Log[2 - x] + 2*(d - e + f)*Lo
g[1 + x] - (d - 2*e + 4*f)*Log[2 + x])/12
```

3.11.3 Rubi [A] (verified)

Time = 0.26 (sec) , antiderivative size = 55, normalized size of antiderivative = 1.08, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.304, Rules used = {2202, 27, 1432, 1081, 1480, 220, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2}{x^4-5x^2+4} \, dx$$

$$\Big\downarrow \text{ 2202}$$

$$\int \frac{fx^2+d}{x^4-5x^2+4}dx+\int \frac{ex}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{ 27}$$

$$\int \frac{fx^2+d}{x^4-5x^2+4}dx+e\int \frac{x}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{ 1432}$$

$$\int \frac{fx^2+d}{x^4-5x^2+4}dx+\frac{1}{2}e\int \frac{1}{x^4-5x^2+4}dx^2$$

$$\Big\downarrow \text{ 1081}$$

$$\int \frac{fx^2+d}{x^4-5x^2+4}dx+\frac{1}{2}e\int \left(\frac{1}{3\left(1-x^2\right)}-\frac{1}{3\left(4-x^2\right)}\right)dx^2$$

$$\Big\downarrow \text{ 1480}$$

$$\frac{1}{3}(d+4f)\int \frac{1}{x^2-4}dx-\frac{1}{3}(d+f)\int \frac{1}{x^2-1}dx+\frac{1}{2}e\int \left(\frac{1}{3\left(1-x^2\right)}-\frac{1}{3\left(4-x^2\right)}\right)dx^2$$

$$\Big\downarrow \text{ 220}$$

$$\frac{1}{2}e\int \left(\frac{1}{3\left(1-x^2\right)}-\frac{1}{3\left(4-x^2\right)}\right)dx^2-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(d+4f)+\frac{1}{3}\text{arctanh}(x)(d+f)$$

$$\Big\downarrow \text{ 2009}$$

$$-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(d+4f)+\frac{1}{3}\text{arctanh}(x)(d+f)+\frac{1}{2}e\left(\frac{1}{3}\log\left(4-x^2\right)-\frac{1}{3}\log\left(1-x^2\right)\right)$$

input
```
Int[(d + e*x + f*x^2)/(4 - 5*x^2 + x^4),x]
```

output
```
-1/6*((d + 4*f)*ArcTanh[x/2]) + ((d + f)*ArcTanh[x])/3 + (e*(-1/3*Log[1 -
x^2] + Log[4 - x^2]/3))/2
```

3.11.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 220
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[b, 2])^(-
1))*ArcTanh[Rt[b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] &&
 (LtQ[a, 0] || GtQ[b, 0])
```

rule 1081
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> With[{q = Rt[b^2
- 4*a*c, 2]}, Simp[c   Int[ExpandIntegrand[1/((b/2 - q/2 + c*x)*(b/2 + q/2
+ c*x)), x], x], x]] /; FreeQ[{a, b, c}, x] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.11.4 Maple [A] (verified)

Time = 0.06 (sec) , antiderivative size = 62, normalized size of antiderivative = 1.22

method	result
default	$\left(-\frac{d}{12}+\frac{e}{6}-\frac{f}{3}\right)\ln\left(x+2\right)+\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}\right)\ln\left(x+1\right)+\left(-\frac{d}{6}-\frac{e}{6}-\frac{f}{6}\right)\ln\left(x-1\right)+\left(\frac{d}{12}+\frac{e}{6}+\right.$
norman	$\left(-\frac{d}{12}+\frac{e}{6}-\frac{f}{3}\right)\ln\left(x+2\right)+\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}\right)\ln\left(x+1\right)+\left(-\frac{d}{6}-\frac{e}{6}-\frac{f}{6}\right)\ln\left(x-1\right)+\left(\frac{d}{12}+\frac{e}{6}+\right.$
parallelrisch	$\frac{\ln(x-2)d}{12}+\frac{\ln(x-2)e}{6}+\frac{\ln(x-2)f}{3}-\frac{\ln(x-1)d}{6}-\frac{\ln(x-1)e}{6}-\frac{\ln(x-1)f}{6}+\frac{\ln(x+1)d}{6}-\frac{\ln(x+1)e}{6}+\frac{\ln(x+1)f}{6}-$
risch	$-\frac{\ln(1-x)d}{6}-\frac{\ln(1-x)e}{6}-\frac{\ln(1-x)f}{6}+\frac{\ln(x+1)d}{6}-\frac{\ln(x+1)e}{6}+\frac{\ln(x+1)f}{6}+\frac{\ln(2-x)d}{12}+\frac{\ln(2-x)e}{6}+\frac{\ln(2-x)f}{3}$

input
```
int((f*x^2+e*x+d)/(x^4-5*x^2+4),x,method=_RETURNVERBOSE)
```

output
```
(-1/12*d+1/6*e-1/3*f)*ln(x+2)+(1/6*d-1/6*e+1/6*f)*ln(x+1)+(-1/6*d-1/6*e-1/
6*f)*ln(x-1)+(1/12*d+1/6*e+1/3*f)*ln(x-2)
```

3.11.5 Fricas [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 51, normalized size of antiderivative = 1.00

$$\int \frac{d+ex+fx^2}{4-5x^2+x^4}\,dx = -\frac{1}{12}\left(d-2\,e+4\,f\right)\log\left(x+2\right)+\frac{1}{6}\left(d-e+f\right)\log\left(x+1\right) - \frac{1}{6}\left(d+e+f\right)\log\left(x-1\right)+\frac{1}{12}\left(d+2\,e+4\,f\right)\log\left(x-2\right)$$

input
```
integrate((f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="fricas")
```

output
```
-1/12*(d - 2*e + 4*f)*log(x + 2) + 1/6*(d - e + f)*log(x + 1) - 1/6*(d + e
 + f)*log(x - 1) + 1/12*(d + 2*e + 4*f)*log(x - 2)
```

3.11.6 Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 2195 vs. $2(44) = 88$.

Time = 95.81 (sec) , antiderivative size = 2195, normalized size of antiderivative = 43.04

$$\int \frac{d+ex+fx^2}{4-5x^2+x^4}\,dx = \text{Too large to display}$$

input
```
integrate((f*x**2+e*x+d)/(x**4-5*x**2+4),x)
```

output
```
-(d - 2*e + 4*f)*log(x + (-35*d**5*e + 51*d**5*(d - 2*e + 4*f)/2 - 820*d**
4*e*f + 90*d**4*f*(d - 2*e + 4*f) - 180*d**3*e**3 - 90*d**3*e**2*(d - 2*e
+ 4*f) - 4100*d**3*e*f**2 + 41*d**3*e*(d - 2*e + 4*f)**2 + 42*d**3*f**2*(d
 - 2*e + 4*f) - 15*d**3*(d - 2*e + 4*f)**3/2 - 432*d**2*e**2*f*(d - 2*e +
4*f) - 8000*d**2*e*f**3 + 240*d**2*e*f*(d - 2*e + 4*f)**2 - 240*d**2*f**3*
(d - 2*e + 4*f) - 12*d**2*f*(d - 2*e + 4*f)**3 + 320*d*e**5 - 96*d*e**4*(d
 - 2*e + 4*f) + 720*d*e**3*f**2 - 80*d*e**3*(d - 2*e + 4*f)**2 - 1080*d*e*
*2*f**2*(d - 2*e + 4*f) + 24*d*e**2*(d - 2*e + 4*f)**3 - 6400*d*e*f**4 + 4
92*d*e*f**2*(d - 2*e + 4*f)**2 - 576*d*f**4*(d - 2*e + 4*f) + 30*d*f**2*(d
 - 2*e + 4*f)**3 + 512*e**5*f - 128*e**3*f*(d - 2*e + 4*f)**2 - 576*e**2*f
**3*(d - 2*e + 4*f) - 1472*e*f**5 + 320*e*f**3*(d - 2*e + 4*f)**2 - 480*f*
*5*(d - 2*e + 4*f) + 48*f**3*(d - 2*e + 4*f)**3)/(9*d**6 + 45*d**5*f - 160
*d**4*e**2 - 36*d**4*f**2 - 1312*d**3*e**2*f - 360*d**3*f**3 + 256*d**2*e*
*4 - 3840*d**2*e**2*f**2 - 144*d**2*f**4 + 1280*d*e**4*f - 5248*d*e**2*f**
3 + 720*d*f**5 + 1024*e**4*f**2 - 2560*e**2*f**4 + 576*f**6))/12 + (d - e
+ f)*log(x + (-35*d**5*e - 51*d**5*(d - e + f) - 820*d**4*e*f - 180*d**4*f
*(d - e + f) - 180*d**3*e**3 + 180*d**3*e**2*(d - e + f) - 4100*d**3*e*f**
2 + 164*d**3*e*(d - e + f)**2 - 84*d**3*f**2*(d - e + f) + 60*d**3*(d - e
+ f)**3 + 864*d**2*e**2*f*(d - e + f) - 8000*d**2*e*f**3 + 960*d**2*e*f*(d
 - e + f)**2 + 480*d**2*f**3*(d - e + f) + 96*d**2*f*(d - e + f)**3 + 3...
```

3.11.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 51, normalized size of antiderivative = 1.00

$$\int \frac{d+ex+fx^2}{4-5x^2+x^4}\,dx = -\frac{1}{12}\,(d-2\,e+4\,f)\log{(x+2)}+\frac{1}{6}\,(d-e+f)\log{(x+1)} - \frac{1}{6}\,(d+e+f)\log{(x-1)}+\frac{1}{12}\,(d+2\,e+4\,f)\log{(x-2)}$$

input
```
integrate((f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="maxima")
```

output
```
-1/12*(d - 2*e + 4*f)*log(x + 2) + 1/6*(d - e + f)*log(x + 1) - 1/6*(d + e
 + f)*log(x - 1) + 1/12*(d + 2*e + 4*f)*log(x - 2)
```

3.11.8 Giac [A] (verification not implemented)

Time = 0.36 (sec) , antiderivative size = 55, normalized size of antiderivative = 1.08

$$\int \frac{d+ex+fx^2}{4-5x^2+x^4}\,dx = -\frac{1}{12}\,(d-2\,e+4\,f)\log\left(|x+2|\right)+\frac{1}{6}\,(d-e+f)\log\left(|x+1|\right) - \frac{1}{6}\,(d+e+f)\log\left(|x-1|\right)+\frac{1}{12}\,(d+2\,e+4\,f)\log\left(|x-2|\right)$$

input
```
integrate((f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="giac")
```

output
```
-1/12*(d - 2*e + 4*f)*log(abs(x + 2)) + 1/6*(d - e + f)*log(abs(x + 1)) -
1/6*(d + e + f)*log(abs(x - 1)) + 1/12*(d + 2*e + 4*f)*log(abs(x - 2))
```

3.11.9 Mupad [B] (verification not implemented)

Time = 7.76 (sec) , antiderivative size = 63, normalized size of antiderivative = 1.24

$$\int \frac{d+ex+fx^2}{4-5x^2+x^4}\,dx = \ln(x+1)\,\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}\right)-\ln(x-1)\,\left(\frac{d}{6}+\frac{e}{6}+\frac{f}{6}\right) + \ln(x-2)\,\left(\frac{d}{12}+\frac{e}{6}+\frac{f}{3}\right)-\ln(x+2)\,\left(\frac{d}{12}-\frac{e}{6}+\frac{f}{3}\right)$$

input
```
int((d + e*x + f*x^2)/(x^4 - 5*x^2 + 4),x)
```

output
```
log(x + 1)*(d/6 - e/6 + f/6) - log(x - 1)*(d/6 + e/6 + f/6) + log(x - 2)*(
d/12 + e/6 + f/3) - log(x + 2)*(d/12 - e/6 + f/3)
```

3.12 $\int \frac{d+ex+fx^2+gx^3}{4-5x^2+x^4}\,dx$

3.12.1 Optimal result

Integrand size = 28, antiderivative size = 57

$$\int \frac{d+ex+fx^2+gx^3}{4-5x^2+x^4}\,dx = -\frac{1}{6}(d+4f)\text{arctanh}\left(\frac{x}{2}\right)+\frac{1}{3}(d+f)\text{arctanh}(x) -\frac{1}{6}(e+g)\log\left(1-x^2\right)+\frac{1}{6}(e+4g)\log\left(4-x^2\right)$$

output
```
-1/6*(d+4*f)*arctanh(1/2*x)+1/3*(d+f)*arctanh(x)-1/6*(e+g)*ln(-x^2+1)+1/6*
(e+4*g)*ln(-x^2+4)
```

3.12.2 Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 68, normalized size of antiderivative = 1.19

$$\int \frac{d+ex+fx^2+gx^3}{4-5x^2+x^4}\,dx = \frac{1}{12}(-2(d+e+f+g)\log(1-x)+(d+2e+4f+8g)\log(2-x) +2(d-e+f-g)\log(1+x)-(d-2e+4f-8g)\log(2+x))$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3)/(4 - 5*x^2 + x^4),x]
```

output
```
(-2*(d + e + f + g)*Log[1 - x] + (d + 2*e + 4*f + 8*g)*Log[2 - x] + 2*(d -
 e + f - g)*Log[1 + x] - (d - 2*e + 4*f - 8*g)*Log[2 + x])/12
```

3.12.3 Rubi [A] (verified)

Time = 0.29 (sec) , antiderivative size = 62, normalized size of antiderivative = 1.09, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.214, Rules used = {2202, 1480, 220, 1576, 1141, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3}{x^4-5x^2+4}\,dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{fx^2+d}{x^4-5x^2+4}dx+\int \frac{x(gx^2+e)}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{1480}$$

$$\frac{1}{3}(d+4f)\int \frac{1}{x^2-4}dx-\frac{1}{3}(d+f)\int \frac{1}{x^2-1}dx+\int \frac{x(gx^2+e)}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{220}$$

$$\int \frac{x(gx^2+e)}{x^4-5x^2+4}dx-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(d+4f)+\frac{1}{3}\text{arctanh}(x)(d+f)$$

$$\Big\downarrow \text{1576}$$

$$\frac{1}{2}\int \frac{gx^2+e}{x^4-5x^2+4}dx^2-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(d+4f)+\frac{1}{3}\text{arctanh}(x)(d+f)$$

$$\Big\downarrow \text{1141}$$

$$\frac{1}{2}\int \left(\frac{e+g}{3(1-x^2)}-\frac{e+4g}{3(4-x^2)}\right)dx^2-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(d+4f)+\frac{1}{3}\text{arctanh}(x)(d+f)$$

$$\Big\downarrow \text{2009}$$

$$-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(d+4f)+\frac{1}{3}\text{arctanh}(x)(d+f)+\frac{1}{2}\left(\frac{1}{3}(e+4g)\log(4-x^2)-\frac{1}{3}(e+g)\log(1-x^2)\right)$$

input
```
Int[(d + e*x + f*x^2 + g*x^3)/(4 - 5*x^2 + x^4),x]
```

output
```
-1/6*((d + 4*f)*ArcTanh[x/2]) + ((d + f)*ArcTanh[x])/3 + (-1/3*((e + g)*Lo
g[1 - x^2]) + ((e + 4*g)*Log[4 - x^2])/3)/2
```

3.12.3.1 Defintions of rubi rules used

rule 220
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[b, 2])^(-
1))*ArcTanh[Rt[b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] &&
 (LtQ[a, 0] || GtQ[b, 0])
```

rule 1141
```
Int[((d_.) + (e_.)*(x_))^(m_.)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_
Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[1/c^p   Int[ExpandIntegrand[
(d + e*x)^m*(b/2 - q/2 + c*x)^p*(b/2 + q/2 + c*x)^p, x], x], x] /; EqQ[p, -
1] ||  !FractionalPowerFactorQ[q]] /; FreeQ[{a, b, c, d, e}, x] && ILtQ[p,
0] && IntegerQ[m] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.12.4 Maple [A] (verified)

Time = 0.07 (sec) , antiderivative size = 74, normalized size of antiderivative = 1.30

method	result
default	$\left(-\frac{d}{12}+\frac{e}{6}-\frac{f}{3}+\frac{2g}{3}\right)\ln\left(x+2\right)+\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}\right)\ln\left(x+1\right)+\left(-\frac{d}{6}-\frac{e}{6}-\frac{f}{6}-\frac{g}{6}\right)\ln\left(x-1\right)$
norman	$\left(-\frac{d}{12}+\frac{e}{6}-\frac{f}{3}+\frac{2g}{3}\right)\ln\left(x+2\right)+\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}\right)\ln\left(x+1\right)+\left(-\frac{d}{6}-\frac{e}{6}-\frac{f}{6}-\frac{g}{6}\right)\ln\left(x-1\right)$
parallelrisch	$\frac{\ln(x-2)d}{12}+\frac{\ln(x-2)e}{6}+\frac{\ln(x-2)f}{3}+\frac{2\ln(x-2)g}{3}-\frac{\ln(x-1)d}{6}-\frac{\ln(x-1)e}{6}-\frac{\ln(x-1)f}{6}-\frac{\ln(x-1)g}{6}+\frac{\ln(x+1)d}{6}$
risch	$-\frac{\ln(x+2)d}{12}+\frac{\ln(x+2)e}{6}-\frac{\ln(x+2)f}{3}+\frac{2\ln(x+2)g}{3}+\frac{\ln(2-x)d}{12}+\frac{\ln(2-x)e}{6}+\frac{\ln(2-x)f}{3}+\frac{2\ln(2-x)g}{3}+\frac{\ln(x+1)}{6}$

input
```
int((g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x,method=_RETURNVERBOSE)
```

output
```
(-1/12*d+1/6*e-1/3*f+2/3*g)*ln(x+2)+(1/6*d-1/6*e+1/6*f-1/6*g)*ln(x+1)+(-1/
6*d-1/6*e-1/6*f-1/6*g)*ln(x-1)+(1/12*d+1/6*e+1/3*f+2/3*g)*ln(x-2)
```

3.12.5 Fricas [A] (verification not implemented)

Time = 0.42 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.07

$$\int \frac{d+ex+fx^2+gx^3}{4-5x^2+x^4}\,dx = -\frac{1}{12}\left(d-2\,e+4\,f-8\,g\right)\log\left(x+2\right) + \frac{1}{6}\left(d-e+f-g\right)\log\left(x+1\right) - \frac{1}{6}\left(d+e+f+g\right)\log\left(x-1\right) + \frac{1}{12}\left(d+2\,e+4\,f+8\,g\right)\log\left(x-2\right)$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="fricas")
```

output
```
-1/12*(d - 2*e + 4*f - 8*g)*log(x + 2) + 1/6*(d - e + f - g)*log(x + 1) -
1/6*(d + e + f + g)*log(x - 1) + 1/12*(d + 2*e + 4*f + 8*g)*log(x - 2)
```

3.12.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3}{4-5x^2+x^4}\,dx = \text{Timed out}$$

input
```
integrate((g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4),x)
```

output
```
Timed out
```

3.12.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 61, normalized size of antiderivative = 1.07

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3}{4-5x^2+x^4}\,dx = &-\frac{1}{12}\,(d-2\,e+4\,f-8\,g)\log{(x+2)}\\ &+\frac{1}{6}\,(d-e+f-g)\log{(x+1)}-\frac{1}{6}\,(d+e+f+g)\log{(x-1)}\\ &+\frac{1}{12}\,(d+2\,e+4\,f+8\,g)\log{(x-2)}\end{aligned}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="maxima")
```

output
```
-1/12*(d - 2*e + 4*f - 8*g)*log(x + 2) + 1/6*(d - e + f - g)*log(x + 1) -
1/6*(d + e + f + g)*log(x - 1) + 1/12*(d + 2*e + 4*f + 8*g)*log(x - 2)
```

3.12.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 65, normalized size of antiderivative = 1.14

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3}{4-5x^2+x^4}\,dx = &-\frac{1}{12}\,(d-2\,e+4\,f-8\,g)\log{(|x+2|)}\\ &+\frac{1}{6}\,(d-e+f-g)\log{(|x+1|)}\\ &-\frac{1}{6}\,(d+e+f+g)\log{(|x-1|)}\\ &+\frac{1}{12}\,(d+2\,e+4\,f+8\,g)\log{(|x-2|)}\end{aligned}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="giac")
```

output
```
-1/12*(d - 2*e + 4*f - 8*g)*log(abs(x + 2)) + 1/6*(d - e + f - g)*log(abs(
x + 1)) - 1/6*(d + e + f + g)*log(abs(x - 1)) + 1/12*(d + 2*e + 4*f + 8*g)
*log(abs(x - 2))
```

3.12.9 Mupad [B] (verification not implemented)

Time = 7.80 (sec) , antiderivative size = 75, normalized size of antiderivative = 1.32

$$\int \frac{d+ex+fx^2+gx^3}{4-5x^2+x^4}\,dx = \ln(x+1)\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}\right)-\ln(x-1)\left(\frac{d}{6}+\frac{e}{6}+\frac{f}{6}+\frac{g}{6}\right) +\ln(x-2)\left(\frac{d}{12}+\frac{e}{6}+\frac{f}{3}+\frac{2g}{3}\right)-\ln(x+2)\left(\frac{d}{12}-\frac{e}{6}+\frac{f}{3}-\frac{2g}{3}\right)$$

input
```
int((d + e*x + f*x^2 + g*x^3)/(x^4 - 5*x^2 + 4),x)
```

output
```
log(x + 1)*(d/6 - e/6 + f/6 - g/6) - log(x - 1)*(d/6 + e/6 + f/6 + g/6) +
log(x - 2)*(d/12 + e/6 + f/3 + (2*g)/3) - log(x + 2)*(d/12 - e/6 + f/3 - (
2*g)/3)
```

3.13 $\int \frac{d+ex+fx^2+gx^3+hx^4}{4-5x^2+x^4} dx$

3.13.1 Optimal result

Integrand size = 33, antiderivative size = 64

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{4-5x^2+x^4}\,dx = hx-\frac{1}{6}(d+4f+16h)\text{arctanh}\left(\frac{x}{2}\right)+\frac{1}{3}(d+f+h)\text{arctanh}(x) - \frac{1}{6}(e+g)\log\left(1-x^2\right)+\frac{1}{6}(e+4g)\log\left(4-x^2\right)$$

output
```
h*x-1/6*(d+4*f+16*h)*arctanh(1/2*x)+1/3*(d+f+h)*arctanh(x)-1/6*(e+g)*ln(-x
^2+1)+1/6*(e+4*g)*ln(-x^2+4)
```

3.13.2 Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 81, normalized size of antiderivative = 1.27

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{4-5x^2+x^4}\,dx = \frac{1}{12}(12hx-2(d+e+f+g+h)\log(1-x) + (d+2(e+2f+4g+8h))\log(2-x) + 2(d-e+f-g+h)\log(1+x) - (d-2e+4f-8g+16h)\log(2+x))$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4)/(4 - 5*x^2 + x^4),x]
```

output
```
(12*h*x - 2*(d + e + f + g + h)*Log[1 - x] + (d + 2*(e + 2*f + 4*g + 8*h))
*Log[2 - x] + 2*(d - e + f - g + h)*Log[1 + x] - (d - 2*e + 4*f - 8*g + 16
*h)*Log[2 + x])/12
```

3.13.3 Rubi [A] (verified)

Time = 0.40 (sec) , antiderivative size = 69, normalized size of antiderivative = 1.08, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.182, Rules used = {2202, 1576, 1141, 2009, 2205, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{ 2202}$$

$$\int \frac{hx^4+fx^2+d}{x^4-5x^2+4}dx+\int \frac{x\left(gx^2+e\right)}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{ 1576}$$

$$\int \frac{hx^4+fx^2+d}{x^4-5x^2+4}dx+\frac{1}{2}\int \frac{gx^2+e}{x^4-5x^2+4}dx^2$$

$$\Big\downarrow \text{ 1141}$$

$$\int \frac{hx^4+fx^2+d}{x^4-5x^2+4}dx+\frac{1}{2}\int \left(\frac{e+g}{3\left(1-x^2\right)}-\frac{e+4g}{3\left(4-x^2\right)}\right)dx^2$$

$$\Big\downarrow \text{ 2009}$$

$$\int \frac{hx^4+fx^2+d}{x^4-5x^2+4}dx+\frac{1}{2}\left(\frac{1}{3}(e+4g)\log\left(4-x^2\right)-\frac{1}{3}(e+g)\log\left(1-x^2\right)\right)$$

$$\Big\downarrow \text{ 2205}$$

$$\int \left(h+\frac{(f+5h)x^2+d-4h}{x^4-5x^2+4}\right)dx+\frac{1}{2}\left(\frac{1}{3}(e+4g)\log\left(4-x^2\right)-\frac{1}{3}(e+g)\log\left(1-x^2\right)\right)$$

$$\Big\downarrow \text{ 2009}$$

$$-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(d+4f+16h)+\frac{1}{3}\text{arctanh}(x)(d+f+h)+\frac{1}{2}\left(\frac{1}{3}(e+4g)\log\left(4-x^2\right)-\frac{1}{3}(e+g)\log\left(1-x^2\right)\right)+hx$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4)/(4 - 5*x^2 + x^4),x]
```

output
```
h*x - ((d + 4*f + 16*h)*ArcTanh[x/2])/6 + ((d + f + h)*ArcTanh[x])/3 + (-1
/3*((e + g)*Log[1 - x^2]) + ((e + 4*g)*Log[4 - x^2])/3)/2
```

3.13.3.1 Defintions of rubi rules used

rule 1141
```
Int[((d_.) + (e_.)*(x_))^(m_.)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_
Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[1/c^p   Int[ExpandIntegrand[
(d + e*x)^m*(b/2 - q/2 + c*x)^p*(b/2 + q/2 + c*x)^p, x], x], x] /; EqQ[p, -
1] ||  !FractionalPowerFactorQ[q]] /; FreeQ[{a, b, c, d, e}, x] && ILtQ[p,
0] && IntegerQ[m] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2205
```
Int[(Px_)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :> Int[ExpandInte
grand[Px/(a + b*x^2 + c*x^4), x], x] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x^
2] && Expon[Px, x^2] > 1
```

3.13.4 Maple [A] (verified)

Time = 0.08 (sec) , antiderivative size = 89, normalized size of antiderivative = 1.39

method	result
default	$hx+\left(-\frac{d}{12}+\frac{e}{6}-\frac{f}{3}+\frac{2g}{3}-\frac{4h}{3}\right)\ln\left(x+2\right)+\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}+\frac{h}{6}\right)\ln\left(x+1\right)+\left(-\frac{d}{6}-\frac{e}{6}-\frac{f}{6}\right.$
norman	$hx+\left(-\frac{d}{12}+\frac{e}{6}-\frac{f}{3}+\frac{2g}{3}-\frac{4h}{3}\right)\ln\left(x+2\right)+\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}+\frac{h}{6}\right)\ln\left(x+1\right)+\left(-\frac{d}{6}-\frac{e}{6}-\frac{f}{6}\right.$
parallelrisch	$hx+\frac{\ln(x-2)d}{12}+\frac{\ln(x-2)e}{6}+\frac{\ln(x-2)f}{3}+\frac{2\ln(x-2)g}{3}+\frac{4\ln(x-2)h}{3}-\frac{\ln(x-1)d}{6}-\frac{\ln(x-1)e}{6}-\frac{\ln(x-1)f}{6}-\ln$
risch	$hx-\frac{\ln(1-x)d}{6}-\frac{\ln(1-x)e}{6}-\frac{\ln(1-x)f}{6}-\frac{\ln(1-x)g}{6}-\frac{\ln(1-x)h}{6}+\frac{\ln(2-x)d}{12}+\frac{\ln(2-x)e}{6}+\frac{\ln(2-x)f}{3}+2\ln(2$

input
```
int((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x,method=_RETURNVERBOSE)
```

output
```
h*x+(-1/12*d+1/6*e-1/3*f+2/3*g-4/3*h)*ln(x+2)+(1/6*d-1/6*e+1/6*f-1/6*g+1/6
*h)*ln(x+1)+(-1/6*d-1/6*e-1/6*f-1/6*g-1/6*h)*ln(x-1)+(1/12*d+1/6*e+1/3*f+2
/3*g+4/3*h)*ln(x-2)
```

3.13.5 Fricas [A] (verification not implemented)

Time = 1.14 (sec) , antiderivative size = 72, normalized size of antiderivative = 1.12

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{4-5x^2+x^4}\,dx = hx - \frac{1}{12}(d-2e+4f-8g+16h)\log(x+2) + \frac{1}{6}(d-e+f-g+h)\log(x+1) - \frac{1}{6}(d+e+f+g+h)\log(x-1) + \frac{1}{12}(d+2e+4f+8g+16h)\log(x-2)$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="fricas")
```

output
```
h*x - 1/12*(d - 2*e + 4*f - 8*g + 16*h)*log(x + 2) + 1/6*(d - e + f - g +
h)*log(x + 1) - 1/6*(d + e + f + g + h)*log(x - 1) + 1/12*(d + 2*e + 4*f +
 8*g + 16*h)*log(x - 2)
```

3.13.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{4-5x^2+x^4}\,dx = \text{Timed out}$$

input
```
integrate((h*x**4+g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4),x)
```

output
```
Timed out
```

3.13.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 72, normalized size of antiderivative = 1.12

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{4-5x^2+x^4}\,dx = hx - \frac{1}{12}\,(d-2\,e+4\,f-8\,g+16\,h)\log{(x+2)} + \frac{1}{6}\,(d-e+f-g+h)\log{(x+1)} - \frac{1}{6}\,(d+e+f+g+h)\log{(x-1)} + \frac{1}{12}\,(d+2\,e+4\,f+8\,g+16\,h)\log{(x-2)}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="maxima")
```

output
```
h*x - 1/12*(d - 2*e + 4*f - 8*g + 16*h)*log(x + 2) + 1/6*(d - e + f - g +
h)*log(x + 1) - 1/6*(d + e + f + g + h)*log(x - 1) + 1/12*(d + 2*e + 4*f +
 8*g + 16*h)*log(x - 2)
```

3.13.8 Giac [A] (verification not implemented)

Time = 0.31 (sec) , antiderivative size = 76, normalized size of antiderivative = 1.19

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{4-5x^2+x^4}\,dx = hx - \frac{1}{12}\,(d-2\,e+4\,f-8\,g+16\,h)\log{(|x+2|)} + \frac{1}{6}\,(d-e+f-g+h)\log{(|x+1|)} - \frac{1}{6}\,(d+e+f+g+h)\log{(|x-1|)} + \frac{1}{12}\,(d+2\,e+4\,f+8\,g+16\,h)\log{(|x-2|)}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="giac")
```

output
```
h*x - 1/12*(d - 2*e + 4*f - 8*g + 16*h)*log(abs(x + 2)) + 1/6*(d - e + f -
 g + h)*log(abs(x + 1)) - 1/6*(d + e + f + g + h)*log(abs(x - 1)) + 1/12*(
d + 2*e + 4*f + 8*g + 16*h)*log(abs(x - 2))
```

3.13.9 Mupad [B] (verification not implemented)

Time = 7.90 (sec) , antiderivative size = 90, normalized size of antiderivative = 1.41

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{4-5x^2+x^4}\,dx = h\,x-\ln(x-1)\left(\frac{d}{6}+\frac{e}{6}+\frac{f}{6}+\frac{g}{6}+\frac{h}{6}\right) + \ln(x+1)\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}+\frac{h}{6}\right) + \ln(x-2)\left(\frac{d}{12}+\frac{e}{6}+\frac{f}{3}+\frac{2\,g}{3}+\frac{4\,h}{3}\right) - \ln(x+2)\left(\frac{d}{12}-\frac{e}{6}+\frac{f}{3}-\frac{2\,g}{3}+\frac{4\,h}{3}\right)$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4)/(x^4 - 5*x^2 + 4),x)
```

output
```
h*x - log(x - 1)*(d/6 + e/6 + f/6 + g/6 + h/6) + log(x + 1)*(d/6 - e/6 + f
/6 - g/6 + h/6) + log(x - 2)*(d/12 + e/6 + f/3 + (2*g)/3 + (4*h)/3) - log(
x + 2)*(d/12 - e/6 + f/3 - (2*g)/3 + (4*h)/3)
```

3.14 $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{4-5x^2+x^4}\,dx$

3.14.1 Optimal result

Integrand size = 38, antiderivative size = 76

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{4-5x^2+x^4}\,dx = hx+\frac{ix^2}{2}-\frac{1}{6}(d+4f+16h)\text{arctanh}\left(\frac{x}{2}\right)+\frac{1}{3}(d+f+h)\text{arctanh}(x)-\frac{1}{6}(e+g+i)\log\left(1-x^2\right)+\frac{1}{6}(e+4g+16i)\log\left(4-x^2\right)$$

output
```
h*x+1/2*i*x^2-1/6*(d+4*f+16*h)*arctanh(1/2*x)+1/3*(d+f+h)*arctanh(x)-1/6*(
e+g+i)*ln(-x^2+1)+1/6*(e+4*g+16*i)*ln(-x^2+4)
```

3.14.2 Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 98, normalized size of antiderivative = 1.29

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{4-5x^2+x^4}\,dx = \frac{1}{12}\big(12hx+6ix^2-2(d+e+f+g+h+i)\log(1-x)+(d+2e+4(f+2g+4h+8i))\log(2-x)+2(d-e+f-g+h-i)\log(1+x)-(d-2(e-2f+4g-8h+16i))\log(2+x)\big)$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(4 - 5*x^2 + x^4),x]
```

output
```
(12*h*x + 6*i*x^2 - 2*(d + e + f + g + h + i)*Log[1 - x] + (d + 2*e + 4*(f
 + 2*g + 4*h + 8*i))*Log[2 - x] + 2*(d - e + f - g + h - i)*Log[1 + x] - (
d - 2*(e - 2*f + 4*g - 8*h + 16*i))*Log[2 + x])/12
```

3.14.3 Rubi [A] (verified)

Time = 0.49 (sec) , antiderivative size = 78, normalized size of antiderivative = 1.03, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.158, Rules used = {2202, 2194, 2188, 2009, 2205, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{hx^4+fx^2+d}{x^4-5x^2+4}dx+\int \frac{x(ix^4+gx^2+e)}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{2194}$$

$$\int \frac{hx^4+fx^2+d}{x^4-5x^2+4}dx+\frac{1}{2}\int \frac{ix^4+gx^2+e}{x^4-5x^2+4}dx^2$$

$$\Big\downarrow \text{2188}$$

$$\int \frac{hx^4+fx^2+d}{x^4-5x^2+4}dx+\frac{1}{2}\int \left(i+\frac{(g+5i)x^2+e-4i}{x^4-5x^2+4}\right)dx^2$$

$$\Big\downarrow \text{2009}$$

$$\int \frac{hx^4+fx^2+d}{x^4-5x^2+4}dx+\frac{1}{2}\left(-\frac{1}{3}\log\left(1-x^2\right)(e+g+i)+\frac{1}{3}\log\left(4-x^2\right)(e+4g+16i)+ix^2\right)$$

$$\Big\downarrow \text{2205}$$

$$\int \left(h+\frac{(f+5h)x^2+d-4h}{x^4-5x^2+4}\right)dx+$$
$$\frac{1}{2}\left(-\frac{1}{3}\log\left(1-x^2\right)(e+g+i)+\frac{1}{3}\log\left(4-x^2\right)(e+4g+16i)+ix^2\right)$$

$$\Big\downarrow \text{2009}$$

$$-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(d+4f+16h)+\frac{1}{3}\text{arctanh}(x)(d+f+h)+$$
$$\frac{1}{2}\left(-\frac{1}{3}\log\left(1-x^2\right)(e+g+i)+\frac{1}{3}\log\left(4-x^2\right)(e+4g+16i)+ix^2\right)+hx$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(4 - 5*x^2 + x^4),x]
```

output
```
h*x - ((d + 4*f + 16*h)*ArcTanh[x/2])/6 + ((d + f + h)*ArcTanh[x])/3 + (i*
x^2 - ((e + g + i)*Log[1 - x^2])/3 + ((e + 4*g + 16*i)*Log[4 - x^2])/3)/2
```

3.14.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2188
```
Int[(Pq_)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_.), x_Symbol] :> Int[Expand
Integrand[Pq*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c}, x] && PolyQ[Pq
, x] && IGtQ[p, -2]
```

rule 2194
```
Int[(Pq_)*(x_)^(m_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :
> Simp[1/2   Subst[Int[x^((m - 1)/2)*SubstFor[x^2, Pq, x]*(a + b*x + c*x^2)
^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pq, x^2] && IntegerQ
[(m - 1)/2]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2205
```
Int[(Px_)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :> Int[ExpandInte
grand[Px/(a + b*x^2 + c*x^4), x], x] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x^
2] && Expon[Px, x^2] > 1
```

3.14.4 Maple [A] (verified)

Time = 0.09 (sec) , antiderivative size = 107, normalized size of antiderivative = 1.41

method	result
default	$\frac{i x^2}{2}+hx+\left(-\frac{d}{12}+\frac{e}{6}-\frac{f}{3}+\frac{2g}{3}-\frac{4h}{3}+\frac{8i}{3}\right)\ln\left(x+2\right)+\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}+\frac{h}{6}-\frac{i}{6}\right)\ln\left(x+1\right)$ -
norman	$\frac{i x^2}{2}+hx+\left(-\frac{d}{12}+\frac{e}{6}-\frac{f}{3}+\frac{2g}{3}-\frac{4h}{3}+\frac{8i}{3}\right)\ln\left(x+2\right)+\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}+\frac{h}{6}-\frac{i}{6}\right)\ln\left(x+1\right)$ -
parallelrisch	$\frac{i x^2}{2}+\frac{\ln(x-2)d}{12}+\frac{\ln(x-2)e}{6}-\frac{\ln(x-1)d}{6}-\frac{\ln(x-1)e}{6}-\frac{\ln(x+1)i}{6}-\frac{\ln(x+2)f}{3}+\frac{\ln(x+1)f}{6}-\frac{\ln(x+2)d}{12}+hx$ -
risch	$\frac{i x^2}{2}-\frac{\ln(x+1)i}{6}-\frac{\ln(x+2)f}{3}-\frac{\ln(1-x)f}{6}+\frac{\ln(x+1)f}{6}+\frac{\ln(2-x)f}{3}+\frac{\ln(2-x)d}{12}+\frac{\ln(2-x)e}{6}-\frac{\ln(x+2)d}{12}+hx$ -

input
```
int((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x,method=_RETURNVERBOSE)
```

output
```
1/2*i*x^2+h*x+(-1/12*d+1/6*e-1/3*f+2/3*g-4/3*h+8/3*i)*ln(x+2)+(1/6*d-1/6*e
+1/6*f-1/6*g+1/6*h-1/6*i)*ln(x+1)+(-1/6*d-1/6*e-1/6*f-1/6*g-1/6*h-1/6*i)*l
n(x-1)+(1/12*d+1/6*e+1/3*f+2/3*g+4/3*h+8/3*i)*ln(x-2)
```

3.14.5 Fricas [A] (verification not implemented)

Time = 5.00 (sec) , antiderivative size = 88, normalized size of antiderivative = 1.16

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{4-5x^2+x^4}\,dx &= \frac{1}{2}\,ix^2+hx\\ &\quad -\frac{1}{12}\,(d-2\,e+4\,f-8\,g+16\,h-32\,i)\log\left(x+2\right)\\ &\quad +\frac{1}{6}\,(d-e+f-g+h-i)\log\left(x+1\right)\\ &\quad -\frac{1}{6}\,(d+e+f+g+h+i)\log\left(x-1\right)\\ &\quad +\frac{1}{12}\,(d+2\,e+4\,f+8\,g+16\,h+32\,i)\log\left(x-2\right)\end{aligned}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="fric
as")
```

output
```
1/2*i*x^2 + h*x - 1/12*(d - 2*e + 4*f - 8*g + 16*h - 32*i)*log(x + 2) + 1/
6*(d - e + f - g + h - i)*log(x + 1) - 1/6*(d + e + f + g + h + i)*log(x -
 1) + 1/12*(d + 2*e + 4*f + 8*g + 16*h + 32*i)*log(x - 2)
```

3.14.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{4-5x^2+x^4}\,dx = \text{Timed out}$$

input
```
integrate((i*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4),x)
```

output
```
Timed out
```

3.14.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 88, normalized size of antiderivative = 1.16

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{4-5x^2+x^4}\,dx = \frac{1}{2}ix^2+hx - \frac{1}{12}(d-2e+4f-8g+16h-32i)\log(x+2) + \frac{1}{6}(d-e+f-g+h-i)\log(x+1) - \frac{1}{6}(d+e+f+g+h+i)\log(x-1) + \frac{1}{12}(d+2e+4f+8g+16h+32i)\log(x-2)$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="maxi
ma")
```

output
```
1/2*i*x^2 + h*x - 1/12*(d - 2*e + 4*f - 8*g + 16*h - 32*i)*log(x + 2) + 1/
6*(d - e + f - g + h - i)*log(x + 1) - 1/6*(d + e + f + g + h + i)*log(x -
 1) + 1/12*(d + 2*e + 4*f + 8*g + 16*h + 32*i)*log(x - 2)
```

3.14.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 92, normalized size of antiderivative = 1.21

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{4-5x^2+x^4}\,dx = \frac{1}{2}\,ix^2+hx - \frac{1}{12}\,(d-2\,e+4\,f-8\,g+16\,h-32\,i)\log\left(|x+2|\right) + \frac{1}{6}\,(d-e+f-g+h-i)\log\left(|x+1|\right) - \frac{1}{6}\,(d+e+f+g+h+i)\log\left(|x-1|\right) + \frac{1}{12}\,(d+2\,e+4\,f+8\,g+16\,h+32\,i)\log\left(|x-2|\right)$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="giac
")
```

output
```
1/2*i*x^2 + h*x - 1/12*(d - 2*e + 4*f - 8*g + 16*h - 32*i)*log(abs(x + 2))
 + 1/6*(d - e + f - g + h - i)*log(abs(x + 1)) - 1/6*(d + e + f + g + h +
i)*log(abs(x - 1)) + 1/12*(d + 2*e + 4*f + 8*g + 16*h + 32*i)*log(abs(x -
2))
```

3.14.9 Mupad [B] (verification not implemented)

Time = 8.15 (sec) , antiderivative size = 108, normalized size of antiderivative = 1.42

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{4-5x^2+x^4}\,dx = h\,x+\frac{i\,x^2}{2}-\ln\left(x-1\right)\,\left(\frac{d}{6}+\frac{e}{6}+\frac{f}{6}+\frac{g}{6}+\frac{h}{6}+\frac{i}{6}\right) + \ln\left(x+1\right)\,\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}+\frac{h}{6}-\frac{i}{6}\right) + \ln\left(x-2\right)\,\left(\frac{d}{12}+\frac{e}{6}+\frac{f}{3}+\frac{2\,g}{3}+\frac{4\,h}{3}+\frac{8\,i}{3}\right) - \ln\left(x+2\right)\,\left(\frac{d}{12}-\frac{e}{6}+\frac{f}{3}-\frac{2\,g}{3}+\frac{4\,h}{3}-\frac{8\,i}{3}\right)$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(x^4 - 5*x^2 + 4),x)
```

output
```
h*x + (i*x^2)/2 - log(x - 1)*(d/6 + e/6 + f/6 + g/6 + h/6 + i/6) + log(x +
 1)*(d/6 - e/6 + f/6 - g/6 + h/6 - i/6) + log(x - 2)*(d/12 + e/6 + f/3 + (
2*g)/3 + (4*h)/3 + (8*i)/3) - log(x + 2)*(d/12 - e/6 + f/3 - (2*g)/3 + (4*
h)/3 - (8*i)/3)
```

3.14. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{4-5x^2+x^4}\,dx$

3.15 $\int \frac{d+ex}{1+x^2+x^4}\,dx$

3.15.1 Optimal result

Integrand size = 16, antiderivative size = 92

$$\int \frac{d+ex}{1+x^2+x^4}\,dx = -\frac{d\arctan\left(\frac{1-2x}{\sqrt{3}}\right)}{2\sqrt{3}}+\frac{d\arctan\left(\frac{1+2x}{\sqrt{3}}\right)}{2\sqrt{3}}+\frac{e\arctan\left(\frac{1+2x^2}{\sqrt{3}}\right)}{\sqrt{3}} - \frac{1}{4}d\log\left(1-x+x^2\right)+\frac{1}{4}d\log\left(1+x+x^2\right)$$

output
```
-1/4*d*ln(x^2-x+1)+1/4*d*ln(x^2+x+1)-1/6*d*arctan(1/3*(1-2*x)*3^(1/2))*3^(
1/2)+1/6*d*arctan(1/3*(1+2*x)*3^(1/2))*3^(1/2)+1/3*e*arctan(1/3*(2*x^2+1)*
3^(1/2))*3^(1/2)
```

3.15.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.17 (sec) , antiderivative size = 98, normalized size of antiderivative = 1.07

$$\int \frac{d+ex}{1+x^2+x^4}\,dx = \frac{1}{6}i\left(\sqrt{6-6i\sqrt{3}}d\arctan\left(\frac{1}{2}\left(-i+\sqrt{3}\right)x\right) - \sqrt{6+6i\sqrt{3}}d\arctan\left(\frac{1}{2}\left(i+\sqrt{3}\right)x\right)+2i\sqrt{3}e\arctan\left(\frac{\sqrt{3}}{1+2x^2}\right)\right)$$

input
```
Integrate[(d + e*x)/(1 + x^2 + x^4),x]
```

output
```
(I/6)*(Sqrt[6 - (6*I)*Sqrt[3]]*d*ArcTan[((-I + Sqrt[3])*x)/2] - Sqrt[6 + (
6*I)*Sqrt[3]]*d*ArcTan[((I + Sqrt[3])*x)/2] + (2*I)*Sqrt[3]*e*ArcTan[Sqrt[
3]/(1 + 2*x^2)])
```

3.15.3 Rubi [A] (verified)

Time = 0.33 (sec) , antiderivative size = 95, normalized size of antiderivative = 1.03, number of steps used = 12, number of rules used = 11, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.688, Rules used = {2202, 27, 1407, 1142, 25, 1083, 217, 1103, 1432, 1083, 217}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{d}{x^4+x^2+1}dx+\int \frac{ex}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{27}$$

$$d\int \frac{1}{x^4+x^2+1}dx+e\int \frac{x}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{1407}$$

$$d\left(\frac{1}{2}\int \frac{1-x}{x^2-x+1}dx+\frac{1}{2}\int \frac{x+1}{x^2+x+1}dx\right)+e\int \frac{x}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{1142}$$

$$d\left(\frac{1}{2}\left(\frac{1}{2}\int \frac{1}{x^2-x+1}dx-\frac{1}{2}\int -\frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}\int \frac{1}{x^2+x+1}dx+\frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx\right)\right)+ e\int \frac{x}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{25}$$

$$d\left(\frac{1}{2}\left(\frac{1}{2}\int \frac{1}{x^2-x+1}dx+\frac{1}{2}\int \frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}\int \frac{1}{x^2+x+1}dx+\frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx\right)\right)+ e\int \frac{x}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{1083}$$

$$d\left(\frac{1}{2}\left(\frac{1}{2}\int \frac{1-2x}{x^2-x+1}dx-\int \frac{1}{-(2x-1)^2-3}d(2x-1)\right)+\frac{1}{2}\left(\frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx-\int \frac{1}{-(2x+1)^2-3}d(2x+1)\right)\right)+e\int \frac{x}{x^4+x^2+1}dx$$

$\Big\downarrow$ 217

$$d\left(\frac{1}{2}\left(\frac{1}{2}\int \frac{1-2x}{x^2-x+1}dx+\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}\right)+\frac{1}{2}\left(\frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx+\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}\right)\right)+e\int \frac{x}{x^4+x^2+1}dx$$

$\Big\downarrow$ 1103

$$e\int \frac{x}{x^4+x^2+1}dx+d\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}-\frac{1}{2}\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}+\frac{1}{2}\log\left(x^2+x+1\right)\right)\right)$$

$\Big\downarrow$ 1432

$$\frac{1}{2}e\int \frac{1}{x^4+x^2+1}dx^2+d\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}-\frac{1}{2}\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}+\frac{1}{2}\log\left(x^2+x+1\right)\right)\right)$$

$\Big\downarrow$ 1083

$$d\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}-\frac{1}{2}\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}+\frac{1}{2}\log\left(x^2+x+1\right)\right)\right)-e\int \frac{1}{-x^4-3}d\left(2x^2+1\right)$$

$\Big\downarrow$ 217

$$d\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}-\frac{1}{2}\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}+\frac{1}{2}\log\left(x^2+x+1\right)\right)\right)+\frac{e\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{\sqrt{3}}$$

input
```
Int[(d + e*x)/(1 + x^2 + x^4),x]
```

3.15. $\int \frac{d+ex}{1+x^2+x^4}\,dx$

output
```
(e*ArcTan[(1 + 2*x^2)/Sqrt[3]])/Sqrt[3] + d*((ArcTan[(-1 + 2*x)/Sqrt[3]]/S
qrt[3] - Log[1 - x + x^2]/2)/2 + (ArcTan[(1 + 2*x)/Sqrt[3]]/Sqrt[3] + Log[
1 + x + x^2]/2)/2)
```

3.15.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 217
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1407
```
Int[((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(-1), x_Symbol] :> With[{q = Rt[a/
c, 2]}, With[{r = Rt[2*q - b/c, 2]}, Simp[1/(2*c*q*r)   Int[(r - x)/(q - r*
x + x^2), x], x] + Simp[1/(2*c*q*r)   Int[(r + x)/(q + r*x + x^2), x], x]]]
 /; FreeQ[{a, b, c}, x] && NeQ[b^2 - 4*a*c, 0] && NegQ[b^2 - 4*a*c]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.15.4 Maple [A] (verified)

Time = 0.12 (sec) , antiderivative size = 68, normalized size of antiderivative = 0.74

method	result
default	$-\frac{d\ln\left(x^2-x+1\right)}{4}+\frac{\left(\frac{d}{2}+e\right)\sqrt{3}\,\arctan\left(\frac{(2x-1)\sqrt{3}}{3}\right)}{3}+\frac{d\ln\left(x^2+x+1\right)}{4}+\frac{\left(\frac{d}{2}-e\right)\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)\sqrt{3}}{3}$
risch	$\frac{d\ln\left(36d^2x^2+48e^2x^2+36d^2x+48e^2x+36d^2+48e^2\right)}{4}+\frac{\sqrt{3}\,d\arctan\left(\frac{8e^2\sqrt{3}\,x}{3\left(3d^2+4e^2\right)}-\frac{4e^2\sqrt{3}}{3\left(3d^2+4e^2\right)}+\frac{2d^2\sqrt{3}\,x}{3d^2+4e^2}-\frac{d^2\sqrt{3}}{3d^2+4e^2}\right)}{6}-\frac{\sqrt{3}\,d\,\mathrm{arcta}}{3}$

input
```
int((e*x+d)/(x^4+x^2+1),x,method=_RETURNVERBOSE)
```

output
```
-1/4*d*ln(x^2-x+1)+1/3*(1/2*d+e)*3^(1/2)*arctan(1/3*(2*x-1)*3^(1/2))+1/4*d
*ln(x^2+x+1)+1/3*(1/2*d-e)*arctan(1/3*(1+2*x)*3^(1/2))*3^(1/2)
```

3.15.5 Fricas [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 65, normalized size of antiderivative = 0.71

$$\int \frac{d+ex}{1+x^2+x^4}\,dx = \frac{1}{6}\sqrt{3}(d-2e)\arctan\left(\frac{1}{3}\sqrt{3}(2x+1)\right) + \frac{1}{6}\sqrt{3}(d+2e)\arctan\left(\frac{1}{3}\sqrt{3}(2x-1)\right) + \frac{1}{4}d\log\left(x^2+x+1\right) - \frac{1}{4}d\log\left(x^2-x+1\right)$$

input
```
integrate((e*x+d)/(x^4+x^2+1),x, algorithm="fricas")
```

output
```
1/6*sqrt(3)*(d - 2*e)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/6*sqrt(3)*(d + 2*e
)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/4*d*log(x^2 + x + 1) - 1/4*d*log(x^2 -
 x + 1)
```

3.15.6 Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 1.66 (sec) , antiderivative size = 923, normalized size of antiderivative = 10.03

$$\int \frac{d+ex}{1+x^2+x^4}\,dx = \text{Too large to display}$$

input
```
integrate((e*x+d)/(x**4+x**2+1),x)
```

output
```
(-d/4 - sqrt(3)*I*(d + 2*e)/12)*log(x + (-7*d**4*e + 6*d**4*(-d/4 - sqrt(3
)*I*(d + 2*e)/12) - 15*d**2*e**3 - 18*d**2*e**2*(-d/4 - sqrt(3)*I*(d + 2*e
)/12) + 60*d**2*e*(-d/4 - sqrt(3)*I*(d + 2*e)/12)**2 + 72*d**2*(-d/4 - sqr
t(3)*I*(d + 2*e)/12)**3 + 4*e**5 + 24*e**4*(-d/4 - sqrt(3)*I*(d + 2*e)/12)
 + 48*e**3*(-d/4 - sqrt(3)*I*(d + 2*e)/12)**2 + 288*e**2*(-d/4 - sqrt(3)*I
*(d + 2*e)/12)**3)/(3*d**5 - 8*d**3*e**2 - 16*d*e**4)) + (-d/4 + sqrt(3)*I
*(d + 2*e)/12)*log(x + (-7*d**4*e + 6*d**4*(-d/4 + sqrt(3)*I*(d + 2*e)/12)
 - 15*d**2*e**3 - 18*d**2*e**2*(-d/4 + sqrt(3)*I*(d + 2*e)/12) + 60*d**2*e
*(-d/4 + sqrt(3)*I*(d + 2*e)/12)**2 + 72*d**2*(-d/4 + sqrt(3)*I*(d + 2*e)/
12)**3 + 4*e**5 + 24*e**4*(-d/4 + sqrt(3)*I*(d + 2*e)/12) + 48*e**3*(-d/4
+ sqrt(3)*I*(d + 2*e)/12)**2 + 288*e**2*(-d/4 + sqrt(3)*I*(d + 2*e)/12)**3
)/(3*d**5 - 8*d**3*e**2 - 16*d*e**4)) + (d/4 - sqrt(3)*I*(d - 2*e)/12)*log
(x + (-7*d**4*e + 6*d**4*(d/4 - sqrt(3)*I*(d - 2*e)/12) - 15*d**2*e**3 - 1
8*d**2*e**2*(d/4 - sqrt(3)*I*(d - 2*e)/12) + 60*d**2*e*(d/4 - sqrt(3)*I*(d
 - 2*e)/12)**2 + 72*d**2*(d/4 - sqrt(3)*I*(d - 2*e)/12)**3 + 4*e**5 + 24*e
**4*(d/4 - sqrt(3)*I*(d - 2*e)/12) + 48*e**3*(d/4 - sqrt(3)*I*(d - 2*e)/12
)**2 + 288*e**2*(d/4 - sqrt(3)*I*(d - 2*e)/12)**3)/(3*d**5 - 8*d**3*e**2 -
 16*d*e**4)) + (d/4 + sqrt(3)*I*(d - 2*e)/12)*log(x + (-7*d**4*e + 6*d**4*
(d/4 + sqrt(3)*I*(d - 2*e)/12) - 15*d**2*e**3 - 18*d**2*e**2*(d/4 + sqrt(3
)*I*(d - 2*e)/12) + 60*d**2*e*(d/4 + sqrt(3)*I*(d - 2*e)/12)**2 + 72*d*...
```

3.15.7 Maxima [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 65, normalized size of antiderivative = 0.71

$$\begin{aligned}\int \frac{d+ex}{1+x^2+x^4}\,dx &= \frac{1}{6}\,\sqrt{3}(d-2\,e)\arctan\left(\frac{1}{3}\,\sqrt{3}(2\,x+1)\right)\\&\quad+\frac{1}{6}\,\sqrt{3}(d+2\,e)\arctan\left(\frac{1}{3}\,\sqrt{3}(2\,x-1)\right)\\&\quad+\frac{1}{4}\,d\log\left(x^2+x+1\right)-\frac{1}{4}\,d\log\left(x^2-x+1\right)\end{aligned}$$

input
```
integrate((e*x+d)/(x^4+x^2+1),x, algorithm="maxima")
```

output
```
1/6*sqrt(3)*(d - 2*e)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/6*sqrt(3)*(d + 2*e
)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/4*d*log(x^2 + x + 1) - 1/4*d*log(x^2 -
 x + 1)
```

3.15.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 65, normalized size of antiderivative = 0.71

$$\int \frac{d+ex}{1+x^2+x^4}\,dx = \frac{1}{6}\sqrt{3}(d-2e)\arctan\left(\frac{1}{3}\sqrt{3}(2x+1)\right) + \frac{1}{6}\sqrt{3}(d+2e)\arctan\left(\frac{1}{3}\sqrt{3}(2x-1)\right) + \frac{1}{4}d\log\left(x^2+x+1\right) - \frac{1}{4}d\log\left(x^2-x+1\right)$$

input
```
integrate((e*x+d)/(x^4+x^2+1),x, algorithm="giac")
```

output
```
1/6*sqrt(3)*(d - 2*e)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/6*sqrt(3)*(d + 2*e
)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/4*d*log(x^2 + x + 1) - 1/4*d*log(x^2 -
 x + 1)
```

3.15.9 Mupad [B] (verification not implemented)

Time = 0.13 (sec) , antiderivative size = 118, normalized size of antiderivative = 1.28

$$\int \frac{d+ex}{1+x^2+x^4}\,dx = -\ln\left(x-\frac{1}{2}-\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{d}{4}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{12}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{6}\right) + \ln\left(x+\frac{1}{2}-\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{d}{4}-\frac{\sqrt{3}\,d\,1\mathrm{i}}{12}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{6}\right) + \ln\left(x-\frac{1}{2}+\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(-\frac{d}{4}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{12}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{6}\right) + \ln\left(x+\frac{1}{2}+\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{d}{4}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{12}-\frac{\sqrt{3}\,e\,1\mathrm{i}}{6}\right)$$

input
```
int((d + e*x)/(x^2 + x^4 + 1),x)
```

output
```
log(x - (3^(1/2)*1i)/2 + 1/2)*(d/4 - (3^(1/2)*d*1i)/12 + (3^(1/2)*e*1i)/6)
 - log(x - (3^(1/2)*1i)/2 - 1/2)*(d/4 + (3^(1/2)*d*1i)/12 + (3^(1/2)*e*1i)
/6) + log(x + (3^(1/2)*1i)/2 - 1/2)*((3^(1/2)*d*1i)/12 - d/4 + (3^(1/2)*e*
1i)/6) + log(x + (3^(1/2)*1i)/2 + 1/2)*(d/4 + (3^(1/2)*d*1i)/12 - (3^(1/2)
*e*1i)/6)
```

3.15. $\int \frac{d+ex}{1+x^2+x^4} \, dx$

3.16 $\int \frac{d+ex+fx^2}{1+x^2+x^4} dx$

3.16.1 Optimal result

Integrand size = 21, antiderivative size = 104

$$\int \frac{d+ex+fx^2}{1+x^2+x^4}\,dx=-\frac{(d+f)\arctan\left(\frac{1-2x}{\sqrt{3}}\right)}{2\sqrt{3}}+\frac{(d+f)\arctan\left(\frac{1+2x}{\sqrt{3}}\right)}{2\sqrt{3}}+\frac{e\arctan\left(\frac{1+2x^2}{\sqrt{3}}\right)}{\sqrt{3}}-\frac{1}{4}(d-f)\log\left(1-x+x^2\right)+\frac{1}{4}(d-f)\log\left(1+x+x^2\right)$$

output
```
-1/4*(d-f)*ln(x^2-x+1)+1/4*(d-f)*ln(x^2+x+1)-1/6*(d+f)*arctan(1/3*(1-2*x)*
3^(1/2))*3^(1/2)+1/6*(d+f)*arctan(1/3*(1+2*x)*3^(1/2))*3^(1/2)+1/3*e*arcta
n(1/3*(2*x^2+1)*3^(1/2))*3^(1/2)
```

3.16.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.11 (sec) , antiderivative size = 121, normalized size of antiderivative = 1.16

$$\int \frac{d+ex+fx^2}{1+x^2+x^4}\,dx=\frac{\left(2id+\left(-i+\sqrt{3}\right)f\right)\arctan\left(\frac{1}{2}\left(-i+\sqrt{3}\right)x\right)}{\sqrt{6+6i\sqrt{3}}}+\frac{\left(-2id+\left(i+\sqrt{3}\right)f\right)\arctan\left(\frac{1}{2}\left(i+\sqrt{3}\right)x\right)}{\sqrt{6-6i\sqrt{3}}}-\frac{e\arctan\left(\frac{\sqrt{3}}{1+2x^2}\right)}{\sqrt{3}}$$

input
```
Integrate[(d + e*x + f*x^2)/(1 + x^2 + x^4),x]
```

output
```
(((2*I)*d + (-I + Sqrt[3])*f)*ArcTan[((-I + Sqrt[3])*x)/2])/Sqrt[6 + (6*I)
*Sqrt[3]] + (((-2*I)*d + (I + Sqrt[3])*f)*ArcTan[((I + Sqrt[3])*x)/2])/Sqr
t[6 - (6*I)*Sqrt[3]] - (e*ArcTan[Sqrt[3]/(1 + 2*x^2)])/Sqrt[3]
```

3.16.3 Rubi [A] (verified)

Time = 0.34 (sec) , antiderivative size = 108, normalized size of antiderivative = 1.04, number of steps used = 12, number of rules used = 11, $\frac{\text{number of rules}}{\text{integrand size}} = 0.524$, Rules used = {2202, 27, 1432, 1083, 217, 1483, 1142, 25, 1083, 217, 1103}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2}{x^4+x^2+1}\,dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{fx^2+d}{x^4+x^2+1}dx+\int \frac{ex}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{27}$$

$$\int \frac{fx^2+d}{x^4+x^2+1}dx+e\int \frac{x}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{1432}$$

$$\int \frac{fx^2+d}{x^4+x^2+1}dx+\frac{1}{2}e\int \frac{1}{x^4+x^2+1}dx^2$$

$$\Big\downarrow \text{1083}$$

$$\int \frac{fx^2+d}{x^4+x^2+1}dx-e\int \frac{1}{-x^4-3}d(2x^2+1)$$

$$\Big\downarrow \text{217}$$

$$\int \frac{fx^2+d}{x^4+x^2+1}dx+\frac{e\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{\sqrt{3}}$$

$$\Big\downarrow \text{1483}$$

$$\frac{1}{2}\int \frac{d-(d-f)x}{x^2-x+1}dx+\frac{1}{2}\int \frac{d+(d-f)x}{x^2+x+1}dx+\frac{e\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{\sqrt{3}}$$

$$\Big\downarrow \textcolor{blue}{1142}$$

$$\frac{1}{2}\left(\frac{1}{2}(d+f)\int \frac{1}{x^2-x+1}dx-\frac{1}{2}(d-f)\int -\frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(d+f)\int \frac{1}{x^2+x+1}dx+\frac{1}{2}(d-f)\int \frac{2x+1}{x^2+x+1}dx\right)+\frac{e\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{\sqrt{3}}$$

$$\Big\downarrow \textcolor{blue}{25}$$

$$\frac{1}{2}\left(\frac{1}{2}(d+f)\int \frac{1}{x^2-x+1}dx+\frac{1}{2}(d-f)\int \frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(d+f)\int \frac{1}{x^2+x+1}dx+\frac{1}{2}(d-f)\int \frac{2x+1}{x^2+x+1}dx\right)+\frac{e\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{\sqrt{3}}$$

$$\Big\downarrow \textcolor{blue}{1083}$$

$$\frac{1}{2}\left(\frac{1}{2}(d-f)\int \frac{1-2x}{x^2-x+1}dx-(d+f)\int \frac{1}{-(2x-1)^2-3}d(2x-1)\right)+\frac{1}{2}\left(\frac{1}{2}(d-f)\int \frac{2x+1}{x^2+x+1}dx-(d+f)\int \frac{1}{-(2x+1)^2-3}d(2x+1)\right)+\frac{e\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{\sqrt{3}}$$

$$\Big\downarrow \textcolor{blue}{217}$$

$$\frac{1}{2}\left(\frac{1}{2}(d-f)\int \frac{1-2x}{x^2-x+1}dx+\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(d+f)}{\sqrt{3}}\right)+\frac{1}{2}\left(\frac{1}{2}(d-f)\int \frac{2x+1}{x^2+x+1}dx+\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(d+f)}{\sqrt{3}}\right)+\frac{e\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{\sqrt{3}}$$

$$\Big\downarrow \textcolor{blue}{1103}$$

$$\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(d+f)}{\sqrt{3}}-\frac{1}{2}(d-f)\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(d+f)}{\sqrt{3}}+\frac{1}{2}(d-f)\log\left(x^2+x+1\right)\right)+\frac{e\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{\sqrt{3}}$$

input
```
Int[(d + e*x + f*x^2)/(1 + x^2 + x^4),x]
```

output
```
(e*ArcTan[(1 + 2*x^2)/Sqrt[3]])/Sqrt[3] + (((d + f)*ArcTan[(-1 + 2*x)/Sqrt
[3]])/Sqrt[3] - ((d - f)*Log[1 - x + x^2])/2)/2 + (((d + f)*ArcTan[(1 + 2*
x)/Sqrt[3]])/Sqrt[3] + ((d - f)*Log[1 + x + x^2])/2)/2
```

3.16. $\int \frac{d+ex+fx^2}{1+x^2+x^4}\,dx$

3.16.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 217
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 1483
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[a/c, 2]}, With[{r = Rt[2*q - b/c, 2]}, Simp[1/(2*c*q*r)   In
t[(d*r - (d - e*q)*x)/(q - r*x + x^2), x], x] + Simp[1/(2*c*q*r)   Int[(d*r
 + (d - e*q)*x)/(q + r*x + x^2), x], x]]] /; FreeQ[{a, b, c, d, e}, x] && N
eQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NegQ[b^2 - 4*a*c]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.16.4 Maple [A] (verified)

Time = 0.23 (sec) , antiderivative size = 82, normalized size of antiderivative = 0.79

method	result	size
default	$\frac{(f-d)\ln(x^2-x+1)}{4}+\frac{\left(\frac{d}{2}+e+\frac{f}{2}\right)\sqrt{3}\,\arctan\left(\frac{(2x-1)\sqrt{3}}{3}\right)}{3}+\frac{(d-f)\ln(x^2+x+1)}{4}+\frac{\left(\frac{d}{2}-e+\frac{f}{2}\right)\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)\sqrt{3}}{3}$	82
risch	Expression too large to display	7878

input
```
int((f*x^2+e*x+d)/(x^4+x^2+1),x,method=_RETURNVERBOSE)
```

output
```
1/4*(f-d)*ln(x^2-x+1)+1/3*(1/2*d+e+1/2*f)*3^(1/2)*arctan(1/3*(2*x-1)*3^(1/
2))+1/4*(d-f)*ln(x^2+x+1)+1/3*(1/2*d-e+1/2*f)*arctan(1/3*(1+2*x)*3^(1/2))*
3^(1/2)
```

3.16.5 Fricas [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 75, normalized size of antiderivative = 0.72

$$\begin{aligned}\int \frac{d+ex+fx^2}{1+x^2+x^4}\,dx &= \frac{1}{6}\sqrt{3}(d-2e+f)\arctan\left(\frac{1}{3}\sqrt{3}(2x+1)\right)\\ &\quad+\frac{1}{6}\sqrt{3}(d+2e+f)\arctan\left(\frac{1}{3}\sqrt{3}(2x-1)\right)\\ &\quad+\frac{1}{4}(d-f)\log\left(x^2+x+1\right)-\frac{1}{4}(d-f)\log\left(x^2-x+1\right)\end{aligned}$$

input
```
integrate((f*x^2+e*x+d)/(x^4+x^2+1),x, algorithm="fricas")
```

output
```
1/6*sqrt(3)*(d - 2*e + f)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/6*sqrt(3)*(d +
 2*e + f)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/4*(d - f)*log(x^2 + x + 1) - 1
/4*(d - f)*log(x^2 - x + 1)
```

3.16.6 Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 65.87 (sec) , antiderivative size = 3589, normalized size of antiderivative = 34.51

$$\int \frac{d+ex+fx^2}{1+x^2+x^4}\,dx = \text{Too large to display}$$

input
```
integrate((f*x**2+e*x+d)/(x**4+x**2+1),x)
```

output
```
(-d/4 + f/4 - sqrt(3)*I*(d + 2*e + f)/12)*log(x + (-7*d**5*e + 6*d**5*(-d/
4 + f/4 - sqrt(3)*I*(d + 2*e + f)/12) + 25*d**4*e*f + 18*d**4*f*(-d/4 + f/
4 - sqrt(3)*I*(d + 2*e + f)/12) - 15*d**3*e**3 - 18*d**3*e**2*(-d/4 + f/4
- sqrt(3)*I*(d + 2*e + f)/12) - 25*d**3*e*f**2 + 60*d**3*e*(-d/4 + f/4 - s
qrt(3)*I*(d + 2*e + f)/12)**2 - 42*d**3*f**2*(-d/4 + f/4 - sqrt(3)*I*(d +
2*e + f)/12) + 72*d**3*(-d/4 + f/4 - sqrt(3)*I*(d + 2*e + f)/12)**3 + 108*
d**2*e**2*f*(-d/4 + f/4 - sqrt(3)*I*(d + 2*e + f)/12) + 20*d**2*e*f**3 - 1
44*d**2*e*f*(-d/4 + f/4 - sqrt(3)*I*(d + 2*e + f)/12)**2 - 12*d**2*f**3*(-
d/4 + f/4 - sqrt(3)*I*(d + 2*e + f)/12) - 144*d**2*f*(-d/4 + f/4 - sqrt(3)
*I*(d + 2*e + f)/12)**3 + 4*d*e**5 + 24*d*e**4*(-d/4 + f/4 - sqrt(3)*I*(d
+ 2*e + f)/12) + 15*d*e**3*f**2 + 48*d*e**3*(-d/4 + f/4 - sqrt(3)*I*(d + 2
*e + f)/12)**2 - 54*d*e**2*f**2*(-d/4 + f/4 - sqrt(3)*I*(d + 2*e + f)/12)
+ 288*d*e**2*(-d/4 + f/4 - sqrt(3)*I*(d + 2*e + f)/12)**3 - 20*d*e*f**4 +
180*d*e*f**2*(-d/4 + f/4 - sqrt(3)*I*(d + 2*e + f)/12)**2 + 36*d*f**4*(-d/
4 + f/4 - sqrt(3)*I*(d + 2*e + f)/12) - 72*d*f**2*(-d/4 + f/4 - sqrt(3)*I*
(d + 2*e + f)/12)**3 - 8*e**5*f - 96*e**3*f*(-d/4 + f/4 - sqrt(3)*I*(d + 2
*e + f)/12)**2 + 36*e**2*f**3*(-d/4 + f/4 - sqrt(3)*I*(d + 2*e + f)/12) +
11*e*f**5 - 48*e*f**3*(-d/4 + f/4 - sqrt(3)*I*(d + 2*e + f)/12)**2 - 6*f**
5*(-d/4 + f/4 - sqrt(3)*I*(d + 2*e + f)/12) + 144*f**3*(-d/4 + f/4 - sqrt(
3)*I*(d + 2*e + f)/12)**3)/(3*d**6 - 3*d**5*f - 8*d**4*e**2 - 3*d**4*f*...
```

3.16.7 Maxima [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 75, normalized size of antiderivative = 0.72

$$\begin{aligned}\int \frac{d+ex+fx^2}{1+x^2+x^4}\,dx &= \frac{1}{6}\,\sqrt{3}(d-2\,e+f)\arctan\left(\frac{1}{3}\,\sqrt{3}(2\,x+1)\right)\\ &\quad+\frac{1}{6}\,\sqrt{3}(d+2\,e+f)\arctan\left(\frac{1}{3}\,\sqrt{3}(2\,x-1)\right)\\ &\quad+\frac{1}{4}\,(d-f)\log\left(x^2+x+1\right)-\frac{1}{4}\,(d-f)\log\left(x^2-x+1\right)\end{aligned}$$

input
```
integrate((f*x^2+e*x+d)/(x^4+x^2+1),x, algorithm="maxima")
```

output
```
1/6*sqrt(3)*(d - 2*e + f)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/6*sqrt(3)*(d +
 2*e + f)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/4*(d - f)*log(x^2 + x + 1) - 1
/4*(d - f)*log(x^2 - x + 1)
```

3.16.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 75, normalized size of antiderivative = 0.72

$$\int \frac{d+ex+fx^2}{1+x^2+x^4}\,dx = \frac{1}{6}\sqrt{3}(d-2e+f)\arctan\left(\frac{1}{3}\sqrt{3}(2x+1)\right) + \frac{1}{6}\sqrt{3}(d+2e+f)\arctan\left(\frac{1}{3}\sqrt{3}(2x-1)\right) + \frac{1}{4}(d-f)\log\left(x^2+x+1\right) - \frac{1}{4}(d-f)\log\left(x^2-x+1\right)$$

input
```
integrate((f*x^2+e*x+d)/(x^4+x^2+1),x, algorithm="giac")
```

output
```
1/6*sqrt(3)*(d - 2*e + f)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/6*sqrt(3)*(d +
 2*e + f)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/4*(d - f)*log(x^2 + x + 1) - 1
/4*(d - f)*log(x^2 - x + 1)
```

3.16.9 Mupad [B] (verification not implemented)

Time = 7.98 (sec) , antiderivative size = 159, normalized size of antiderivative = 1.53

$$\int \frac{d+ex+fx^2}{1+x^2+x^4}\,dx = -\ln\left(x-\frac{1}{2}-\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{d}{4}-\frac{f}{4}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{12}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{6}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{12}\right) - \ln\left(x+\frac{1}{2}-\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{f}{4}-\frac{d}{4}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{12}-\frac{\sqrt{3}\,e\,1\mathrm{i}}{6}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{12}\right) + \ln\left(x-\frac{1}{2}+\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{f}{4}-\frac{d}{4}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{12}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{6}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{12}\right) + \ln\left(x+\frac{1}{2}+\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{d}{4}-\frac{f}{4}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{12}-\frac{\sqrt{3}\,e\,1\mathrm{i}}{6}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{12}\right)$$

input
```
int((d + e*x + f*x^2)/(x^2 + x^4 + 1),x)
```

output
```
log(x + (3^(1/2)*1i)/2 - 1/2)*(f/4 - d/4 + (3^(1/2)*d*1i)/12 + (3^(1/2)*e*
1i)/6 + (3^(1/2)*f*1i)/12) - log(x - (3^(1/2)*1i)/2 + 1/2)*(f/4 - d/4 + (3
^(1/2)*d*1i)/12 - (3^(1/2)*e*1i)/6 + (3^(1/2)*f*1i)/12) - log(x - (3^(1/2)
*1i)/2 - 1/2)*(d/4 - f/4 + (3^(1/2)*d*1i)/12 + (3^(1/2)*e*1i)/6 + (3^(1/2)
*f*1i)/12) + log(x + (3^(1/2)*1i)/2 + 1/2)*(d/4 - f/4 + (3^(1/2)*d*1i)/12
- (3^(1/2)*e*1i)/6 + (3^(1/2)*f*1i)/12)
```

3.16. $\int \frac{d+ex+fx^2}{1+x^2+x^4}\,dx$

3.17 $\int \frac{d+ex+fx^2+gx^3}{1+x^2+x^4}\,dx$

3.17.1 Optimal result

Integrand size = 26, antiderivative size = 127

$$\int \frac{d+ex+fx^2+gx^3}{1+x^2+x^4}\,dx = -\frac{(d+f)\arctan\left(\frac{1-2x}{\sqrt{3}}\right)}{2\sqrt{3}}+\frac{(d+f)\arctan\left(\frac{1+2x}{\sqrt{3}}\right)}{2\sqrt{3}} + \frac{(2e-g)\arctan\left(\frac{1+2x^2}{\sqrt{3}}\right)}{2\sqrt{3}}-\frac{1}{4}(d-f)\log\left(1-x+x^2\right) + \frac{1}{4}(d-f)\log\left(1+x+x^2\right)+\frac{1}{4}g\log\left(1+x^2+x^4\right)$$

output
```
-1/4*(d-f)*ln(x^2-x+1)+1/4*(d-f)*ln(x^2+x+1)+1/4*g*ln(x^4+x^2+1)-1/6*(d+f)
*arctan(1/3*(1-2*x)*3^(1/2))*3^(1/2)+1/6*(d+f)*arctan(1/3*(1+2*x)*3^(1/2))
*3^(1/2)+1/6*(2*e-g)*arctan(1/3*(2*x^2+1)*3^(1/2))*3^(1/2)
```

3.17.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.36 (sec) , antiderivative size = 150, normalized size of antiderivative = 1.18

$$\int \frac{d+ex+fx^2+gx^3}{1+x^2+x^4}\,dx$$
$$= \frac{2\sqrt{2-2i\sqrt{3}}\left(2id+\left(-i+\sqrt{3}\right)f\right)\arctan\left(\frac{1}{2}\left(-i+\sqrt{3}\right)x\right)+2\left(\sqrt{2+2i\sqrt{3}}\left(-2id+\left(i+\sqrt{3}\right)f\right)\arctan\left(\right.\right.}{8\sqrt{3}}$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3)/(1 + x^2 + x^4),x]
```

output
```
(2*Sqrt[2 - (2*I)*Sqrt[3]]*((2*I)*d + (-I + Sqrt[3])*f)*ArcTan[((-I + Sqrt
[3])*x)/2] + 2*(Sqrt[2 + (2*I)*Sqrt[3]]*((-2*I)*d + (I + Sqrt[3])*f)*ArcTa
n[((I + Sqrt[3])*x)/2] + (-4*e + 2*g)*ArcTan[Sqrt[3]/(1 + 2*x^2)] + Sqrt[3
]*g*Log[1 + x^2 + x^4]))/(8*Sqrt[3])
```

3.17.3 Rubi [A] (verified)

Time = 0.44 (sec) , antiderivative size = 133, normalized size of antiderivative = 1.05, number of steps used = 13, number of rules used = 12, $\frac{\text{number of rules}}{\text{integrand size}} = 0.462$, Rules used = {2202, 1483, 1142, 25, 1083, 217, 1103, 1576, 1142, 1083, 217, 1103}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{fx^2+d}{x^4+x^2+1}dx+\int \frac{x\left(gx^2+e\right)}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{1483}$$

$$\frac{1}{2}\int \frac{d-(d-f)x}{x^2-x+1}dx+\frac{1}{2}\int \frac{d+(d-f)x}{x^2+x+1}dx+\int \frac{x\left(gx^2+e\right)}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{1142}$$

$$\frac{1}{2}\left(\frac{1}{2}(d+f)\int \frac{1}{x^2-x+1}dx-\frac{1}{2}(d-f)\int -\frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(d+f)\int \frac{1}{x^2+x+1}dx+\frac{1}{2}(d-f)\int \frac{2x+1}{x^2+x+1}dx\right)+\int \frac{x\left(gx^2+e\right)}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{25}$$

$$\frac{1}{2}\left(\frac{1}{2}(d+f)\int \frac{1}{x^2-x+1}dx+\frac{1}{2}(d-f)\int \frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(d+f)\int \frac{1}{x^2+x+1}dx+\frac{1}{2}(d-f)\int \frac{2x+1}{x^2+x+1}dx\right)+\int \frac{x\left(gx^2+e\right)}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{1083}$$

$$\frac{1}{2}\left(\frac{1}{2}(d-f)\int \frac{1-2x}{x^2-x+1}dx-(d+f)\int \frac{1}{-(2x-1)^2-3}d(2x-1)\right)+$$
$$\frac{1}{2}\left(\frac{1}{2}(d-f)\int \frac{2x+1}{x^2+x+1}dx-(d+f)\int \frac{1}{-(2x+1)^2-3}d(2x+1)\right)+\int \frac{x\left(gx^2+e\right)}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{217}$$

$$\frac{1}{2}\left(\frac{1}{2}(d-f)\int \frac{1-2x}{x^2-x+1}dx+\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(d+f)}{\sqrt{3}}\right)+$$
$$\frac{1}{2}\left(\frac{1}{2}(d-f)\int \frac{2x+1}{x^2+x+1}dx+\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(d+f)}{\sqrt{3}}\right)+\int \frac{x\left(gx^2+e\right)}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{1103}$$

$$\int \frac{x\left(gx^2+e\right)}{x^4+x^2+1}dx+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(d+f)}{\sqrt{3}}-\frac{1}{2}(d-f)\log\left(x^2-x+1\right)\right)+$$
$$\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(d+f)}{\sqrt{3}}+\frac{1}{2}(d-f)\log\left(x^2+x+1\right)\right)$$

$$\Big\downarrow \text{1576}$$

$$\frac{1}{2}\int \frac{gx^2+e}{x^4+x^2+1}dx^2+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(d+f)}{\sqrt{3}}-\frac{1}{2}(d-f)\log\left(x^2-x+1\right)\right)+$$
$$\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(d+f)}{\sqrt{3}}+\frac{1}{2}(d-f)\log\left(x^2+x+1\right)\right)$$

$$\Big\downarrow \text{1142}$$

$$\frac{1}{2}\left(\frac{1}{2}(2e-g)\int \frac{1}{x^4+x^2+1}dx^2+\frac{1}{2}g\int \frac{2x^2+1}{x^4+x^2+1}dx^2\right)+$$
$$\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(d+f)}{\sqrt{3}}-\frac{1}{2}(d-f)\log\left(x^2-x+1\right)\right)+$$
$$\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(d+f)}{\sqrt{3}}+\frac{1}{2}(d-f)\log\left(x^2+x+1\right)\right)$$

$$\Big\downarrow \text{1083}$$

3.17. $\int \frac{d+ex+fx^2+gx^3}{1+x^2+x^4}\,dx$

$$\frac{1}{2}\left(\frac{1}{2}g\int \frac{2x^2+1}{x^4+x^2+1}dx^2-(2e-g)\int \frac{1}{-x^4-3}d(2x^2+1)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(d+f)}{\sqrt{3}}-\frac{1}{2}(d-f)\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(d+f)}{\sqrt{3}}+\frac{1}{2}(d-f)\log\left(x^2+x+1\right)\right)$$

$$\Big\downarrow \; 217$$

$$\frac{1}{2}\left(\frac{1}{2}g\int \frac{2x^2+1}{x^4+x^2+1}dx^2+\frac{\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g)}{\sqrt{3}}\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(d+f)}{\sqrt{3}}-\frac{1}{2}(d-f)\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(d+f)}{\sqrt{3}}+\frac{1}{2}(d-f)\log\left(x^2+x+1\right)\right)$$

$$\Big\downarrow \; 1103$$

$$\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(d+f)}{\sqrt{3}}-\frac{1}{2}(d-f)\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(d+f)}{\sqrt{3}}+\frac{1}{2}(d-f)\log\left(x^2+x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g)}{\sqrt{3}}+\frac{1}{2}g\log\left(x^4+x^2+1\right)\right)$$

input
```
Int[(d + e*x + f*x^2 + g*x^3)/(1 + x^2 + x^4),x]
```

output
```
(((d + f)*ArcTan[(-1 + 2*x)/Sqrt[3]])/Sqrt[3] - ((d - f)*Log[1 - x + x^2])
/2)/2 + (((d + f)*ArcTan[(1 + 2*x)/Sqrt[3]])/Sqrt[3] + ((d - f)*Log[1 + x
+ x^2])/2)/2 + (((2*e - g)*ArcTan[(1 + 2*x^2)/Sqrt[3]])/Sqrt[3] + (g*Log[1
 + x^2 + x^4])/2)/2
```

3.17. $\int \frac{d+ex+fx^2+gx^3}{1+x^2+x^4}\,dx$

3.17.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 217
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1483
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[a/c, 2]}, With[{r = Rt[2*q - b/c, 2]}, Simp[1/(2*c*q*r)   In
t[(d*r - (d - e*q)*x)/(q - r*x + x^2), x], x] + Simp[1/(2*c*q*r)   Int[(d*r
 + (d - e*q)*x)/(q + r*x + x^2), x], x]]] /; FreeQ[{a, b, c, d, e}, x] && N
eQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NegQ[b^2 - 4*a*c]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.17.4 Maple [A] (verified)

Time = 0.27 (sec) , antiderivative size = 90, normalized size of antiderivative = 0.71

method	result
default	$\frac{(f-d+g)\ln(x^2-x+1)}{4}+\frac{\left(\frac{d}{2}+e+\frac{f}{2}-\frac{g}{2}\right)\sqrt{3}\arctan\left(\frac{(2x-1)\sqrt{3}}{3}\right)}{3}+\frac{(d-f+g)\ln(x^2+x+1)}{4}+\frac{\left(\frac{d}{2}-e+\frac{f}{2}+\frac{g}{2}\right)\arctan\left(\frac{(1+2x)\sqrt{3}}{3}\right)\sqrt{}}{3}$
risch	Expression too large to display

input
```
int((g*x^3+f*x^2+e*x+d)/(x^4+x^2+1),x,method=_RETURNVERBOSE)
```

output
```
1/4*(f-d+g)*ln(x^2-x+1)+1/3*(1/2*d+e+1/2*f-1/2*g)*3^(1/2)*arctan(1/3*(2*x-
1)*3^(1/2))+1/4*(d-f+g)*ln(x^2+x+1)+1/3*(1/2*d-e+1/2*f+1/2*g)*arctan(1/3*(
1+2*x)*3^(1/2))*3^(1/2)
```

3.17.5 Fricas [A] (verification not implemented)

Time = 0.41 (sec) , antiderivative size = 83, normalized size of antiderivative = 0.65

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3}{1+x^2+x^4}\,dx &= \frac{1}{6}\sqrt{3}(d-2e+f+g)\arctan\left(\frac{1}{3}\sqrt{3}(2x+1)\right)\\ &\quad+\frac{1}{6}\sqrt{3}(d+2e+f-g)\arctan\left(\frac{1}{3}\sqrt{3}(2x-1)\right)\\ &\quad+\frac{1}{4}(d-f+g)\log\left(x^2+x+1\right)-\frac{1}{4}(d-f-g)\log\left(x^2-x+1\right)\end{aligned}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(x^4+x^2+1),x, algorithm="fricas")
```

output
```
1/6*sqrt(3)*(d - 2*e + f + g)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/6*sqrt(3)*
(d + 2*e + f - g)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/4*(d - f + g)*log(x^2
+ x + 1) - 1/4*(d - f - g)*log(x^2 - x + 1)
```

3.17.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d + ex + fx^2 + gx^3}{1 + x^2 + x^4} \, dx = \text{Timed out}$$

input
```
integrate((g*x**3+f*x**2+e*x+d)/(x**4+x**2+1),x)
```

output
```
Timed out
```

3.17.7 Maxima [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 83, normalized size of antiderivative = 0.65

$$\begin{aligned}\int \frac{d + ex + fx^2 + gx^3}{1 + x^2 + x^4} \, dx = {} & \frac{1}{6}\sqrt{3}(d - 2e + f + g)\arctan\left(\frac{1}{3}\sqrt{3}(2x + 1)\right) \\ & + \frac{1}{6}\sqrt{3}(d + 2e + f - g)\arctan\left(\frac{1}{3}\sqrt{3}(2x - 1)\right) \\ & + \frac{1}{4}(d - f + g)\log\left(x^2 + x + 1\right) - \frac{1}{4}(d - f - g)\log\left(x^2 - x + 1\right)\end{aligned}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(x^4+x^2+1),x, algorithm="maxima")
```

output
```
1/6*sqrt(3)*(d - 2*e + f + g)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/6*sqrt(3)*
(d + 2*e + f - g)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/4*(d - f + g)*log(x^2
+ x + 1) - 1/4*(d - f - g)*log(x^2 - x + 1)
```

3.17.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 83, normalized size of antiderivative = 0.65

$$\begin{aligned}\int \frac{d + ex + fx^2 + gx^3}{1 + x^2 + x^4} \, dx = {} & \frac{1}{6}\sqrt{3}(d - 2e + f + g)\arctan\left(\frac{1}{3}\sqrt{3}(2x + 1)\right) \\ & + \frac{1}{6}\sqrt{3}(d + 2e + f - g)\arctan\left(\frac{1}{3}\sqrt{3}(2x - 1)\right) \\ & + \frac{1}{4}(d - f + g)\log\left(x^2 + x + 1\right) - \frac{1}{4}(d - f - g)\log\left(x^2 - x + 1\right)\end{aligned}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(x^4+x^2+1),x, algorithm="giac")
```

output
```
1/6*sqrt(3)*(d - 2*e + f + g)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/6*sqrt(3)*
(d + 2*e + f - g)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/4*(d - f + g)*log(x^2
+ x + 1) - 1/4*(d - f - g)*log(x^2 - x + 1)
```

3.17.9 Mupad [B] (verification not implemented)

Time = 8.11 (sec) , antiderivative size = 199, normalized size of antiderivative = 1.57

$$\int \frac{d+ex+fx^2+gx^3}{1+x^2+x^4}\,dx = -\ln\left(x-\frac{1}{2}-\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{d}{4}-\frac{f}{4}-\frac{g}{4}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{12}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{6}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{12}-\frac{\sqrt{3}\,g\,1\mathrm{i}}{12}\right)-\ln\left(x+\frac{1}{2}-\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{f}{4}-\frac{d}{4}-\frac{g}{4}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{12}-\frac{\sqrt{3}\,e\,1\mathrm{i}}{6}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{12}+\frac{\sqrt{3}\,g\,1\mathrm{i}}{12}\right)+\ln\left(x-\frac{1}{2}+\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{f}{4}-\frac{d}{4}+\frac{g}{4}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{12}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{6}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{12}-\frac{\sqrt{3}\,g\,1\mathrm{i}}{12}\right)+\ln\left(x+\frac{1}{2}+\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{d}{4}-\frac{f}{4}+\frac{g}{4}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{12}-\frac{\sqrt{3}\,e\,1\mathrm{i}}{6}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{12}+\frac{\sqrt{3}\,g\,1\mathrm{i}}{12}\right)$$

input
```
int((d + e*x + f*x^2 + g*x^3)/(x^2 + x^4 + 1),x)
```

output
```
log(x + (3^(1/2)*1i)/2 - 1/2)*(f/4 - d/4 + g/4 + (3^(1/2)*d*1i)/12 + (3^(1
/2)*e*1i)/6 + (3^(1/2)*f*1i)/12 - (3^(1/2)*g*1i)/12) - log(x - (3^(1/2)*1i
)/2 + 1/2)*(f/4 - d/4 - g/4 + (3^(1/2)*d*1i)/12 - (3^(1/2)*e*1i)/6 + (3^(1
/2)*f*1i)/12 + (3^(1/2)*g*1i)/12) - log(x - (3^(1/2)*1i)/2 - 1/2)*(d/4 - f
/4 - g/4 + (3^(1/2)*d*1i)/12 + (3^(1/2)*e*1i)/6 + (3^(1/2)*f*1i)/12 - (3^(
1/2)*g*1i)/12) + log(x + (3^(1/2)*1i)/2 + 1/2)*(d/4 - f/4 + g/4 + (3^(1/2)
*d*1i)/12 - (3^(1/2)*e*1i)/6 + (3^(1/2)*f*1i)/12 + (3^(1/2)*g*1i)/12)
```

3.18 $\int \frac{d+ex+fx^2+gx^3+hx^4}{1+x^2+x^4}\,dx$

3.18.1 Optimal result

Integrand size = 31, antiderivative size = 136

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{1+x^2+x^4}\,dx = hx-\frac{(d+f-2h)\arctan\left(\frac{1-2x}{\sqrt{3}}\right)}{2\sqrt{3}}+\frac{(d+f-2h)\arctan\left(\frac{1+2x}{\sqrt{3}}\right)}{2\sqrt{3}}+\frac{(2e-g)\arctan\left(\frac{1+2x^2}{\sqrt{3}}\right)}{2\sqrt{3}}-\frac{1}{4}(d-f)\log\left(1-x+x^2\right)+\frac{1}{4}(d-f)\log\left(1+x+x^2\right)+\frac{1}{4}g\log\left(1+x^2+x^4\right)$$

output
```
h*x-1/4*(d-f)*ln(x^2-x+1)+1/4*(d-f)*ln(x^2+x+1)+1/4*g*ln(x^4+x^2+1)-1/6*(d
+f-2*h)*arctan(1/3*(1-2*x)*3^(1/2))*3^(1/2)+1/6*(d+f-2*h)*arctan(1/3*(1+2*
x)*3^(1/2))*3^(1/2)+1/6*(2*e-g)*arctan(1/3*(2*x^2+1)*3^(1/2))*3^(1/2)
```

3.18.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.47 (sec) , antiderivative size = 165, normalized size of antiderivative = 1.21

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{1+x^2+x^4}\,dx = \frac{1}{24}\Bigg(24hx+4\Big(\left(3i+\sqrt{3}\right)d+\left(-3i+\sqrt{3}\right)f - 2\sqrt{3}h\Big)\arctan\left(\frac{1}{2}\left(-i+\sqrt{3}\right)x\right)+4\Big(\left(-3i+\sqrt{3}\right)d + \left(3i+\sqrt{3}\right)f-2\sqrt{3}h\Big)\arctan\left(\frac{1}{2}\left(i+\sqrt{3}\right)x\right) - 8\sqrt{3}e\arctan\left(\frac{\sqrt{3}}{1+2x^2}\right)+4\sqrt{3}g\arctan\left(\frac{\sqrt{3}}{1+2x^2}\right) + 6g\log\left(1+x^2+x^4\right)\Bigg)$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4)/(1 + x^2 + x^4),x]
```

output
```
(24*h*x + 4*((3*I + Sqrt[3])*d + (-3*I + Sqrt[3])*f - 2*Sqrt[3]*h)*ArcTan[
((-I + Sqrt[3])*x)/2] + 4*((-3*I + Sqrt[3])*d + (3*I + Sqrt[3])*f - 2*Sqrt
[3]*h)*ArcTan[((I + Sqrt[3])*x)/2] - 8*Sqrt[3]*e*ArcTan[Sqrt[3]/(1 + 2*x^2
)] + 4*Sqrt[3]*g*ArcTan[Sqrt[3]/(1 + 2*x^2)] + 6*g*Log[1 + x^2 + x^4])/24
```

3.18.3 Rubi [A] (verified)

Time = 0.42 (sec) , antiderivative size = 138, normalized size of antiderivative = 1.01, number of steps used = 9, number of rules used = 8, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.258, Rules used = {2202, 1576, 1142, 1083, 217, 1103, 2205, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{ 2202}$$

$$\int \frac{hx^4+fx^2+d}{x^4+x^2+1}dx+\int \frac{x\left(gx^2+e\right)}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{1576}$$

$$\int \frac{hx^4+fx^2+d}{x^4+x^2+1}dx+\frac{1}{2}\int \frac{gx^2+e}{x^4+x^2+1}dx^2$$

$$\Big\downarrow \text{1142}$$

$$\int \frac{hx^4+fx^2+d}{x^4+x^2+1}dx+\frac{1}{2}\left(\frac{1}{2}(2e-g)\int \frac{1}{x^4+x^2+1}dx^2+\frac{1}{2}g\int \frac{2x^2+1}{x^4+x^2+1}dx^2\right)$$

$$\Big\downarrow \text{1083}$$

$$\int \frac{hx^4+fx^2+d}{x^4+x^2+1}dx+\frac{1}{2}\left(\frac{1}{2}g\int \frac{2x^2+1}{x^4+x^2+1}dx^2-(2e-g)\int \frac{1}{-x^4-3}d(2x^2+1)\right)$$

$$\Big\downarrow \text{217}$$

$$\frac{1}{2}\left(\frac{1}{2}g\int \frac{2x^2+1}{x^4+x^2+1}dx^2+\frac{\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g)}{\sqrt{3}}\right)+\int \frac{hx^4+fx^2+d}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{1103}$$

$$\int \frac{hx^4+fx^2+d}{x^4+x^2+1}dx+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g)}{\sqrt{3}}+\frac{1}{2}g\log\left(x^4+x^2+1\right)\right)$$

$$\Big\downarrow \text{2205}$$

$$\int \left(h+\frac{(f-h)x^2+d-h}{x^4+x^2+1}\right)dx+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g)}{\sqrt{3}}+\frac{1}{2}g\log\left(x^4+x^2+1\right)\right)$$

$$\Big\downarrow \text{2009}$$

$$-\frac{\arctan\left(\frac{1-2x}{\sqrt{3}}\right)(d+f-2h)}{2\sqrt{3}}+\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(d+f-2h)}{2\sqrt{3}}+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g)}{\sqrt{3}}+\frac{1}{2}g\log\left(x^4+x^2+1\right)\right)-\frac{1}{4}(d-f)\log\left(x^2-x+1\right)+\frac{1}{4}(d-f)\log\left(x^2+x+1\right)+hx$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4)/(1 + x^2 + x^4),x]
```

3.18. $\int \frac{d+ex+fx^2+gx^3+hx^4}{1+x^2+x^4}\,dx$

output
```
h*x - ((d + f - 2*h)*ArcTan[(1 - 2*x)/Sqrt[3]])/(2*Sqrt[3]) + ((d + f - 2*
h)*ArcTan[(1 + 2*x)/Sqrt[3]])/(2*Sqrt[3]) - ((d - f)*Log[1 - x + x^2])/4 +
 ((d - f)*Log[1 + x + x^2])/4 + (((2*e - g)*ArcTan[(1 + 2*x^2)/Sqrt[3]])/S
qrt[3] + (g*Log[1 + x^2 + x^4])/2)/2
```

3.18.3.1 Defintions of rubi rules used

rule 217
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2205
```
Int[(Px_)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :> Int[ExpandInte
grand[Px/(a + b*x^2 + c*x^4), x], x] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x^
2] && Expon[Px, x^2] > 1
```

3.18.4 Maple [A] (verified)

Time = 0.39 (sec) , antiderivative size = 99, normalized size of antiderivative = 0.73

method	result
default	$hx+\frac{(f-d+g)\ln(x^2-x+1)}{4}+\frac{\left(\frac{d}{2}+e+\frac{f}{2}-\frac{g}{2}-h\right)\sqrt{3}\,\arctan\left(\frac{(2x-1)\sqrt{3}}{3}\right)}{3}+\frac{(d-f+g)\ln(x^2+x+1)}{4}+\frac{\left(\frac{d}{2}-e+\frac{f}{2}+\frac{g}{2}-h\right)\text{arctan}}{3}$
risch	Expression too large to display

input
```
int((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1),x,method=_RETURNVERBOSE)
```

output
```
h*x+1/4*(f-d+g)*ln(x^2-x+1)+1/3*(1/2*d+e+1/2*f-1/2*g-h)*3^(1/2)*arctan(1/3
*(2*x-1)*3^(1/2))+1/4*(d-f+g)*ln(x^2+x+1)+1/3*(1/2*d-e+1/2*f+1/2*g-h)*arct
an(1/3*(1+2*x)*3^(1/2))*3^(1/2)
```

3.18.5 Fricas [A] (verification not implemented)

Time = 1.12 (sec) , antiderivative size = 92, normalized size of antiderivative = 0.68

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3+hx^4}{1+x^2+x^4}\,dx &= \frac{1}{6}\,\sqrt{3}(d-2\,e+f+g-2\,h)\arctan\left(\frac{1}{3}\,\sqrt{3}(2\,x+1)\right)\\&\quad+\frac{1}{6}\,\sqrt{3}(d+2\,e+f-g-2\,h)\arctan\left(\frac{1}{3}\,\sqrt{3}(2\,x-1)\right)\\&\quad+hx+\frac{1}{4}\,(d-f+g)\log\left(x^2+x+1\right)\\&\quad-\frac{1}{4}\,(d-f-g)\log\left(x^2-x+1\right)\end{aligned}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1),x, algorithm="fricas")
```

output
```
1/6*sqrt(3)*(d - 2*e + f + g - 2*h)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/6*sq
rt(3)*(d + 2*e + f - g - 2*h)*arctan(1/3*sqrt(3)*(2*x - 1)) + h*x + 1/4*(d
 - f + g)*log(x^2 + x + 1) - 1/4*(d - f - g)*log(x^2 - x + 1)
```

3.18.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{1+x^2+x^4}\,dx = \text{Timed out}$$

input
```
integrate((h*x**4+g*x**3+f*x**2+e*x+d)/(x**4+x**2+1),x)
```

output
```
Timed out
```

3.18.7 Maxima [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 92, normalized size of antiderivative = 0.68

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{1+x^2+x^4}\,dx = \frac{1}{6}\sqrt{3}(d-2e+f+g-2h)\arctan\left(\frac{1}{3}\sqrt{3}(2x+1)\right) + \frac{1}{6}\sqrt{3}(d+2e+f-g-2h)\arctan\left(\frac{1}{3}\sqrt{3}(2x-1)\right) + hx + \frac{1}{4}(d-f+g)\log\left(x^2+x+1\right) - \frac{1}{4}(d-f-g)\log\left(x^2-x+1\right)$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1),x, algorithm="maxima")
```

output
```
1/6*sqrt(3)*(d - 2*e + f + g - 2*h)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/6*sq
rt(3)*(d + 2*e + f - g - 2*h)*arctan(1/3*sqrt(3)*(2*x - 1)) + h*x + 1/4*(d
 - f + g)*log(x^2 + x + 1) - 1/4*(d - f - g)*log(x^2 - x + 1)
```

3.18.8 Giac [A] (verification not implemented)

Time = 0.32 (sec) , antiderivative size = 92, normalized size of antiderivative = 0.68

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{1+x^2+x^4}\,dx = \frac{1}{6}\sqrt{3}(d-2e+f+g-2h)\arctan\left(\frac{1}{3}\sqrt{3}(2x+1)\right) + \frac{1}{6}\sqrt{3}(d+2e+f-g-2h)\arctan\left(\frac{1}{3}\sqrt{3}(2x-1)\right) + hx + \frac{1}{4}(d-f+g)\log\left(x^2+x+1\right) - \frac{1}{4}(d-f-g)\log\left(x^2-x+1\right)$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1),x, algorithm="giac")
```

output
```
1/6*sqrt(3)*(d - 2*e + f + g - 2*h)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/6*sq
rt(3)*(d + 2*e + f - g - 2*h)*arctan(1/3*sqrt(3)*(2*x - 1)) + h*x + 1/4*(d
 - f + g)*log(x^2 + x + 1) - 1/4*(d - f - g)*log(x^2 - x + 1)
```

3.18.9 Mupad [B] (verification not implemented)

Time = 11.14 (sec) , antiderivative size = 1209, normalized size of antiderivative = 8.89

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{1+x^2+x^4}\,dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4)/(x^2 + x^4 + 1),x)
```

output
```
log(d*f*9i - d*e*6i + d*g*3i - d*h*3i + e*h*6i + f*h*3i - g*h*3i - 3*3^(1/
2)*d^2 - d^2*x*6i - f^2*x*3i - d^2*3i - f^2*6i + 2*3^(1/2)*d*e + 3*3^(1/2)
*d*f - 3^(1/2)*d*g - 4*3^(1/2)*e*f + 3*3^(1/2)*d*h + 2*3^(1/2)*e*h + 2*3^(
1/2)*f*g - 3*3^(1/2)*f*h - 3^(1/2)*g*h + d*f*x*9i - e*f*x*6i + d*h*x*3i +
e*h*x*6i + f*g*x*3i - f*h*x*3i - g*h*x*3i + 3*3^(1/2)*f^2*x - 3*3^(1/2)*d*
f*x - 2*3^(1/2)*d*g*x - 2*3^(1/2)*e*f*x + 3*3^(1/2)*d*h*x - 2*3^(1/2)*e*h*
x + 3^(1/2)*f*g*x - 3*3^(1/2)*f*h*x + 3^(1/2)*g*h*x + 4*3^(1/2)*d*e*x)*(d/
4 - f/4 + g/4 - (3^(1/2)*d*1i)/12 + (3^(1/2)*e*1i)/6 - (3^(1/2)*f*1i)/12 -
 (3^(1/2)*g*1i)/12 + (3^(1/2)*h*1i)/6) - log(d*g*3i - d*f*9i - d*e*6i + d*
h*3i + e*h*6i - f*h*3i - g*h*3i - 3*3^(1/2)*d^2 - d^2*x*6i - f^2*x*3i + d^
2*3i + f^2*6i - 2*3^(1/2)*d*e + 3*3^(1/2)*d*f + 3^(1/2)*d*g + 4*3^(1/2)*e*
f + 3*3^(1/2)*d*h - 2*3^(1/2)*e*h - 2*3^(1/2)*f*g - 3*3^(1/2)*f*h + 3^(1/2
)*g*h + d*f*x*9i + e*f*x*6i + d*h*x*3i - e*h*x*6i - f*g*x*3i - f*h*x*3i +
g*h*x*3i - 3*3^(1/2)*f^2*x + 3*3^(1/2)*d*f*x - 2*3^(1/2)*d*g*x - 2*3^(1/2)
*e*f*x - 3*3^(1/2)*d*h*x - 2*3^(1/2)*e*h*x + 3^(1/2)*f*g*x + 3*3^(1/2)*f*h
*x + 3^(1/2)*g*h*x + 4*3^(1/2)*d*e*x)*(d/4 - f/4 - g/4 + (3^(1/2)*d*1i)/12
 + (3^(1/2)*e*1i)/6 + (3^(1/2)*f*1i)/12 - (3^(1/2)*g*1i)/12 - (3^(1/2)*h*1
i)/6) + log(d*f*9i - d*e*6i + d*g*3i - d*h*3i + e*h*6i + f*h*3i - g*h*3i +
 3*3^(1/2)*d^2 - d^2*x*6i - f^2*x*3i - d^2*3i - f^2*6i - 2*3^(1/2)*d*e - 3
*3^(1/2)*d*f + 3^(1/2)*d*g + 4*3^(1/2)*e*f - 3*3^(1/2)*d*h - 2*3^(1/2)*...
```

3.19 $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{1+x^2+x^4}\,dx$

3.19.1 Optimal result

Integrand size = 36, antiderivative size = 151

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{1+x^2+x^4}\,dx = hx+\frac{ix^2}{2}-\frac{(d+f-2h)\arctan\left(\frac{1-2x}{\sqrt{3}}\right)}{2\sqrt{3}} + \frac{(d+f-2h)\arctan\left(\frac{1+2x}{\sqrt{3}}\right)}{2\sqrt{3}} + \frac{(2e-g-i)\arctan\left(\frac{1+2x^2}{\sqrt{3}}\right)}{2\sqrt{3}} - \frac{1}{4}(d-f)\log\left(1-x+x^2\right) + \frac{1}{4}(d-f)\log\left(1+x+x^2\right)+\frac{1}{4}(g-i)\log\left(1+x^2+x^4\right)$$

output
```
h*x+1/2*i*x^2-1/4*(d-f)*ln(x^2-x+1)+1/4*(d-f)*ln(x^2+x+1)+1/4*(g-i)*ln(x^4
+x^2+1)-1/6*(d+f-2*h)*arctan(1/3*(1-2*x)*3^(1/2))*3^(1/2)+1/6*(d+f-2*h)*ar
ctan(1/3*(1+2*x)*3^(1/2))*3^(1/2)+1/6*(2*e-g-i)*arctan(1/3*(2*x^2+1)*3^(1/
2))*3^(1/2)
```

3.19.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.47 (sec) , antiderivative size = 187, normalized size of antiderivative = 1.24

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{1+x^2+x^4}\,dx = \frac{1}{12}\Bigg(6x(2h+ix)+\left(1+i\sqrt{3}\right)\left(2\sqrt{3}d-\left(3i+\sqrt{3}\right)f - \left(-3i+\sqrt{3}\right)h\right)\arctan\left(\frac{1}{2}\left(-i+\sqrt{3}\right)x\right) + \left(i+\sqrt{3}\right)\left(-2i\sqrt{3}d+\left(3+i\sqrt{3}\right)f + i\left(3i+\sqrt{3}\right)h\right)\arctan\left(\frac{1}{2}\left(i+\sqrt{3}\right)x\right) - 2\sqrt{3}(2e-g-i)\arctan\left(\frac{\sqrt{3}}{1+2x^2}\right) + 3(g-i)\log\left(1+x^2+x^4\right)\Bigg)$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(1 + x^2 + x^4),x]
```

output
```
(6*x*(2*h + i*x) + (1 + I*Sqrt[3])*(2*Sqrt[3]*d - (3*I + Sqrt[3])*f - (-3*
I + Sqrt[3])*h)*ArcTan[((-I + Sqrt[3])*x)/2] + (I + Sqrt[3])*((-2*I)*Sqrt[
3]*d + (3 + I*Sqrt[3])*f + I*(3*I + Sqrt[3])*h)*ArcTan[((I + Sqrt[3])*x)/2
] - 2*Sqrt[3]*(2*e - g - i)*ArcTan[Sqrt[3]/(1 + 2*x^2)] + 3*(g - i)*Log[1
+ x^2 + x^4])/12
```

3.19.3 Rubi [A] (verified)

Time = 0.49 (sec) , antiderivative size = 150, normalized size of antiderivative = 0.99, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}} = 0.167$, Rules used = {2202, 2194, 2188, 2009, 2205, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{x^4+x^2+1}\,dx$$

$$\Big\downarrow \; 2202$$

$$\int \frac{hx^4+fx^2+d}{x^4+x^2+1}dx+\int \frac{x\left(ix^4+gx^2+e\right)}{x^4+x^2+1}dx$$

$$\Big\downarrow \text{2194}$$

$$\int \frac{hx^4+fx^2+d}{x^4+x^2+1}dx+\frac{1}{2}\int \frac{ix^4+gx^2+e}{x^4+x^2+1}dx^2$$

$$\Big\downarrow \text{2188}$$

$$\int \frac{hx^4+fx^2+d}{x^4+x^2+1}dx+\frac{1}{2}\int \left(i+\frac{(g-i)x^2+e-i}{x^4+x^2+1}\right)dx^2$$

$$\Big\downarrow \text{2009}$$

$$\int \frac{hx^4+fx^2+d}{x^4+x^2+1}dx+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g-i)}{\sqrt{3}}+\frac{1}{2}(g-i)\log\left(x^4+x^2+1\right)+ix^2\right)$$

$$\Big\downarrow \text{2205}$$

$$\int \left(h+\frac{(f-h)x^2+d-h}{x^4+x^2+1}\right)dx+$$
$$\frac{1}{2}\left(\frac{\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g-i)}{\sqrt{3}}+\frac{1}{2}(g-i)\log\left(x^4+x^2+1\right)+ix^2\right)$$

$$\Big\downarrow \text{2009}$$

$$-\frac{\arctan\left(\frac{1-2x}{\sqrt{3}}\right)(d+f-2h)}{2\sqrt{3}}+\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(d+f-2h)}{2\sqrt{3}}+$$
$$\frac{1}{2}\left(\frac{\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g-i)}{\sqrt{3}}+\frac{1}{2}(g-i)\log\left(x^4+x^2+1\right)+ix^2\right)-\frac{1}{4}(d-f)\log\left(x^2-x+1\right)+$$
$$\frac{1}{4}(d-f)\log\left(x^2+x+1\right)+hx$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(1 + x^2 + x^4),x]
```

output
```
h*x - ((d + f - 2*h)*ArcTan[(1 - 2*x)/Sqrt[3]])/(2*Sqrt[3]) + ((d + f - 2*
h)*ArcTan[(1 + 2*x)/Sqrt[3]])/(2*Sqrt[3]) - ((d - f)*Log[1 - x + x^2])/4 +
 ((d - f)*Log[1 + x + x^2])/4 + (i*x^2 + ((2*e - g - i)*ArcTan[(1 + 2*x^2)
/Sqrt[3]])/Sqrt[3] + ((g - i)*Log[1 + x^2 + x^4])/2)/2
```

3.19. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{1+x^2+x^4}\,dx$

3.19.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2188
```
Int[(Pq_)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_.), x_Symbol] :> Int[Expand
Integrand[Pq*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c}, x] && PolyQ[Pq
, x] && IGtQ[p, -2]
```

rule 2194
```
Int[(Pq_)*(x_)^(m_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :
> Simp[1/2   Subst[Int[x^((m - 1)/2)*SubstFor[x^2, Pq, x]*(a + b*x + c*x^2)
^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pq, x^2] && IntegerQ
[(m - 1)/2]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2205
```
Int[(Px_)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :> Int[ExpandInte
grand[Px/(a + b*x^2 + c*x^4), x], x] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x^
2] && Expon[Px, x^2] > 1
```

3.19.4 Maple [A] (verified)

Time = 0.46 (sec) , antiderivative size = 117, normalized size of antiderivative = 0.77

method	result
default	$\frac{i x^2}{2}+hx+\frac{(g-i+f-d)\ln(x^2-x+1)}{4}+\frac{\left(\frac{d}{2}+e+\frac{f}{2}-\frac{g}{2}-h-\frac{i}{2}\right)\sqrt{3}\arctan\left(\frac{(2x-1)\sqrt{3}}{3}\right)}{3}+\frac{(d-f+g-i)\ln(x^2+x+1)}{4}+\left(\frac{d}{2}-e+\right.$
risch	Expression too large to display

input
```
int((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1),x,method=_RETURNVERBOSE)
```

output
```
1/2*i*x^2+h*x+1/4*(g-i+f-d)*ln(x^2-x+1)+1/3*(1/2*d+e+1/2*f-1/2*g-h-1/2*i)*
3^(1/2)*arctan(1/3*(2*x-1)*3^(1/2))+1/4*(d-f+g-i)*ln(x^2+x+1)+1/3*(1/2*d-e
+1/2*f+1/2*g-h+1/2*i)*arctan(1/3*(1+2*x)*3^(1/2))*3^(1/2)
```

3.19.5 Fricas [A] (verification not implemented)

Time = 4.39 (sec) , antiderivative size = 106, normalized size of antiderivative = 0.70

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{1+x^2+x^4}\,dx = \frac{1}{2}\,ix^2+\frac{1}{6}\,\sqrt{3}(d-2\,e+f+g-2\,h+i)\arctan\left(\frac{1}{3}\,\sqrt{3}(2\,x+1)\right) + \frac{1}{6}\,\sqrt{3}(d+2\,e+f-g-2\,h-i)\arctan\left(\frac{1}{3}\,\sqrt{3}(2\,x-1)\right)+hx + \frac{1}{4}\,(d-f+g-i)\log\left(x^2+x+1\right)-\frac{1}{4}\,(d-f-g+i)\log\left(x^2-x+1\right)$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1),x, algorithm="fricas
")
```

output
```
1/2*i*x^2 + 1/6*sqrt(3)*(d - 2*e + f + g - 2*h + i)*arctan(1/3*sqrt(3)*(2*
x + 1)) + 1/6*sqrt(3)*(d + 2*e + f - g - 2*h - i)*arctan(1/3*sqrt(3)*(2*x
- 1)) + h*x + 1/4*(d - f + g - i)*log(x^2 + x + 1) - 1/4*(d - f - g + i)*l
og(x^2 - x + 1)
```

3.19.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{1+x^2+x^4}\,dx = \text{Timed out}$$

input
```
integrate((i*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(x**4+x**2+1),x)
```

output
```
Timed out
```

3.19.7 Maxima [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 106, normalized size of antiderivative = 0.70

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{1+x^2+x^4}\,dx = \frac{1}{2}\,ix^2+\frac{1}{6}\,\sqrt{3}(d-2\,e+f+g-2\,h+i)\arctan\left(\frac{1}{3}\,\sqrt{3}(2\,x+1)\right) + \frac{1}{6}\,\sqrt{3}(d+2\,e+f-g-2\,h-i)\arctan\left(\frac{1}{3}\,\sqrt{3}(2\,x-1)\right)+hx + \frac{1}{4}\,(d-f+g-i)\log\left(x^2+x+1\right)-\frac{1}{4}\,(d-f-g+i)\log\left(x^2-x+1\right)$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1),x, algorithm="maxima
")
```

output
```
1/2*i*x^2 + 1/6*sqrt(3)*(d - 2*e + f + g - 2*h + i)*arctan(1/3*sqrt(3)*(2*
x + 1)) + 1/6*sqrt(3)*(d + 2*e + f - g - 2*h - i)*arctan(1/3*sqrt(3)*(2*x
- 1)) + h*x + 1/4*(d - f + g - i)*log(x^2 + x + 1) - 1/4*(d - f - g + i)*l
og(x^2 - x + 1)
```

3.19.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 106, normalized size of antiderivative = 0.70

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{1+x^2+x^4}\,dx = \frac{1}{2}\,ix^2+\frac{1}{6}\,\sqrt{3}(d-2\,e+f+g-2\,h+i)\arctan\left(\frac{1}{3}\,\sqrt{3}(2\,x+1)\right) + \frac{1}{6}\,\sqrt{3}(d+2\,e+f-g-2\,h-i)\arctan\left(\frac{1}{3}\,\sqrt{3}(2\,x-1)\right)+hx + \frac{1}{4}\,(d-f+g-i)\log\left(x^2+x+1\right)-\frac{1}{4}\,(d-f-g+i)\log\left(x^2-x+1\right)$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1),x, algorithm="giac")
```

output
```
1/2*i*x^2 + 1/6*sqrt(3)*(d - 2*e + f + g - 2*h + i)*arctan(1/3*sqrt(3)*(2*
x + 1)) + 1/6*sqrt(3)*(d + 2*e + f - g - 2*h - i)*arctan(1/3*sqrt(3)*(2*x
- 1)) + h*x + 1/4*(d - f + g - i)*log(x^2 + x + 1) - 1/4*(d - f - g + i)*l
og(x^2 - x + 1)
```

3.19.9 Mupad [B] (verification not implemented)

Time = 12.66 (sec) , antiderivative size = 1509, normalized size of antiderivative = 9.99

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{1+x^2+x^4}\,dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(x^2 + x^4 + 1),x)
```

output
```
h*x - log(d*g*3i - d*f*9i - d*e*6i + d*h*3i + d*i*3i + e*h*6i - f*h*3i - g
*h*3i - h*i*3i - 3*3^(1/2)*d^2 - d^2*x*6i - f^2*x*3i + d^2*3i + f^2*6i - 2
*3^(1/2)*d*e + 3*3^(1/2)*d*f + 3^(1/2)*d*g + 4*3^(1/2)*e*f + 3*3^(1/2)*d*h
 + 3^(1/2)*d*i - 2*3^(1/2)*e*h - 2*3^(1/2)*f*g - 3*3^(1/2)*f*h - 2*3^(1/2)
*f*i + 3^(1/2)*g*h + 3^(1/2)*h*i + d*f*x*9i + e*f*x*6i + d*h*x*3i - e*h*x*
6i - f*g*x*3i - f*h*x*3i - f*i*x*3i + g*h*x*3i + h*i*x*3i - 3*3^(1/2)*f^2*
x + 3*3^(1/2)*d*f*x - 2*3^(1/2)*d*g*x - 2*3^(1/2)*e*f*x - 3*3^(1/2)*d*h*x
- 2*3^(1/2)*d*i*x - 2*3^(1/2)*e*h*x + 3^(1/2)*f*g*x + 3*3^(1/2)*f*h*x + 3^
(1/2)*f*i*x + 3^(1/2)*g*h*x + 3^(1/2)*h*i*x + 4*3^(1/2)*d*e*x)*(d/4 - f/4
- g/4 + i/4 + (3^(1/2)*d*1i)/12 + (3^(1/2)*e*1i)/6 + (3^(1/2)*f*1i)/12 - (
3^(1/2)*g*1i)/12 - (3^(1/2)*h*1i)/6 - (3^(1/2)*i*1i)/12) - log(d*e*6i + d*
f*9i - d*g*3i - d*h*3i - d*i*3i - e*h*6i + f*h*3i + g*h*3i + h*i*3i - 3*3^
(1/2)*d^2 + d^2*x*6i + f^2*x*3i - d^2*3i - f^2*6i - 2*3^(1/2)*d*e + 3*3^(1
/2)*d*f + 3^(1/2)*d*g + 4*3^(1/2)*e*f + 3*3^(1/2)*d*h + 3^(1/2)*d*i - 2*3^
(1/2)*e*h - 2*3^(1/2)*f*g - 3*3^(1/2)*f*h - 2*3^(1/2)*f*i + 3^(1/2)*g*h +
3^(1/2)*h*i - d*f*x*9i - e*f*x*6i - d*h*x*3i + e*h*x*6i + f*g*x*3i + f*h*x
*3i + f*i*x*3i - g*h*x*3i - h*i*x*3i - 3*3^(1/2)*f^2*x + 3*3^(1/2)*d*f*x -
 2*3^(1/2)*d*g*x - 2*3^(1/2)*e*f*x - 3*3^(1/2)*d*h*x - 2*3^(1/2)*d*i*x - 2
*3^(1/2)*e*h*x + 3^(1/2)*f*g*x + 3*3^(1/2)*f*h*x + 3^(1/2)*f*i*x + 3^(1/2)
*g*h*x + 3^(1/2)*h*i*x + 4*3^(1/2)*d*e*x)*(d/4 - f/4 - g/4 + i/4 - (3^(...
```

3.20 $\int \frac{d+ex}{a+bx^2+cx^4}\,dx$

3.20.1 Optimal result

Integrand size = 20, antiderivative size = 189

$$\int \frac{d+ex}{a+bx^2+cx^4}\,dx = \frac{\sqrt{2}\sqrt{c}d\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{\sqrt{b^2-4ac}\sqrt{b-\sqrt{b^2-4ac}}} - \frac{\sqrt{2}\sqrt{c}d\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{\sqrt{b^2-4ac}\sqrt{b+\sqrt{b^2-4ac}}} - \frac{e\,\mathrm{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\sqrt{b^2-4ac}}$$

output
```
-e*arctanh((2*c*x^2+b)/(-4*a*c+b^2)^(1/2))/(-4*a*c+b^2)^(1/2)+d*arctan(x*2
^(1/2)*c^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2))*2^(1/2)*c^(1/2)/(-4*a*c+b^2)^
(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2)-d*arctan(x*2^(1/2)*c^(1/2)/(b+(-4*a*c+b
^2)^(1/2))^(1/2))*2^(1/2)*c^(1/2)/(-4*a*c+b^2)^(1/2)/(b+(-4*a*c+b^2)^(1/2)
)^(1/2)
```

3.20.2 Mathematica [A] (verified)

Time = 0.17 (sec) , antiderivative size = 194, normalized size of antiderivative = 1.03

$$\int \frac{d+ex}{a+bx^2+cx^4}\,dx = \frac{\frac{2\sqrt{2}\sqrt{c}d\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{\sqrt{b-\sqrt{b^2-4ac}}} - \frac{2\sqrt{2}\sqrt{c}d\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{\sqrt{b+\sqrt{b^2-4ac}}} + e\left(\log\left(-b+\sqrt{b^2-4ac}-2cx^2\right) - \log\left(b+\sqrt{b^2-4\right.\right.}{2\sqrt{b^2-4ac}}$$

input
```
Integrate[(d + e*x)/(a + b*x^2 + c*x^4),x]
```

output
```
((2*Sqrt[2]*Sqrt[c]*d*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c
]]])/Sqrt[b - Sqrt[b^2 - 4*a*c]] - (2*Sqrt[2]*Sqrt[c]*d*ArcTan[(Sqrt[2]*Sq
rt[c]*x)/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/Sqrt[b + Sqrt[b^2 - 4*a*c]] + e*(Lo
g[-b + Sqrt[b^2 - 4*a*c] - 2*c*x^2] - Log[b + Sqrt[b^2 - 4*a*c] + 2*c*x^2]
))/(2*Sqrt[b^2 - 4*a*c])
```

3.20.3 Rubi [A] (verified)

Time = 0.39 (sec) , antiderivative size = 190, normalized size of antiderivative = 1.01, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.350$, Rules used = {2202, 27, 1406, 218, 1432, 1083, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex}{a+bx^2+cx^4}dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{d}{cx^4+bx^2+a}dx+\int \frac{ex}{cx^4+bx^2+a}dx$$

$$\Big\downarrow \text{27}$$

$$d\int \frac{1}{cx^4+bx^2+a}dx+e\int \frac{x}{cx^4+bx^2+a}dx$$

$$\Big\downarrow \text{1406}$$

$$d\left(\frac{c\int \frac{1}{cx^2+\frac{1}{2}\left(b-\sqrt{b^2-4ac}\right)}dx}{\sqrt{b^2-4ac}}-\frac{c\int \frac{1}{cx^2+\frac{1}{2}\left(b+\sqrt{b^2-4ac}\right)}dx}{\sqrt{b^2-4ac}}\right)+e\int \frac{x}{cx^4+bx^2+a}dx$$

$$\Big\downarrow \text{218}$$

$$e\int \frac{x}{cx^4+bx^2+a}dx+d\left(\frac{\sqrt{2}\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{\sqrt{b^2-4ac}\sqrt{b-\sqrt{b^2-4ac}}}-\frac{\sqrt{2}\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)}{\sqrt{b^2-4ac}\sqrt{\sqrt{b^2-4ac}+b}}\right)$$

$$\Big\downarrow \text{1432}$$

$$\frac{1}{2}e\int \frac{1}{cx^4+bx^2+a}dx^2+d\left(\frac{\sqrt{2}\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{\sqrt{b^2-4ac}\sqrt{b-\sqrt{b^2-4ac}}}-\frac{\sqrt{2}\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)}{\sqrt{b^2-4ac}\sqrt{\sqrt{b^2-4ac}+b}}\right)$$

$$\Big\downarrow \text{1083}$$

$$d\left(\frac{\sqrt{2}\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{\sqrt{b^2-4ac}\sqrt{b-\sqrt{b^2-4ac}}}-\frac{\sqrt{2}\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)}{\sqrt{b^2-4ac}\sqrt{\sqrt{b^2-4ac}+b}}\right)-e\int \frac{1}{-x^4+b^2-4ac}d(2cx^2+b)$$

$$\Big\downarrow \text{219}$$

$$d\left(\frac{\sqrt{2}\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{\sqrt{b^2-4ac}\sqrt{b-\sqrt{b^2-4ac}}}-\frac{\sqrt{2}\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)}{\sqrt{b^2-4ac}\sqrt{\sqrt{b^2-4ac}+b}}\right)-\frac{e\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\sqrt{b^2-4ac}}$$

input
```
Int[(d + e*x)/(a + b*x^2 + c*x^4),x]
```

output
```
d*((Sqrt[2]*Sqrt[c]*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]
])/(Sqrt[b^2 - 4*a*c]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) - (Sqrt[2]*Sqrt[c]*ArcT
an[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(Sqrt[b^2 - 4*a*c]*Sq
rt[b + Sqrt[b^2 - 4*a*c]])) - (e*ArcTanh[(b + 2*c*x^2)/Sqrt[b^2 - 4*a*c]])
/Sqrt[b^2 - 4*a*c]
```

3.20.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1406
```
Int[((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(-1), x_Symbol] :> With[{q = Rt[b^
2 - 4*a*c, 2]}, Simp[c/q   Int[1/(b/2 - q/2 + c*x^2), x], x] - Simp[c/q   I
nt[1/(b/2 + q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c}, x] && NeQ[b^2 - 4*a*c
, 0] && PosQ[b^2 - 4*a*c]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.20.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.08 (sec) , antiderivative size = 43, normalized size of antiderivative = 0.23

method	result
risch	$\frac{\left(\sum_{\_R=\mathrm{RootOf}\left(c\_Z^4+\_Z^2 b+a\right)}\frac{\left(\_Re+d\right)\ln\left(x-\_R\right)}{2c\_R^3+\_Rb}\right)}{2}$
default	$4c\left(\frac{\sqrt{-4ac+b^2}\left(\frac{e\ln\left(2cx^2+\sqrt{-4ac+b^2}+b\right)}{4c}+\frac{d\sqrt{2}\arctan\left(\frac{cx\sqrt{2}}{\sqrt{\left(b+\sqrt{-4ac+b^2}\right)c}}\right)}{2\sqrt{\left(b+\sqrt{-4ac+b^2}\right)c}}\right)}{8ac-2b^2}-\frac{\sqrt{-4ac+b^2}\left(\frac{e\ln\left(-2cx^2+\sqrt{-4ac+b^2}-b\right)}{4c}-\frac{d\sqrt{2}\ldots}{\ldots}\right.}{8ac-2b^2}\right.$

input
```
int((e*x+d)/(c*x^4+b*x^2+a),x,method=_RETURNVERBOSE)
```

output
```
1/2*sum((_R*e+d)/(2*_R^3*c+_R*b)*ln(x-_R),_R=RootOf(_Z^4*c+_Z^2*b+a))
```

3.20.5 Fricas [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 3.16 (sec) , antiderivative size = 398481, normalized size of antiderivative = 2108.37

$$\int \frac{d+ex}{a+bx^2+cx^4}\,dx = \text{Too large to display}$$

input
```
integrate((e*x+d)/(c*x^4+b*x^2+a),x, algorithm="fricas")
```

output
```
Too large to include
```

3.20.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex}{a+bx^2+cx^4}\,dx = \text{Timed out}$$

input
```
integrate((e*x+d)/(c*x**4+b*x**2+a),x)
```

output
```
Timed out
```

3.20.7 Maxima [F]

$$\int \frac{d+ex}{a+bx^2+cx^4}\,dx = \int \frac{ex+d}{cx^4+bx^2+a}\,dx$$

input
```
integrate((e*x+d)/(c*x^4+b*x^2+a),x, algorithm="maxima")
```

output
```
integrate((e*x + d)/(c*x^4 + b*x^2 + a), x)
```

3.20.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 1342 vs. $2(149) = 298$.

Time = 1.40 (sec) , antiderivative size = 1342, normalized size of antiderivative = 7.10

$$\int \frac{d+ex}{a+bx^2+cx^4}\,dx = \text{Too large to display}$$

input
```
integrate((e*x+d)/(c*x^4+b*x^2+a),x, algorithm="giac")
```

output
```
1/4*(sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^4 - 8*sqrt(2)*sqrt(b*c + sq
rt(b^2 - 4*a*c)*c)*a*b^2*c - 2*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^3
*c - 2*b^4*c + 16*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*c^2 + 8*sqrt
(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c^2 + sqrt(2)*sqrt(b*c + sqrt(b^2
- 4*a*c)*c)*b^2*c^2 + 16*a*b^2*c^2 + 2*b^3*c^2 - 4*sqrt(2)*sqrt(b*c + sqrt
(b^2 - 4*a*c)*c)*a*c^3 - 32*a^2*c^3 - 8*a*b*c^3 - sqrt(2)*sqrt(b^2 - 4*a*c
)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^3 + 4*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b
*c + sqrt(b^2 - 4*a*c)*c)*a*b*c + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + s
qrt(b^2 - 4*a*c)*c)*b^2*c - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2
- 4*a*c)*c)*b*c^2 + 2*(b^2 - 4*a*c)*b^2*c - 8*(b^2 - 4*a*c)*a*c^2 - 2*(b^2
 - 4*a*c)*b*c^2)*d*arctan(2*sqrt(1/2)*x/sqrt((b + sqrt(b^2 - 4*a*c))/c))/(
(a*b^4 - 8*a^2*b^2*c - 2*a*b^3*c + 16*a^3*c^2 + 8*a^2*b*c^2 + a*b^2*c^2 -
4*a^2*c^3)*abs(c)) + 1/4*(sqrt(2)*sqrt(b*c - sqrt(b^2 - 4*a*c)*c)*b^4 - 8*
sqrt(2)*sqrt(b*c - sqrt(b^2 - 4*a*c)*c)*a*b^2*c - 2*sqrt(2)*sqrt(b*c - sqr
t(b^2 - 4*a*c)*c)*b^3*c + 2*b^4*c + 16*sqrt(2)*sqrt(b*c - sqrt(b^2 - 4*a*c
)*c)*a^2*c^2 + 8*sqrt(2)*sqrt(b*c - sqrt(b^2 - 4*a*c)*c)*a*b*c^2 + sqrt(2)
*sqrt(b*c - sqrt(b^2 - 4*a*c)*c)*b^2*c^2 - 16*a*b^2*c^2 + 2*b^3*c^2 - 4*sq
rt(2)*sqrt(b*c - sqrt(b^2 - 4*a*c)*c)*a*c^3 + 32*a^2*c^3 - 8*a*b*c^3 - sqr
t(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c - sqrt(b^2 - 4*a*c)*c)*b^3 + 4*sqrt(2)*sqr
t(b^2 - 4*a*c)*sqrt(b*c - sqrt(b^2 - 4*a*c)*c)*a*b*c + 2*sqrt(2)*sqrt(b...
```

3.20.9 Mupad [B] (verification not implemented)

Time = 8.36 (sec) , antiderivative size = 1308, normalized size of antiderivative = 6.92

$$\int \frac{d+ex}{a+bx^2+cx^4}\,dx = \sum_{k=1}^{4} \ln\Big(c^2\Big(d\,e^2+e^3\,x$$
$$+\mathrm{root}\big(128\,a^2\,b^2\,c\,z^4-256\,a^3\,c^2\,z^4-16\,a\,b^4\,z^4+16\,a\,b\,c\,d^2\,z^2-32\,a^2\,c\,e^2\,z^2+8\,a\,b^2\,e^2\,z^2-4\,b^3\,d^2\,z^2+16$$
$$-\mathrm{root}\big(128\,a^2\,b^2\,c\,z^4-256\,a^3\,c^2\,z^4-16\,a\,b^4\,z^4+16\,a\,b\,c\,d^2\,z^2-32\,a^2\,c\,e^2\,z^2+8\,a\,b^2\,e^2\,z^2-4\,b^3\,d^2\,z^2+16$$
$$-\mathrm{root}\big(128\,a^2\,b^2\,c\,z^4-256\,a^3\,c^2\,z^4-16\,a\,b^4\,z^4+16\,a\,b\,c\,d^2\,z^2-32\,a^2\,c\,e^2\,z^2+8\,a\,b^2\,e^2\,z^2-4\,b^3\,d^2\,z^2+16$$
$$+\mathrm{root}\big(128\,a^2\,b^2\,c\,z^4-256\,a^3\,c^2\,z^4-16\,a\,b^4\,z^4+16\,a\,b\,c\,d^2\,z^2-32\,a^2\,c\,e^2\,z^2+8\,a\,b^2\,e^2\,z^2-4\,b^3\,d^2\,z^2+16\,a\,c\,d^2$$
$$-\mathrm{root}\big(128\,a^2\,b^2\,c\,z^4-256\,a^3\,c^2\,z^4-16\,a\,b^4\,z^4+16\,a\,b\,c\,d^2\,z^2-32\,a^2\,c\,e^2\,z^2+8\,a\,b^2\,e^2\,z^2-4\,b^3\,d^2\,z^2+16\,a\,c\,d^2$$
$$-\mathrm{root}\big(128\,a^2\,b^2\,c\,z^4-256\,a^3\,c^2\,z^4-16\,a\,b^4\,z^4+16\,a\,b\,c\,d^2\,z^2-32\,a^2\,c\,e^2\,z^2+8\,a\,b^2\,e^2\,z^2-4\,b^3\,d^2\,z^2+16$$
$$+\mathrm{root}\big(128\,a^2\,b^2\,c\,z^4-256\,a^3\,c^2\,z^4-16\,a\,b^4\,z^4+16\,a\,b\,c\,d^2\,z^2-32\,a^2\,c\,e^2\,z^2+8\,a\,b^2\,e^2\,z^2-4\,b^3\,d^2\,z^2+16\,a\,c\,d^2$$
$$+\mathrm{root}\big(128\,a^2\,b^2\,c\,z^4-256\,a^3\,c^2\,z^4-16\,a\,b^4\,z^4+16\,a\,b\,c\,d^2\,z^2-32\,a^2\,c\,e^2\,z^2+8\,a\,b^2\,e^2\,z^2-4\,b^3\,d^2\,z^2+16$$
$$+\mathrm{root}\big(128\,a^2\,b^2\,c\,z^4-256\,a^3\,c^2\,z^4-16\,a\,b^4\,z^4+16\,a\,b\,c\,d^2\,z^2-32\,a^2\,c\,e^2\,z^2+8\,a\,b^2\,e^2\,z^2-4\,b^3\,d^2\,z^2+16$$
$$-256\,a^3\,c^2\,z^4-16\,a\,b^4\,z^4+16\,a\,b\,c\,d^2\,z^2-32\,a^2\,c\,e^2\,z^2+8\,a\,b^2\,e^2\,z^2-4\,b^3\,d^2\,z^2$$
$$+16\,a\,c\,d^2\,e\,z-4\,b^2\,d^2\,e\,z-b\,d^2\,e^2-c\,d^4-a\,e^4,z,k\big)$$

input
```
int((d + e*x)/(a + b*x^2 + c*x^4),x)
```

output

```
symsum(log(c^2*(d*e^2 + e^3*x + 4*root(128*a^2*b^2*c*z^4 - 256*a^3*c^2*z^4
 - 16*a*b^4*z^4 + 16*a*b*c*d^2*z^2 - 32*a^2*c*e^2*z^2 + 8*a*b^2*e^2*z^2 -
4*b^3*d^2*z^2 + 16*a*c*d^2*e*z - 4*b^2*d^2*e*z - b*d^2*e^2 - c*d^4 - a*e^4
, z, k)^2*b^2*d - 8*root(128*a^2*b^2*c*z^4 - 256*a^3*c^2*z^4 - 16*a*b^4*z^
4 + 16*a*b*c*d^2*z^2 - 32*a^2*c*e^2*z^2 + 8*a*b^2*e^2*z^2 - 4*b^3*d^2*z^2
+ 16*a*c*d^2*e*z - 4*b^2*d^2*e*z - b*d^2*e^2 - c*d^4 - a*e^4, z, k)^3*b^3*
x - 16*root(128*a^2*b^2*c*z^4 - 256*a^3*c^2*z^4 - 16*a*b^4*z^4 + 16*a*b*c*
d^2*z^2 - 32*a^2*c*e^2*z^2 + 8*a*b^2*e^2*z^2 - 4*b^3*d^2*z^2 + 16*a*c*d^2*
e*z - 4*b^2*d^2*e*z - b*d^2*e^2 - c*d^4 - a*e^4, z, k)^2*a*c*d + 2*root(12
8*a^2*b^2*c*z^4 - 256*a^3*c^2*z^4 - 16*a*b^4*z^4 + 16*a*b*c*d^2*z^2 - 32*a
^2*c*e^2*z^2 + 8*a*b^2*e^2*z^2 - 4*b^3*d^2*z^2 + 16*a*c*d^2*e*z - 4*b^2*d^
2*e*z - b*d^2*e^2 - c*d^4 - a*e^4, z, k)*b*e^2*x - 4*root(128*a^2*b^2*c*z^
4 - 256*a^3*c^2*z^4 - 16*a*b^4*z^4 + 16*a*b*c*d^2*z^2 - 32*a^2*c*e^2*z^2 +
 8*a*b^2*e^2*z^2 - 4*b^3*d^2*z^2 + 16*a*c*d^2*e*z - 4*b^2*d^2*e*z - b*d^2*
e^2 - c*d^4 - a*e^4, z, k)*c*d^2*x - 4*root(128*a^2*b^2*c*z^4 - 256*a^3*c^
2*z^4 - 16*a*b^4*z^4 + 16*a*b*c*d^2*z^2 - 32*a^2*c*e^2*z^2 + 8*a*b^2*e^2*z
^2 - 4*b^3*d^2*z^2 + 16*a*c*d^2*e*z - 4*b^2*d^2*e*z - b*d^2*e^2 - c*d^4 -
a*e^4, z, k)^2*b^2*e*x + 4*root(128*a^2*b^2*c*z^4 - 256*a^3*c^2*z^4 - 16*a
*b^4*z^4 + 16*a*b*c*d^2*z^2 - 32*a^2*c*e^2*z^2 + 8*a*b^2*e^2*z^2 - 4*b^3*d
^2*z^2 + 16*a*c*d^2*e*z - 4*b^2*d^2*e*z - b*d^2*e^2 - c*d^4 - a*e^4, z,...
```

3.20. $\int \frac{d+ex}{a+bx^2+cx^4}\,dx$

3.21 $\int \frac{d+ex+fx^2}{a+bx^2+cx^4}\,dx$

3.21.1 Optimal result

Integrand size = 25, antiderivative size = 211

$$\int \frac{d+ex+fx^2}{a+bx^2+cx^4}\,dx = \frac{\left(f+\frac{2cd-bf}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{\sqrt{2}\sqrt{c}\sqrt{b-\sqrt{b^2-4ac}}} + \frac{\left(f-\frac{2cd-bf}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{\sqrt{2}\sqrt{c}\sqrt{b+\sqrt{b^2-4ac}}} - \frac{e\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\sqrt{b^2-4ac}}$$

output
```
-e*arctanh((2*c*x^2+b)/(-4*a*c+b^2)^(1/2))/(-4*a*c+b^2)^(1/2)+1/2*arctan(x
*2^(1/2)*c^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2))*(f+(-b*f+2*c*d)/(-4*a*c+b^2
)^(1/2))*2^(1/2)/c^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2)+1/2*arctan(x*2^(1/2)
*c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2))*(f+(b*f-2*c*d)/(-4*a*c+b^2)^(1/2))*
2^(1/2)/c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2)
```

3.21.2 Mathematica [A] (verified)

Time = 0.14 (sec) , antiderivative size = 234, normalized size of antiderivative = 1.11

$$\int \frac{d+ex+fx^2}{a+bx^2+cx^4}\,dx = \frac{\frac{\sqrt{2}\left(2cd+\left(-b+\sqrt{b^2-4ac}\right)f\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{\sqrt{c}\sqrt{b-\sqrt{b^2-4ac}}} + \frac{\sqrt{2}\left(-2cd+\left(b+\sqrt{b^2-4ac}\right)f\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{\sqrt{c}\sqrt{b+\sqrt{b^2-4ac}}} + e\log\left(-b+\sqrt{b^2-4a\right.}{2\sqrt{b^2-4ac}}$$

input
```
Integrate[(d + e*x + f*x^2)/(a + b*x^2 + c*x^4),x]
```

output
```
((Sqrt[2]*(2*c*d + (-b + Sqrt[b^2 - 4*a*c])*f)*ArcTan[(Sqrt[2]*Sqrt[c]*x)/
Sqrt[b - Sqrt[b^2 - 4*a*c]]])/(Sqrt[c]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + (Sqr
t[2]*(-2*c*d + (b + Sqrt[b^2 - 4*a*c])*f)*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[
b + Sqrt[b^2 - 4*a*c]]])/(Sqrt[c]*Sqrt[b + Sqrt[b^2 - 4*a*c]]) + e*Log[-b
+ Sqrt[b^2 - 4*a*c] - 2*c*x^2] - e*Log[b + Sqrt[b^2 - 4*a*c] + 2*c*x^2])/(
2*Sqrt[b^2 - 4*a*c])
```

3.21.3 Rubi [A] (verified)

Time = 0.43 (sec) , antiderivative size = 211, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.280$, Rules used = {2202, 27, 1432, 1083, 219, 1480, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2}{a+bx^2+cx^4}dx$$

$$\Big\downarrow \text{ 2202}$$

$$\int \frac{fx^2+d}{cx^4+bx^2+a}dx+\int \frac{ex}{cx^4+bx^2+a}dx$$

$$\Big\downarrow \text{ 27}$$

$$\int \frac{fx^2+d}{cx^4+bx^2+a}dx+e\int \frac{x}{cx^4+bx^2+a}dx$$

$$\Big\downarrow \text{ 1432}$$

$$\int \frac{fx^2+d}{cx^4+bx^2+a}dx+\frac{1}{2}e\int \frac{1}{cx^4+bx^2+a}dx^2$$

$$\Big\downarrow \text{ 1083}$$

$$\int \frac{fx^2+d}{cx^4+bx^2+a}dx-e\int \frac{1}{-x^4+b^2-4ac}d\left(2cx^2+b\right)$$

$$\Big\downarrow \text{ 219}$$

$$\int \frac{fx^2+d}{cx^4+bx^2+a}dx-\frac{e\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\sqrt{b^2-4ac}}$$

$$\Big\downarrow \text{ 1480}$$

$$\frac{1}{2}\left(\frac{2cd-bf}{\sqrt{b^2-4ac}}+f\right)\int \frac{1}{cx^2+\frac{1}{2}\left(b-\sqrt{b^2-4ac}\right)}dx+$$
$$\frac{1}{2}\left(f-\frac{2cd-bf}{\sqrt{b^2-4ac}}\right)\int \frac{1}{cx^2+\frac{1}{2}\left(b+\sqrt{b^2-4ac}\right)}dx-\frac{e\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\sqrt{b^2-4ac}}$$
$$\Big\downarrow \text{ 218}$$
$$\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{2cd-bf}{\sqrt{b^2-4ac}}+f\right)}{\sqrt{2}\sqrt{c}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(f-\frac{2cd-bf}{\sqrt{b^2-4ac}}\right)}{\sqrt{2}\sqrt{c}\sqrt{\sqrt{b^2-4ac}+b}}-\frac{e\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\sqrt{b^2-4ac}}$$

input
```
Int[(d + e*x + f*x^2)/(a + b*x^2 + c*x^4),x]
```

output
```
((f + (2*c*d - b*f)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b -
 Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[c]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + ((f
- (2*c*d - b*f)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqr
t[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[c]*Sqrt[b + Sqrt[b^2 - 4*a*c]]) - (e*ArcTa
nh[(b + 2*c*x^2)/Sqrt[b^2 - 4*a*c]])/Sqrt[b^2 - 4*a*c]
```

3.21.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.21.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.07 (sec) , antiderivative size = 48, normalized size of antiderivative = 0.23

method	result
risch	$\frac{\left(\sum_{\_R=\operatorname{RootOf}\left(c\_Z^4+\_Z^2 b+a\right)}\frac{\left(\_R^2 f+\_R e+d\right)\ln\left(x-\_R\right)}{2c\_R^3+\_R b}\right)}{2}$
default	$4c\left(-\frac{\sqrt{-4ac+b^2}\left(-\frac{e\ln\left(2c\,x^2+\sqrt{-4ac+b^2}+b\right)}{2}+\frac{\left(f\sqrt{-4ac+b^2}+bf-2cd\right)\sqrt{2}\,\arctan\left(\frac{cx\sqrt{2}}{\sqrt{\left(b+\sqrt{-4ac+b^2}\right)c}}\right)}{2\sqrt{\left(b+\sqrt{-4ac+b^2}\right)c}}\right)}{4c(4ac-b^2)}-\frac{\sqrt{-4ac+b^2}\left(\frac{e\ln\left(-2\ldots\right.\right.}{}\right.}{}\right.$

input
```
int((f*x^2+e*x+d)/(c*x^4+b*x^2+a),x,method=_RETURNVERBOSE)
```

output
```
1/2*sum((_R^2*f+_R*e+d)/(2*_R^3*c+_R*b)*ln(x-_R),_R=RootOf(_Z^4*c+_Z^2*b+a
))
```

3.21.5 Fricas [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 18.02 (sec) , antiderivative size = 723401, normalized size of antiderivative = 3428.44

$$\int \frac{d+ex+fx^2}{a+bx^2+cx^4}\,dx = \text{Too large to display}$$

input
```
integrate((f*x^2+e*x+d)/(c*x^4+b*x^2+a),x, algorithm="fricas")
```

output
```
Too large to include
```

3.21.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2}{a+bx^2+cx^4}\,dx = \text{Timed out}$$

input
```
integrate((f*x**2+e*x+d)/(c*x**4+b*x**2+a),x)
```

output
```
Timed out
```

3.21.7 Maxima [F]

$$\int \frac{d+ex+fx^2}{a+bx^2+cx^4}\,dx = \int \frac{fx^2+ex+d}{cx^4+bx^2+a}\,dx$$

input
```
integrate((f*x^2+e*x+d)/(c*x^4+b*x^2+a),x, algorithm="maxima")
```

output
```
integrate((f*x^2 + e*x + d)/(c*x^4 + b*x^2 + a), x)
```

3.21.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 1714 vs. $2(173) = 346$.

Time = 1.29 (sec) , antiderivative size = 1714, normalized size of antiderivative = 8.12

$$\int \frac{d+ex+fx^2}{a+bx^2+cx^4}\,dx = \text{Too large to display}$$

input
```
integrate((f*x^2+e*x+d)/(c*x^4+b*x^2+a),x, algorithm="giac")
```

output
```
-1/2*(b^2*c^2 - 4*a*c^3 - 2*b*c^3 + c^4)*sqrt(b^2 - 4*a*c)*e*log(x^2 + 1/2
*(b - sqrt(b^2 - 4*a*c))/c)/((b^4 - 8*a*b^2*c - 2*b^3*c + 16*a^2*c^2 + 8*a
*b*c^2 + b^2*c^2 - 4*a*c^3)*c^2) + 1/4*((sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a
*c)*c)*b^4 - 8*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^2*c - 2*sqrt(2)
*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^3*c - 2*b^4*c + 16*sqrt(2)*sqrt(b*c + s
qrt(b^2 - 4*a*c)*c)*a^2*c^2 + 8*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*
b*c^2 + sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^2*c^2 + 16*a*b^2*c^2 + 2
*b^3*c^2 - 4*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*c^3 - 32*a^2*c^3 -
8*a*b*c^3 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^3
+ 4*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c + 2*sq
rt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^2*c - sqrt(2)*sq
rt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b*c^2 + 2*(b^2 - 4*a*c)*b^
2*c - 8*(b^2 - 4*a*c)*a*c^2 - 2*(b^2 - 4*a*c)*b*c^2)*d - 2*(2*a*b^2*c^2 -
8*a^2*c^3 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^
2 + 4*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*c + 2*
sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c - sqrt(2)*
sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*c^2 - 2*(b^2 - 4*a*c)*
a*c^2)*f)*arctan(2*sqrt(1/2)*x/sqrt((b + sqrt(b^2 - 4*a*c))/c))/((a*b^4 -
8*a^2*b^2*c - 2*a*b^3*c + 16*a^3*c^2 + 8*a^2*b*c^2 + a*b^2*c^2 - 4*a^2*c^3
)*abs(c)) + 1/4*((sqrt(2)*sqrt(b*c - sqrt(b^2 - 4*a*c)*c)*b^4 - 8*sqrt(...
```

3.21.9 Mupad [B] (verification not implemented)

Time = 8.64 (sec) , antiderivative size = 3942, normalized size of antiderivative = 18.68

$$\int \frac{d+ex+fx^2}{a+bx^2+cx^4}\,dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2)/(a + b*x^2 + c*x^4),x)
```

output
```
symsum(log(c^2*d*e^2 - c^2*d^2*f + c^2*e^3*x - a*c*f^3 - 8*root(16*a*b^4*c
*z^4 - 128*a^2*b^2*c^2*z^4 + 256*a^3*c^3*z^4 - 16*a*b^2*c*d*f*z^2 + 64*a^2
*c^2*d*f*z^2 - 16*a^2*b*c*f^2*z^2 - 8*a*b^2*c*e^2*z^2 - 16*a*b*c^2*d^2*z^2
 + 32*a^2*c^2*e^2*z^2 + 4*b^3*c*d^2*z^2 + 4*a*b^3*f^2*z^2 + 16*a^2*c*e*f^2
*z + 4*b^2*c*d^2*e*z - 4*a*b^2*e*f^2*z - 16*a*c^2*d^2*e*z - 4*a*c*d*e^2*f
+ 2*a*c*d^2*f^2 - 2*b*c*d^3*f - 2*a*b*d*f^3 + b*c*d^2*e^2 + a*b*e^2*f^2 +
a*c*e^4 + b^2*d^2*f^2 + c^2*d^4 + a^2*f^4, z, k)^3*b^3*c^2*x + b*c*d*f^2 -
 16*root(16*a*b^4*c*z^4 - 128*a^2*b^2*c^2*z^4 + 256*a^3*c^3*z^4 - 16*a*b^2
*c*d*f*z^2 + 64*a^2*c^2*d*f*z^2 - 16*a^2*b*c*f^2*z^2 - 8*a*b^2*c*e^2*z^2 -
 16*a*b*c^2*d^2*z^2 + 32*a^2*c^2*e^2*z^2 + 4*b^3*c*d^2*z^2 + 4*a*b^3*f^2*z
^2 + 16*a^2*c*e*f^2*z + 4*b^2*c*d^2*e*z - 4*a*b^2*e*f^2*z - 16*a*c^2*d^2*e
*z - 4*a*c*d*e^2*f + 2*a*c*d^2*f^2 - 2*b*c*d^3*f - 2*a*b*d*f^3 + b*c*d^2*e
^2 + a*b*e^2*f^2 + a*c*e^4 + b^2*d^2*f^2 + c^2*d^4 + a^2*f^4, z, k)^2*a*c^
3*d - 4*root(16*a*b^4*c*z^4 - 128*a^2*b^2*c^2*z^4 + 256*a^3*c^3*z^4 - 16*a
*b^2*c*d*f*z^2 + 64*a^2*c^2*d*f*z^2 - 16*a^2*b*c*f^2*z^2 - 8*a*b^2*c*e^2*z
^2 - 16*a*b*c^2*d^2*z^2 + 32*a^2*c^2*e^2*z^2 + 4*b^3*c*d^2*z^2 + 4*a*b^3*f
^2*z^2 + 16*a^2*c*e*f^2*z + 4*b^2*c*d^2*e*z - 4*a*b^2*e*f^2*z - 16*a*c^2*d
^2*e*z - 4*a*c*d*e^2*f + 2*a*c*d^2*f^2 - 2*b*c*d^3*f - 2*a*b*d*f^3 + b*c*d
^2*e^2 + a*b*e^2*f^2 + a*c*e^4 + b^2*d^2*f^2 + c^2*d^4 + a^2*f^4, z, k)*c^
3*d^2*x + 4*root(16*a*b^4*c*z^4 - 128*a^2*b^2*c^2*z^4 + 256*a^3*c^3*z^4...
```

3.21. $\int \frac{d+ex+fx^2}{a+bx^2+cx^4}\,dx$

3.22 $\int \frac{d+ex+fx^2+gx^3}{a+bx^2+cx^4}\,dx$

3.22.1 Optimal result

Integrand size = 30, antiderivative size = 245

$$\int \frac{d+ex+fx^2+gx^3}{a+bx^2+cx^4}\,dx = \frac{\left(f+\frac{2cd-bf}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{\sqrt{2}\sqrt{c}\sqrt{b-\sqrt{b^2-4ac}}} + \frac{\left(f-\frac{2cd-bf}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{\sqrt{2}\sqrt{c}\sqrt{b+\sqrt{b^2-4ac}}} - \frac{(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{2c\sqrt{b^2-4ac}} + \frac{g\log\left(a+bx^2+cx^4\right)}{4c}$$

output
```
1/4*g*ln(c*x^4+b*x^2+a)/c-1/2*(-b*g+2*c*e)*arctanh((2*c*x^2+b)/(-4*a*c+b^2
)^(1/2))/c/(-4*a*c+b^2)^(1/2)+1/2*arctan(x*2^(1/2)*c^(1/2)/(b-(-4*a*c+b^2)
^(1/2))^(1/2))*(f+(-b*f+2*c*d)/(-4*a*c+b^2)^(1/2))*2^(1/2)/c^(1/2)/(b-(-4*
a*c+b^2)^(1/2))^(1/2)+1/2*arctan(x*2^(1/2)*c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^
(1/2))*(f+(b*f-2*c*d)/(-4*a*c+b^2)^(1/2))*2^(1/2)/c^(1/2)/(b+(-4*a*c+b^2)^
(1/2))^(1/2)
```

3.22.2 Mathematica [A] (verified)

Time = 0.18 (sec) , antiderivative size = 280, normalized size of antiderivative = 1.14

$$\int \frac{d+ex+fx^2+gx^3}{a+bx^2+cx^4}\,dx$$
$$= \frac{\frac{2\sqrt{2}\sqrt{c}\left(2cd+\left(-b+\sqrt{b^2-4ac}\right)f\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{\sqrt{b-\sqrt{b^2-4ac}}}+\frac{2\sqrt{2}\sqrt{c}\left(-2cd+\left(b+\sqrt{b^2-4ac}\right)f\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{\sqrt{b+\sqrt{b^2-4ac}}}+(2ce+(-b+}{4c\sqrt{b^2-}}$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4),x]
```

output
```
((2*Sqrt[2]*Sqrt[c]*(2*c*d + (-b + Sqrt[b^2 - 4*a*c])*f)*ArcTan[(Sqrt[2]*S
qrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]])/Sqrt[b - Sqrt[b^2 - 4*a*c]] + (2*S
qrt[2]*Sqrt[c]*(-2*c*d + (b + Sqrt[b^2 - 4*a*c])*f)*ArcTan[(Sqrt[2]*Sqrt[c
]*x)/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/Sqrt[b + Sqrt[b^2 - 4*a*c]] + (2*c*e +
(-b + Sqrt[b^2 - 4*a*c])*g)*Log[-b + Sqrt[b^2 - 4*a*c] - 2*c*x^2] + (-2*c*
e + (b + Sqrt[b^2 - 4*a*c])*g)*Log[b + Sqrt[b^2 - 4*a*c] + 2*c*x^2])/(4*c*
Sqrt[b^2 - 4*a*c])
```

3.22.3 Rubi [A] (verified)

Time = 0.48 (sec) , antiderivative size = 248, normalized size of antiderivative = 1.01, number of steps used = 9, number of rules used = 8, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.267, Rules used = {2202, 1480, 218, 1576, 1142, 1083, 219, 1103}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3}{a+bx^2+cx^4}\,dx$$
$$\Big\downarrow \text{2202}$$
$$\int \frac{fx^2+d}{cx^4+bx^2+a}dx+\int \frac{x\left(gx^2+e\right)}{cx^4+bx^2+a}dx$$
$$\Big\downarrow \text{1480}$$

$$\frac{1}{2}\left(\frac{2cd-bf}{\sqrt{b^2-4ac}}+f\right)\int \frac{1}{cx^2+\frac{1}{2}\left(b-\sqrt{b^2-4ac}\right)}dx+\frac{1}{2}\left(f-\frac{2cd-bf}{\sqrt{b^2-4ac}}\right)\int \frac{1}{cx^2+\frac{1}{2}\left(b+\sqrt{b^2-4ac}\right)}dx+\int \frac{x\left(gx^2+e\right)}{cx^4+bx^2+a}dx$$

$$\Big\downarrow \; 218$$

$$\int \frac{x\left(gx^2+e\right)}{cx^4+bx^2+a}dx+\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{2cd-bf}{\sqrt{b^2-4ac}}+f\right)}{\sqrt{2}\sqrt{c}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(f-\frac{2cd-bf}{\sqrt{b^2-4ac}}\right)}{\sqrt{2}\sqrt{c}\sqrt{\sqrt{b^2-4ac}+b}}$$

$$\Big\downarrow \; 1576$$

$$\frac{1}{2}\int \frac{gx^2+e}{cx^4+bx^2+a}dx^2+\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{2cd-bf}{\sqrt{b^2-4ac}}+f\right)}{\sqrt{2}\sqrt{c}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(f-\frac{2cd-bf}{\sqrt{b^2-4ac}}\right)}{\sqrt{2}\sqrt{c}\sqrt{\sqrt{b^2-4ac}+b}}$$

$$\Big\downarrow \; 1142$$

$$\frac{1}{2}\left(\frac{(2ce-bg)\int \frac{1}{cx^4+bx^2+a}dx^2}{2c}+\frac{g\int \frac{2cx^2+b}{cx^4+bx^2+a}dx^2}{2c}\right)+\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{2cd-bf}{\sqrt{b^2-4ac}}+f\right)}{\sqrt{2}\sqrt{c}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(f-\frac{2cd-bf}{\sqrt{b^2-4ac}}\right)}{\sqrt{2}\sqrt{c}\sqrt{\sqrt{b^2-4ac}+b}}$$

$$\Big\downarrow \; 1083$$

$$\frac{1}{2}\left(\frac{g\int \frac{2cx^2+b}{cx^4+bx^2+a}dx^2}{2c}-\frac{(2ce-bg)\int \frac{1}{-x^4+b^2-4ac}d\left(2cx^2+b\right)}{c}\right)+\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{2cd-bf}{\sqrt{b^2-4ac}}+f\right)}{\sqrt{2}\sqrt{c}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(f-\frac{2cd-bf}{\sqrt{b^2-4ac}}\right)}{\sqrt{2}\sqrt{c}\sqrt{\sqrt{b^2-4ac}+b}}$$

$$\Big\downarrow \; 219$$

$$\frac{1}{2}\left(\frac{g\int \frac{2cx^2+b}{cx^4+bx^2+a}dx^2}{2c}-\frac{(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{c\sqrt{b^2-4ac}}\right)+\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{2cd-bf}{\sqrt{b^2-4ac}}+f\right)}{\sqrt{2}\sqrt{c}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(f-\frac{2cd-bf}{\sqrt{b^2-4ac}}\right)}{\sqrt{2}\sqrt{c}\sqrt{\sqrt{b^2-4ac}+b}}$$

$$\Big\downarrow \; 1103$$

3.22. $\int \frac{d+ex+fx^2+gx^3}{a+bx^2+cx^4}\,dx$

$$\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{2cd-bf}{\sqrt{b^2-4ac}}+f\right)}{\sqrt{2}\sqrt{c}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(f-\frac{2cd-bf}{\sqrt{b^2-4ac}}\right)}{\sqrt{2}\sqrt{c}\sqrt{\sqrt{b^2-4ac}+b}}+\frac{1}{2}\left(\frac{g\log\left(a+bx^2+cx^4\right)}{2c}-\frac{(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{c\sqrt{b^2-4ac}}\right)$$

input
```
Int[(d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4),x]
```

output
```
((f + (2*c*d - b*f)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b -
 Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[c]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + ((f
- (2*c*d - b*f)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqr
t[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[c]*Sqrt[b + Sqrt[b^2 - 4*a*c]]) + (-(((2*c
*e - b*g)*ArcTanh[(b + 2*c*x^2)/Sqrt[b^2 - 4*a*c]])/(c*Sqrt[b^2 - 4*a*c]))
 + (g*Log[a + b*x^2 + c*x^4])/(2*c))/2
```

3.22.3.1 Defintions of rubi rules used

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.22.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.07 (sec) , antiderivative size = 53, normalized size of antiderivative = 0.22

method	result
risch	$\frac{\left(\sum_{\_R=\mathrm{RootOf}\left(c\_Z^4+\_Z^2 b+a\right)}\frac{\left(\_R^3 g+\_R^2 f+\_Re+d\right)\ln\left(x-\_R\right)}{2c\_R^3+\_Rb}\right)}{2}$
default	$4c\left(-\frac{\sqrt{-4ac+b^2}\left(\frac{\left(\sqrt{-4ac+b^2}\,g+bg-2ec\right)\ln\left(2c\,x^2+\sqrt{-4ac+b^2}+b\right)}{4c}+\frac{\left(f\sqrt{-4ac+b^2}+bf-2cd\right)\sqrt{2}\arctan\left(\frac{cx\sqrt{2}}{\sqrt{\left(b+\sqrt{-4ac+b^2}\right)c}}\right)}{2\sqrt{\left(b+\sqrt{-4ac+b^2}\right)c}}\right)}{4c(4ac-b^2)}-\right.$

input
```
int((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a),x,method=_RETURNVERBOSE)
```

output
```
1/2*sum((_R^3*g+_R^2*f+_R*e+d)/(2*_R^3*c+_R*b)*ln(x-_R),_R=RootOf(_Z^4*c+_
Z^2*b+a))
```

3.22.5 Fricas [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 249.08 (sec) , antiderivative size = 2136355, normalized size of antiderivative = 8719.82

$$\int \frac{d+ex+fx^2+gx^3}{a+bx^2+cx^4}\,dx = \text{Too large to display}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a),x, algorithm="fricas")
```

output
```
Too large to include
```

3.22.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3}{a+bx^2+cx^4}\,dx = \text{Timed out}$$

input
```
integrate((g*x**3+f*x**2+e*x+d)/(c*x**4+b*x**2+a),x)
```

output
```
Timed out
```

3.22.7 Maxima [F]

$$\int \frac{d+ex+fx^2+gx^3}{a+bx^2+cx^4}\,dx = \int \frac{gx^3+fx^2+ex+d}{cx^4+bx^2+a}\,dx$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a),x, algorithm="maxima")
```

output
```
integrate((g*x^3 + f*x^2 + e*x + d)/(c*x^4 + b*x^2 + a), x)
```

3.22.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 3270 vs. $2(203) = 406$.

Time = 1.33 (sec) , antiderivative size = 3270, normalized size of antiderivative = 13.35

$$\int \frac{d+ex+fx^2+gx^3}{a+bx^2+cx^4}\,dx = \text{Too large to display}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a),x, algorithm="giac")
```

output
```
1/4*g*log(abs(c*x^4 + b*x^2 + a))/c + 1/8*((2*b^4*c^2 - 16*a*b^2*c^3 + 32*
a^2*c^4 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^4 +
8*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^2*c + 2*sq
rt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^3*c - 16*sqrt(2)
*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*c^2 - 8*sqrt(2)*sqr
t(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c^2 - sqrt(2)*sqrt(b^2
- 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^2*c^2 + 4*sqrt(2)*sqrt(b^2 - 4*
a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*c^3 - 2*(b^2 - 4*a*c)*b^2*c^2 + 8*(
b^2 - 4*a*c)*a*c^3)*c^2*f + 2*(sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^4
*c^2 - 8*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^2*c^3 - 2*sqrt(2)*sqr
t(b*c + sqrt(b^2 - 4*a*c)*c)*b^3*c^3 - 2*b^4*c^3 + 16*sqrt(2)*sqrt(b*c + s
qrt(b^2 - 4*a*c)*c)*a^2*c^4 + 8*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*
b*c^4 + sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^2*c^4 + 16*a*b^2*c^4 - 4
*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*c^5 - 32*a^2*c^5 + 2*(b^2 - 4*a
*c)*b^2*c^3 - 8*(b^2 - 4*a*c)*a*c^4)*d*abs(c) + 2*(2*b^3*c^5 - 8*a*b*c^6 -
 sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^3*c^3 + 4*sqr
t(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c^4 + 2*sqrt(2)
*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^2*c^4 - sqrt(2)*sqrt(
b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b*c^5 - 2*(b^2 - 4*a*c)*b*c^5
)*d - (2*b^4*c^4 - 8*a*b^2*c^5 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + s...
```

3.22.9 Mupad [B] (verification not implemented)

Time = 8.94 (sec) , antiderivative size = 15179, normalized size of antiderivative = 61.96

$$\int \frac{d+ex+fx^2+gx^3}{a+bx^2+cx^4}\,dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4),x)
```

output
```
symsum(log(c^2*d*e^2 + b^2*d*g^2 - c^2*d^2*f + c^2*e^3*x - a*c*f^3 - 8*roo
t(128*a^2*b^2*c^3*z^4 - 16*a*b^4*c^2*z^4 - 256*a^3*c^4*z^4 - 128*a^2*b^2*c
^2*g*z^3 + 16*a*b^4*c*g*z^3 + 256*a^3*c^3*g*z^3 + 32*a^2*b*c^2*e*g*z^2 + 1
6*a*b^2*c^2*d*f*z^2 - 8*a*b^3*c*e*g*z^2 + 40*a^2*b^2*c*g^2*z^2 + 16*a^2*b*
c^2*f^2*z^2 + 8*a*b^2*c^2*e^2*z^2 - 64*a^2*c^3*d*f*z^2 - 4*a*b^3*c*f^2*z^2
 + 16*a*b*c^3*d^2*z^2 - 96*a^3*c^2*g^2*z^2 - 32*a^2*c^3*e^2*z^2 - 4*b^3*c^
2*d^2*z^2 - 4*a*b^4*g^2*z^2 - 8*a*b^2*c*d*f*g*z + 32*a^2*c^2*d*f*g*z - 16*
a^2*b*c*e*g^2*z - 4*a*b^2*c*e^2*g*z - 16*a*b*c^2*d^2*g*z + 4*a*b^2*c*e*f^2
*z + 16*a^2*c^2*e^2*g*z - 16*a^2*c^2*e*f^2*z - 4*b^2*c^2*d^2*e*z + 4*b^3*c
*d^2*g*z + 4*a*b^3*e*g^2*z + 16*a*c^3*d^2*e*z + 16*a^3*c*g^3*z - 4*a^2*b^2
*g^3*z - 4*a*b*c*d*e*f*g + 2*a*b*c*e^3*g + 2*a*b*c*d*f^3 + 4*a^2*c*e*f^2*g
 - 4*a^2*c*d*f*g^2 + 2*b^2*c*d^2*e*g - 4*a*c^2*d^2*e*g + 2*a*b^2*d*f*g^2 +
 4*a*c^2*d*e^2*f + 3*a*b*c*d^2*g^2 + 2*a^2*b*e*g^3 + 2*b*c^2*d^3*f - a*b*c
*e^2*f^2 - 2*a^2*c*e^2*g^2 - 2*a*c^2*d^2*f^2 - a^2*b*f^2*g^2 - b^2*c*d^2*f
^2 - a*b^2*e^2*g^2 - b*c^2*d^2*e^2 - b^3*d^2*g^2 - a^2*c*f^4 - a*c^2*e^4 -
 a^3*g^4 - c^3*d^4, z, k)^3*b^3*c^2*x - a*c*d*g^2 + b*c*d*f^2 - a*b*f*g^2
- a*b*g^3*x - 16*root(128*a^2*b^2*c^3*z^4 - 16*a*b^4*c^2*z^4 - 256*a^3*c^4
*z^4 - 128*a^2*b^2*c^2*g*z^3 + 16*a*b^4*c*g*z^3 + 256*a^3*c^3*g*z^3 + 32*a
^2*b*c^2*e*g*z^2 + 16*a*b^2*c^2*d*f*z^2 - 8*a*b^3*c*e*g*z^2 + 40*a^2*b^2*c
*g^2*z^2 + 16*a^2*b*c^2*f^2*z^2 + 8*a*b^2*c^2*e^2*z^2 - 64*a^2*c^3*d*f*...
```

3.22. $\int \frac{d+ex+fx^2+gx^3}{a+bx^2+cx^4}\,dx$

3.23 $\int \frac{d+ex+fx^2+gx^3+hx^4}{a+bx^2+cx^4}\,dx$

3.23.1 Optimal result

Integrand size = 35, antiderivative size = 290

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{a+bx^2+cx^4}\,dx=\frac{hx}{c}+\frac{\left(cf-bh+\frac{2c^2d+b^2h-c(bf+2ah)}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{\sqrt{2}c^{3/2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\left(cf-bh-\frac{2c^2d-bcf+b^2h-2ach}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{\sqrt{2}c^{3/2}\sqrt{b+\sqrt{b^2-4ac}}}-\frac{(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{2c\sqrt{b^2-4ac}}+\frac{g\log\left(a+bx^2+cx^4\right)}{4c}$$

output
```
h*x/c+1/4*g*ln(c*x^4+b*x^2+a)/c-1/2*(-b*g+2*c*e)*arctanh((2*c*x^2+b)/(-4*a
*c+b^2)^(1/2))/c/(-4*a*c+b^2)^(1/2)+1/2*arctan(x*2^(1/2)*c^(1/2)/(b-(-4*a*
c+b^2)^(1/2))^(1/2))*(c*f-b*h+(2*c^2*d+b^2*h-c*(2*a*h+b*f))/(-4*a*c+b^2)^(
1/2))/c^(3/2)*2^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2)+1/2*arctan(x*2^(1/2)*c^
(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2))*(c*f-b*h+(2*a*c*h-b^2*h+b*c*f-2*c^2*d)
/(-4*a*c+b^2)^(1/2))/c^(3/2)*2^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2)
```

3.23.2 Mathematica [A] (verified)

Time = 0.31 (sec) , antiderivative size = 383, normalized size of antiderivative = 1.32

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{a+bx^2+cx^4}\,dx$$
$$= \frac{4\sqrt{c}hx + \frac{2\sqrt{2}\left(2c^2d+b\left(b-\sqrt{b^2-4ac}\right)h+c\left(-bf+\sqrt{b^2-4ac}f-2ah\right)\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{\sqrt{b^2-4ac}\sqrt{b-\sqrt{b^2-4ac}}} - \frac{2\sqrt{2}\left(2c^2d+b\left(b+\sqrt{b^2-4ac}\right)h-c\left(bf+\sqrt{b^2-\cdots\right.\right.}{\sqrt{b^2-4ac}\sqrt{b+\cdots}}}{}$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4)/(a + b*x^2 + c*x^4),x]
```

output
```
(4*Sqrt[c]*h*x + (2*Sqrt[2]*(2*c^2*d + b*(b - Sqrt[b^2 - 4*a*c])*h + c*(-(
b*f) + Sqrt[b^2 - 4*a*c]*f - 2*a*h))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - S
qrt[b^2 - 4*a*c]]])/(Sqrt[b^2 - 4*a*c]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) - (2*S
qrt[2]*(2*c^2*d + b*(b + Sqrt[b^2 - 4*a*c])*h - c*(b*f + Sqrt[b^2 - 4*a*c]
*f + 2*a*h))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(Sqr
t[b^2 - 4*a*c]*Sqrt[b + Sqrt[b^2 - 4*a*c]]) + (Sqrt[c]*(2*c*e + (-b + Sqrt
[b^2 - 4*a*c])*g)*Log[-b + Sqrt[b^2 - 4*a*c] - 2*c*x^2])/Sqrt[b^2 - 4*a*c]
 + (Sqrt[c]*(-2*c*e + (b + Sqrt[b^2 - 4*a*c])*g)*Log[b + Sqrt[b^2 - 4*a*c]
 + 2*c*x^2])/Sqrt[b^2 - 4*a*c])/(4*c^(3/2))
```

3.23.3 Rubi [A] (verified)

Time = 0.90 (sec) , antiderivative size = 293, normalized size of antiderivative = 1.01, number of steps used = 9, number of rules used = 8, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.229, Rules used = {2202, 1576, 1142, 1083, 219, 1103, 2205, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{a+bx^2+cx^4}\,dx$$
$$\Big\downarrow \text{2202}$$
$$\int \frac{hx^4+fx^2+d}{cx^4+bx^2+a}dx+\int \frac{x\left(gx^2+e\right)}{cx^4+bx^2+a}dx$$
$$\Big\downarrow \text{1576}$$
$$\int \frac{hx^4+fx^2+d}{cx^4+bx^2+a}dx+\frac{1}{2}\int \frac{gx^2+e}{cx^4+bx^2+a}dx^2$$

$$\Big\downarrow \text{1142}$$

$$\int \frac{hx^4+fx^2+d}{cx^4+bx^2+a}dx+\frac{1}{2}\left(\frac{(2ce-bg)\int \frac{1}{cx^4+bx^2+a}dx^2}{2c}+\frac{g\int \frac{2cx^2+b}{cx^4+bx^2+a}dx^2}{2c}\right)$$

$$\Big\downarrow \text{1083}$$

$$\frac{1}{2}\left(\frac{g\int \frac{2cx^2+b}{cx^4+bx^2+a}dx^2}{2c}-\frac{(2ce-bg)\int \frac{1}{-x^4+b^2-4ac}d(2cx^2+b)}{c}\right)+\int \frac{hx^4+fx^2+d}{cx^4+bx^2+a}dx$$

$$\Big\downarrow \text{219}$$

$$\frac{1}{2}\left(\frac{g\int \frac{2cx^2+b}{cx^4+bx^2+a}dx^2}{2c}-\frac{(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{c\sqrt{b^2-4ac}}\right)+\int \frac{hx^4+fx^2+d}{cx^4+bx^2+a}dx$$

$$\Big\downarrow \text{1103}$$

$$\int \frac{hx^4+fx^2+d}{cx^4+bx^2+a}dx+\frac{1}{2}\left(\frac{g\log\left(a+bx^2+cx^4\right)}{2c}-\frac{(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{c\sqrt{b^2-4ac}}\right)$$

$$\Big\downarrow \text{2205}$$

$$\int \left(\frac{h}{c}+\frac{(cf-bh)x^2+cd-ah}{c\left(cx^4+bx^2+a\right)}\right)dx+\frac{1}{2}\left(\frac{g\log\left(a+bx^2+cx^4\right)}{2c}-\frac{(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{c\sqrt{b^2-4ac}}\right)$$

$$\Big\downarrow \text{2009}$$

$$\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{-c(2ah+bf)+b^2h+2c^2d}{\sqrt{b^2-4ac}}-bh+cf\right)}{\sqrt{2}c^{3/2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{-2ach+b^2h-bcf+2c^2d}{\sqrt{b^2-4ac}}-bh+cf\right)}{\sqrt{2}c^{3/2}\sqrt{\sqrt{b^2-4ac}+b}}+\frac{1}{2}\left(\frac{g\log\left(a+bx^2+cx^4\right)}{2c}-\frac{(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{c\sqrt{b^2-4ac}}\right)+\frac{hx}{c}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4)/(a + b*x^2 + c*x^4),x]
```

3.23. $\int \frac{d+ex+fx^2+gx^3+hx^4}{a+bx^2+cx^4}\,dx$

output
```
(h*x)/c + ((c*f - b*h + (2*c^2*d + b^2*h - c*(b*f + 2*a*h))/Sqrt[b^2 - 4*a
*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*c^(
3/2)*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + ((c*f - b*h - (2*c^2*d - b*c*f + b^2*h
 - 2*a*c*h)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqrt[b^
2 - 4*a*c]]])/(Sqrt[2]*c^(3/2)*Sqrt[b + Sqrt[b^2 - 4*a*c]]) + (-(((2*c*e -
 b*g)*ArcTanh[(b + 2*c*x^2)/Sqrt[b^2 - 4*a*c]])/(c*Sqrt[b^2 - 4*a*c])) + (
g*Log[a + b*x^2 + c*x^4])/(2*c))/2
```

3.23.3.1 Defintions of rubi rules used

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2205
```
Int[(Px_)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :> Int[ExpandInte
grand[Px/(a + b*x^2 + c*x^4), x], x] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x^
2] && Expon[Px, x^2] > 1
```

3.23.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.10 (sec) , antiderivative size = 78, normalized size of antiderivative = 0.27

method	result
risch	$\frac{hx}{c}+\frac{\sum_{\_R=\operatorname{RootOf}\left(c\_Z^4+\_Z^2b+a\right)}\frac{\left(cg\_R^3+(-bh+cf)\_R^2+\_Rce-ah+cd\right)\ln\left(x-\_R\right)}{2c\_R^3+\_Rb}}{2c}$
default	$\frac{hx}{c}+\frac{\sqrt{-4ac+b^2}\left(\frac{\left(-\sqrt{-4ac+b^2}\,cg-gbc+2e\,c^2\right)\ln\left(2c\,x^2+\sqrt{-4ac+b^2}+b\right)}{4c}+\frac{\left(\sqrt{-4ac+b^2}\,bh-\sqrt{-4ac+b^2}\,fc-2ach+b^2h-fbc+2c^2d\right)\sqrt{2}\,\text{arcta}}{2\sqrt{\left(b+\sqrt{-4ac+b^2}\right)c}}\right.}{c(4ac-b^2)}$

input
```
int((h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a),x,method=_RETURNVERBOSE)
```

output
```
h*x/c+1/2/c*sum((c*g*_R^3+(-b*h+c*f)*_R^2+_R*c*e-a*h+c*d)/(2*_R^3*c+_R*b)*
ln(x-_R),_R=RootOf(_Z^4*c+_Z^2*b+a))
```

3.23.5 Fricas [F(-1)]

Timed out.

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4}{a + bx^2 + cx^4} \, dx = \text{Timed out}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a),x, algorithm="fricas")
```

output
```
Timed out
```

3.23.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4}{a + bx^2 + cx^4} \, dx = \text{Timed out}$$

input
```
integrate((h*x**4+g*x**3+f*x**2+e*x+d)/(c*x**4+b*x**2+a),x)
```

output
```
Timed out
```

3.23.7 Maxima [F]

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4}{a + bx^2 + cx^4} \, dx = \int \frac{hx^4 + gx^3 + fx^2 + ex + d}{cx^4 + bx^2 + a} \, dx$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a),x, algorithm="maxima")
```

output
```
h*x/c + integrate((c*g*x^3 + c*e*x + (c*f - b*h)*x^2 + c*d - a*h)/(c*x^4 +
 b*x^2 + a), x)/c
```

3.23.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 5199 vs. $2(248) = 496$.

Time = 1.47 (sec) , antiderivative size = 5199, normalized size of antiderivative = 17.93

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4}{a + bx^2 + cx^4} \, dx = \text{Too large to display}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a),x, algorithm="giac")
```

output
```
h*x/c + 1/4*g*log(abs(c*x^4 + b*x^2 + a))/c - 1/8*((2*b^4*c^3 - 16*a*b^2*c
^4 + 32*a^2*c^5 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c
)*b^4*c + 8*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^
2*c^2 + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^3*c^
2 - 16*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*c^3 -
 8*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c^3 - sqr
t(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^2*c^3 + 4*sqrt(2)
*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*c^4 - 2*(b^2 - 4*a*c)
*b^2*c^3 + 8*(b^2 - 4*a*c)*a*c^4)*c^2*f - (2*b^5*c^2 - 16*a*b^3*c^3 + 32*a
^2*b*c^4 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^5 +
 8*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^3*c + 2*s
qrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^4*c - 16*sqrt(2
)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b*c^2 - 8*sqrt(2)*
sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^2*c^2 - sqrt(2)*sqrt
(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^3*c^2 + 4*sqrt(2)*sqrt(b^2
 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c^3 - 2*(b^2 - 4*a*c)*b^3*c^
2 + 8*(b^2 - 4*a*c)*a*b*c^3)*c^2*h - 2*(sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*
c)*c)*b^4*c^3 - 8*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^2*c^4 - 2*sq
rt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^3*c^4 - 2*b^4*c^4 + 16*sqrt(2)*sqr
t(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*c^5 + 8*sqrt(2)*sqrt(b*c + sqrt(b^2 - ...
```

3.23.9 Mupad [B] (verification not implemented)

Time = 8.47 (sec) , antiderivative size = 5981, normalized size of antiderivative = 20.62

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4}{a + bx^2 + cx^4} \, dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4)/(a + b*x^2 + c*x^4),x)
```

output

```
symsum(log((x*(c^3*e^3 + c^3*d^2*g + b^3*e*h^2 - a*b*c*g^3 - 2*c^3*d*e*f +
 a*c^2*e*g^2 + b*c^2*e*f^2 - a*c^2*f^2*g - 2*b*c^2*e^2*g + b^2*c*e*g^2 - a
*b^2*g*h^2 + a^2*c*g*h^2 - 2*a*b*c*e*h^2 + 2*b*c^2*d*e*h - 2*a*c^2*d*g*h +
 2*a*c^2*e*f*h - 2*b^2*c*e*f*h + 2*a*b*c*f*g*h))/c - root(128*a^2*b^2*c^4*
z^4 - 16*a*b^4*c^3*z^4 - 256*a^3*c^5*z^4 - 128*a^2*b^2*c^3*g*z^3 + 16*a*b^
4*c^2*g*z^3 + 256*a^3*c^4*g*z^3 + 32*a^2*b*c^3*e*g*z^2 + 32*a^2*b*c^3*d*h*
z^2 - 8*a*b^3*c^2*e*g*z^2 - 8*a*b^3*c^2*d*h*z^2 + 16*a*b^2*c^3*d*f*z^2 + 8
*a*b^4*c*f*h*z^2 - 48*a^2*b^2*c^2*f*h*z^2 - 48*a^3*b*c^2*h^2*z^2 + 28*a^2*
b^3*c*h^2*z^2 + 16*a^2*b*c^3*f^2*z^2 - 4*a*b^3*c^2*f^2*z^2 + 8*a*b^2*c^3*e
^2*z^2 + 64*a^3*c^3*f*h*z^2 - 64*a^2*c^4*d*f*z^2 - 4*a*b^4*c*g^2*z^2 + 16*
a*b*c^4*d^2*z^2 + 40*a^2*b^2*c^2*g^2*z^2 - 96*a^3*c^3*g^2*z^2 - 32*a^2*c^4
*e^2*z^2 - 4*b^3*c^3*d^2*z^2 - 4*a*b^5*h^2*z^2 + 8*a^2*b^2*c*f*g*h*z + 32*
a^2*b*c^2*e*f*h*z - 8*a*b^2*c^2*d*f*g*z + 8*a*b^2*c^2*d*e*h*z - 8*a*b^3*c*
e*f*h*z - 20*a^2*b^2*c*e*h^2*z - 16*a^2*b*c^2*e*g^2*z - 4*a*b^2*c^2*e^2*g*
z + 4*a*b^2*c^2*e*f^2*z - 32*a^3*c^2*f*g*h*z + 32*a^2*c^3*d*f*g*z - 32*a^2
*c^3*d*e*h*z + 16*a^3*b*c*g*h^2*z + 4*a*b^3*c*e*g^2*z - 16*a*b*c^3*d^2*g*z
 - 4*a^2*b^3*g*h^2*z + 16*a^3*c^2*e*h^2*z + 16*a^2*c^3*e^2*g*z + 4*b^3*c^2
*d^2*g*z - 16*a^2*c^3*e*f^2*z - 4*b^2*c^3*d^2*e*z - 4*a^2*b^2*c*g^3*z + 4*
a*b^4*e*h^2*z + 16*a*c^4*d^2*e*z + 16*a^3*c^2*g^3*z - 4*a^2*b*c*e*f*g*h -
4*a*b*c^2*d*e*f*g + 8*a^2*c^2*d*e*g*h - 2*a^2*b*c*d*g^2*h + 2*a*b^2*c*e...
```

3.23. $\int \frac{d+ex+fx^2+gx^3+hx^4}{a+bx^2+cx^4}\,dx$

3.24 $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{a+bx^2+cx^4}\,dx$

3.24.1 Optimal result

Integrand size = 40, antiderivative size = 321

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{a+bx^2+cx^4}\,dx &= \frac{hx}{c}+\frac{ix^2}{2c}+\frac{\left(cf-bh+\frac{2c^2d+b^2h-c(bf+2ah)}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{\sqrt{2}c^{3/2}\sqrt{b-\sqrt{b^2-4ac}}}\\ &+\frac{\left(cf-bh-\frac{2c^2d-bcf+b^2h-2ach}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{\sqrt{2}c^{3/2}\sqrt{b+\sqrt{b^2-4ac}}}\\ &-\frac{(2c^2e-bcg+b^2i-2aci)\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{2c^2\sqrt{b^2-4ac}}+\frac{(cg-bi)\log\left(a+bx^2+cx^4\right)}{4c^2}\end{aligned}$$

output
```
h*x/c+1/2*i*x^2/c+1/4*(-b*i+c*g)*ln(c*x^4+b*x^2+a)/c^2-1/2*(-2*a*c*i+b^2*i
-b*c*g+2*c^2*e)*arctanh((2*c*x^2+b)/(-4*a*c+b^2)^(1/2))/c^2/(-4*a*c+b^2)^(
1/2)+1/2*arctan(x*2^(1/2)*c^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2))*(c*f-b*h+(
2*c^2*d+b^2*h-c*(2*a*h+b*f))/(-4*a*c+b^2)^(1/2))/c^(3/2)*2^(1/2)/(b-(-4*a*
c+b^2)^(1/2))^(1/2)+1/2*arctan(x*2^(1/2)*c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1
/2))*(c*f-b*h+(2*a*c*h-b^2*h+b*c*f-2*c^2*d)/(-4*a*c+b^2)^(1/2))/c^(3/2)*2^
(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2)
```

3.24.2 Mathematica [A] (verified)

Time = 0.39 (sec) , antiderivative size = 441, normalized size of antiderivative = 1.37

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{a+bx^2+cx^4}\,dx$$
$$= \frac{4chx+2cix^2+\frac{2\sqrt{2}\sqrt{c}\left(2c^2d+b\left(b-\sqrt{b^2-4ac}\right)h+c\left(-bf+\sqrt{b^2-4ac}f-2ah\right)\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{\sqrt{b^2-4ac}\sqrt{b-\sqrt{b^2-4ac}}}-\frac{2\sqrt{2}\sqrt{c}\left(2c^2d+b\left(b+\sqrt{b^2-4ac}\right)\right.}{\sqrt{}}}{}$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(a + b*x^2 + c*x^4),x]
```

output
```
(4*c*h*x + 2*c*i*x^2 + (2*Sqrt[2]*Sqrt[c]*(2*c^2*d + b*(b - Sqrt[b^2 - 4*a
*c])*h + c*(-(b*f) + Sqrt[b^2 - 4*a*c]*f - 2*a*h))*ArcTan[(Sqrt[2]*Sqrt[c]
*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]])/(Sqrt[b^2 - 4*a*c]*Sqrt[b - Sqrt[b^2 - 4
*a*c]]) - (2*Sqrt[2]*Sqrt[c]*(2*c^2*d + b*(b + Sqrt[b^2 - 4*a*c])*h - c*(b
*f + Sqrt[b^2 - 4*a*c]*f + 2*a*h))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqr
t[b^2 - 4*a*c]]])/(Sqrt[b^2 - 4*a*c]*Sqrt[b + Sqrt[b^2 - 4*a*c]]) + ((2*c^
2*e + b*(b - Sqrt[b^2 - 4*a*c])*i + c*(-(b*g) + Sqrt[b^2 - 4*a*c]*g - 2*a*
i))*Log[-b + Sqrt[b^2 - 4*a*c] - 2*c*x^2])/Sqrt[b^2 - 4*a*c] - ((2*c^2*e +
 b*(b + Sqrt[b^2 - 4*a*c])*i - c*(b*g + Sqrt[b^2 - 4*a*c]*g + 2*a*i))*Log[
b + Sqrt[b^2 - 4*a*c] + 2*c*x^2])/Sqrt[b^2 - 4*a*c])/(4*c^2)
```

3.24.3 Rubi [A] (verified)

Time = 0.79 (sec) , antiderivative size = 321, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.150, Rules used = {2202, 2194, 2188, 2009, 2205, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{a+bx^2+cx^4}\,dx$$
$$\Big\downarrow \; 2202$$
$$\int \frac{hx^4+fx^2+d}{cx^4+bx^2+a}dx+\int \frac{x\left(ix^4+gx^2+e\right)}{cx^4+bx^2+a}dx$$
$$\Big\downarrow \; 2194$$

$$\int \frac{hx^4+fx^2+d}{cx^4+bx^2+a}dx+\frac{1}{2}\int \frac{ix^4+gx^2+e}{cx^4+bx^2+a}dx^2$$

$$\Big\downarrow \text{ 2188}$$

$$\int \frac{hx^4+fx^2+d}{cx^4+bx^2+a}dx+\frac{1}{2}\int \left(\frac{i}{c}+\frac{(cg-bi)x^2+ce-ai}{c\left(cx^4+bx^2+a\right)}\right)dx^2$$

$$\Big\downarrow \text{ 2009}$$

$$\int \frac{hx^4+fx^2+d}{cx^4+bx^2+a}dx+$$
$$\frac{1}{2}\left(-\frac{\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-2aci+b^2i-bcg+2c^2e\right)}{c^2\sqrt{b^2-4ac}}+\frac{(cg-bi)\log\left(a+bx^2+cx^4\right)}{2c^2}+\frac{ix^2}{c}\right)$$

$$\Big\downarrow \text{ 2205}$$

$$\int \left(\frac{h}{c}+\frac{(cf-bh)x^2+cd-ah}{c\left(cx^4+bx^2+a\right)}\right)dx+$$
$$\frac{1}{2}\left(-\frac{\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-2aci+b^2i-bcg+2c^2e\right)}{c^2\sqrt{b^2-4ac}}+\frac{(cg-bi)\log\left(a+bx^2+cx^4\right)}{2c^2}+\frac{ix^2}{c}\right)$$

$$\Big\downarrow \text{ 2009}$$

$$\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{-c(2ah+bf)+b^2h+2c^2d}{\sqrt{b^2-4ac}}-bh+cf\right)}{\sqrt{2}c^{3/2}\sqrt{b-\sqrt{b^2-4ac}}}+$$
$$\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{-2ach+b^2h-bcf+2c^2d}{\sqrt{b^2-4ac}}-bh+cf\right)}{\sqrt{2}c^{3/2}\sqrt{\sqrt{b^2-4ac}+b}}+$$
$$\frac{1}{2}\left(-\frac{\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-2aci+b^2i-bcg+2c^2e\right)}{c^2\sqrt{b^2-4ac}}+\frac{(cg-bi)\log\left(a+bx^2+cx^4\right)}{2c^2}+\frac{ix^2}{c}\right)+\frac{hx}{c}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(a + b*x^2 + c*x^4),x]
```

output
```
(h*x)/c + ((c*f - b*h + (2*c^2*d + b^2*h - c*(b*f + 2*a*h))/Sqrt[b^2 - 4*a
*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*c^(
3/2)*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + ((c*f - b*h - (2*c^2*d - b*c*f + b^2*h
 - 2*a*c*h)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqrt[b^
2 - 4*a*c]]])/(Sqrt[2]*c^(3/2)*Sqrt[b + Sqrt[b^2 - 4*a*c]]) + ((i*x^2)/c -
 ((2*c^2*e - b*c*g + b^2*i - 2*a*c*i)*ArcTanh[(b + 2*c*x^2)/Sqrt[b^2 - 4*a
*c]])/(c^2*Sqrt[b^2 - 4*a*c]) + ((c*g - b*i)*Log[a + b*x^2 + c*x^4])/(2*c^
2))/2
```

3.24. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{a+bx^2+cx^4}\,dx$

3.24.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2188
```
Int[(Pq_)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_.), x_Symbol] :> Int[Expand
Integrand[Pq*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c}, x] && PolyQ[Pq
, x] && IGtQ[p, -2]
```

rule 2194
```
Int[(Pq_)*(x_)^(m_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :
> Simp[1/2   Subst[Int[x^((m - 1)/2)*SubstFor[x^2, Pq, x]*(a + b*x + c*x^2)
^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pq, x^2] && IntegerQ
[(m - 1)/2]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2205
```
Int[(Px_)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :> Int[ExpandInte
grand[Px/(a + b*x^2 + c*x^4), x], x] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x^
2] && Expon[Px, x^2] > 1
```

3.24.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.10 (sec) , antiderivative size = 99, normalized size of antiderivative = 0.31

method	result
risch	$\frac{hx}{c}+\frac{i\,x^{2}}{2c}+\frac{\sum_{\_R=\operatorname{RootOf}\left(c\_Z^{4}+\_Z^{2}b+a\right)}\frac{\left(\left(-bi+gc\right)\_R^{3}+\left(-bh+cf\right)\_R^{2}+\left(-ai+ec\right)\_R-ah+cd\right)\ln\left(x-\_R\right)}{2c\_R^{3}+\_Rb}}{2c}$
default	$\frac{hx+\frac{1}{2}i\,x^{2}}{c}+\frac{\sqrt{-4ac+b^{2}}\left(\frac{\left(\sqrt{-4ac+b^{2}}\,bi-\sqrt{-4ac+b^{2}}\,cg-2aci+b^{2}i-gbc+2e\,c^{2}\right)\ln\left(2c\,x^{2}+\sqrt{-4ac+b^{2}}+b\right)}{4c}+\frac{\left(\sqrt{-4ac+b^{2}}\,bh-\sqrt{-4ac+b^{2}}\,fc-\right.}{}\right.}{c\left(4ac-b^{2}\right)}$

input
```
int((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a),x,method=_RETURNVERBOS
E)
```

output
```
h*x/c+1/2*i*x^2/c+1/2/c*sum(((-b*i+c*g)*_R^3+(-b*h+c*f)*_R^2+(-a*i+c*e)*_R
-a*h+c*d)/(2*_R^3*c+_R*b)*ln(x-_R),_R=RootOf(_Z^4*c+_Z^2*b+a))
```

3.24.5 Fricas [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{a+bx^2+cx^4}\,dx = \text{Timed out}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a),x, algorithm="fr
icas")
```

output
```
Timed out
```

3.24.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{a+bx^2+cx^4}\,dx = \text{Timed out}$$

input
```
integrate((i*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(c*x**4+b*x**2+a),x)
```

output
```
Timed out
```

3.24.7 Maxima [F]

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{a+bx^2+cx^4}\,dx = \int \frac{ix^5+hx^4+gx^3+fx^2+ex+d}{cx^4+bx^2+a}\,dx$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a),x, algorithm="ma
xima")
```

output
```
1/2*(i*x^2 + 2*h*x)/c - integrate(-((c*g - b*i)*x^3 + (c*f - b*h)*x^2 + c*
d - a*h + (c*e - a*i)*x)/(c*x^4 + b*x^2 + a), x)/c
```

3.24.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 5941 vs. $2(277) = 554$.

Time = 1.55 (sec) , antiderivative size = 5941, normalized size of antiderivative = 18.51

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4 + ix^5}{a + bx^2 + cx^4} \, dx = \text{Too large to display}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a),x, algorithm="gi
ac")
```

output
```
1/4*(c*g - b*i)*log(abs(c*x^4 + b*x^2 + a))/c^2 + 1/2*(c*i*x^2 + 2*c*h*x)/
c^2 + 1/8*((2*b^4*c^3 - 16*a*b^2*c^4 + 32*a^2*c^5 - sqrt(2)*sqrt(b^2 - 4*a
*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^4*c + 8*sqrt(2)*sqrt(b^2 - 4*a*c)*sq
rt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^2*c^2 + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt
(b*c + sqrt(b^2 - 4*a*c)*c)*b^3*c^2 - 16*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*
c + sqrt(b^2 - 4*a*c)*c)*a^2*c^3 - 8*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c +
sqrt(b^2 - 4*a*c)*c)*a*b*c^3 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b
^2 - 4*a*c)*c)*b^2*c^3 + 4*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 -
 4*a*c)*c)*a*c^4 - 2*(b^2 - 4*a*c)*b^2*c^3 + 8*(b^2 - 4*a*c)*a*c^4)*c^2*f
- (2*b^5*c^2 - 16*a*b^3*c^3 + 32*a^2*b*c^4 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqr
t(b*c + sqrt(b^2 - 4*a*c)*c)*b^5 + 8*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c +
sqrt(b^2 - 4*a*c)*c)*a*b^3*c + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt
(b^2 - 4*a*c)*c)*b^4*c - 16*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2
- 4*a*c)*c)*a^2*b*c^2 - 8*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 -
4*a*c)*c)*a*b^2*c^2 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*
c)*c)*b^3*c^2 + 4*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c
)*a*b*c^3 - 2*(b^2 - 4*a*c)*b^3*c^2 + 8*(b^2 - 4*a*c)*a*b*c^3)*c^2*h + 2*(
sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^4*c^3 - 8*sqrt(2)*sqrt(b*c + sqr
t(b^2 - 4*a*c)*c)*a*b^2*c^4 - 2*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^
3*c^4 - 2*b^4*c^4 + 16*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*c^5 ...
```

3.24.9 Mupad [B] (verification not implemented)

Time = 8.74 (sec) , antiderivative size = 11383, normalized size of antiderivative = 35.46

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4 + ix^5}{a + bx^2 + cx^4} \, dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(a + b*x^2 + c*x^4),x)
```

output
```
symsum(log((x*(c^4*e^3 - a^3*c*i^3 + c^4*d^2*g + b^4*e*i^2 + a^2*b^2*i^3 +
 b^2*c^2*e*g^2 + 3*a^2*c^2*e*i^2 + a^2*c^2*g*h^2 + 2*b^2*c^2*e^2*i - a^2*c
^2*g^2*i - 2*c^4*d*e*f - a*b*c^2*g^3 + a*c^3*e*g^2 + b*c^3*e*f^2 - a*c^3*f
^2*g - 2*b*c^3*e^2*g - 3*a*c^3*e^2*i - b*c^3*d^2*i + b^3*c*e*h^2 - a*b^3*g
*i^2 - 2*a*b*c^2*e*h^2 - 3*a*b^2*c*e*i^2 - a*b^2*c*g*h^2 + 2*a*b^2*c*g^2*i
 + a^2*b*c*h^2*i - 2*b^2*c^2*e*f*h - 2*a^2*c^2*f*h*i + 2*b*c^3*d*e*h + 2*a
*c^3*d*f*i - 2*a*c^3*d*g*h + 2*a*c^3*e*f*h - 2*b^3*c*e*g*i + 2*a*b*c^2*e*g
*i + 2*a*b*c^2*f*g*h))/c^2 - (a*c^3*f^3 - c^4*d*e^2 + c^4*d^2*f - b^4*d*i^
2 - b^2*c^2*d*g^2 - a^2*c^2*d*i^2 + a^2*c^2*f*h^2 - a^2*c^2*g^2*h - a^2*b^
2*h*i^2 - a^2*b*c*h^3 + a*c^3*d*g^2 - b*c^3*d*f^2 + a*c^3*e^2*h - b*c^3*d^
2*h - b^3*c*d*h^2 + a*b^3*f*i^2 + a^3*c*h*i^2 + 2*a*b*c^2*d*h^2 + a*b*c^2*
f*g^2 + 3*a*b^2*c*d*i^2 - 2*a*b*c^2*f^2*h + a*b^2*c*f*h^2 - 2*a^2*b*c*f*i^
2 - 2*b^2*c^2*d*e*i + 2*b^2*c^2*d*f*h - 2*a^2*c^2*e*h*i + 2*a^2*c^2*f*g*i
+ 2*b*c^3*d*e*g + 2*a*c^3*d*e*i - 2*a*c^3*d*f*h - 2*a*c^3*e*f*g + 2*b^3*c*
d*g*i - 4*a*b*c^2*d*g*i + 2*a*b*c^2*e*f*i - 2*a*b^2*c*f*g*i + 2*a^2*b*c*g*
h*i)/c^2 - root(128*a^2*b^2*c^5*z^4 - 16*a*b^4*c^4*z^4 - 256*a^3*c^6*z^4 +
 128*a^2*b^3*c^3*i*z^3 - 128*a^2*b^2*c^4*g*z^3 - 256*a^3*b*c^4*i*z^3 - 16*
a*b^5*c^2*i*z^3 + 16*a*b^4*c^3*g*z^3 + 256*a^3*c^5*g*z^3 + 160*a^3*b*c^3*g
*i*z^2 + 8*a*b^4*c^2*f*h*z^2 + 8*a*b^4*c^2*e*i*z^2 + 32*a^2*b*c^4*e*g*z^2
+ 32*a^2*b*c^4*d*h*z^2 - 8*a*b^3*c^3*e*g*z^2 - 8*a*b^3*c^3*d*h*z^2 + 16...
```

3.25 $\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{a+bx^2+cx^4} dx$

3.25.1 Optimal result

Integrand size = 55, antiderivative size = 545

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{a+bx^2+cx^4} dx$$

$$= \frac{(c^2h+b^2m-c(bk+am))\,x}{c^3} + \frac{(cj-bl)x^2}{2c^2} + \frac{(ck-bm)x^3}{3c^2} + \frac{lx^4}{4c} + \frac{mx^5}{5c}$$

$$+ \frac{\left(c^3f-c^2(bh+ak)-b^3m+bc(bk+2am)+\frac{2c^4d-c^3(bf+2ah)+b^4m-b^2c(bk+4am)+c^2(b^2h+3abk+2a^2m)}{\sqrt{b^2-4ac}}\right)\arctan\Big(}{\sqrt{2}c^{7/2}\sqrt{b-\sqrt{b^2-4ac}}}$$

$$+ \frac{\left(c^3f-c^2(bh+ak)-b^3m+bc(bk+2am)-\frac{2c^4d-c^3(bf+2ah)+b^4m-b^2c(bk+4am)+c^2(b^2h+3abk+2a^2m)}{\sqrt{b^2-4ac}}\right)\arctan\Big(}{\sqrt{2}c^{7/2}\sqrt{b+\sqrt{b^2-4ac}}}$$

$$- \frac{(2c^3e-c^2(bg+2aj)-b^3l+bc(bj+3al))\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{2c^3\sqrt{b^2-4ac}}$$

$$+ \frac{(c^2g+b^2l-c(bj+al))\log(a+bx^2+cx^4)}{4c^3}$$

output
```
(c^2*h+b^2*m-c*(a*m+b*k))*x/c^3+1/2*(-b*l+c*j)*x^2/c^2+1/3*(-b*m+c*k)*x^3/
c^2+1/4*l*x^4/c+1/5*m*x^5/c+1/4*(c^2*g+b^2*l-c*(a*l+b*j))*ln(c*x^4+b*x^2+a
)/c^3-1/2*(2*c^3*e-c^2*(2*a*j+b*g)-b^3*l+b*c*(3*a*l+b*j))*arctanh((2*c*x^2
+b)/(-4*a*c+b^2)^(1/2))/c^3/(-4*a*c+b^2)^(1/2)+1/2*arctan(x*2^(1/2)*c^(1/2
)/(b-(-4*a*c+b^2)^(1/2))^(1/2))*(c^3*f-c^2*(a*k+b*h)-b^3*m+b*c*(2*a*m+b*k)
+(2*c^4*d-c^3*(2*a*h+b*f)+b^4*m-b^2*c*(4*a*m+b*k)+c^2*(2*a^2*m+3*a*b*k+b^2
*h))/(-4*a*c+b^2)^(1/2))/c^(7/2)*2^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2)+1/2*
arctan(x*2^(1/2)*c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2))*(c^3*f-c^2*(a*k+b*h
)-b^3*m+b*c*(2*a*m+b*k)+(-2*c^4*d+c^3*(2*a*h+b*f)-b^4*m+b^2*c*(4*a*m+b*k)-
c^2*(2*a^2*m+3*a*b*k+b^2*h))/(-4*a*c+b^2)^(1/2))/c^(7/2)*2^(1/2)/(b+(-4*a*
c+b^2)^(1/2))^(1/2)
```

3.25.2 Mathematica [A] (verified)

Time = 0.73 (sec) , antiderivative size = 816, normalized size of antiderivative = 1.50

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{a+bx^2+cx^4}\,dx$$

$$= \frac{(c^2h+b^2m-c(bk+am))\,x}{c^3} + \frac{(cj-bl)x^2}{2c^2} + \frac{(ck-bm)x^3}{3c^2} + \frac{lx^4}{4c} + \frac{mx^5}{5c}$$

$$+ \frac{\left(2c^4d+c^3\left(-bf+\sqrt{b^2-4ac}f-2ah\right)+b^3\left(b-\sqrt{b^2-4ac}\right)m+c^2\left(b^2h-b\sqrt{b^2-4ac}h+3abk-a\sqrt{b^2\right.\right.}{\sqrt{2}c^{7/2}\sqrt{b^2-4ac}\sqrt{b}}$$

$$- \frac{\left(2c^4d-c^3\left(bf+\sqrt{b^2-4ac}f+2ah\right)+b^3\left(b+\sqrt{b^2-4ac}\right)m+c^2\left(b^2h+b\sqrt{b^2-4ac}h+3abk+a\sqrt{b^2-\right.\right.}{\sqrt{2}c^{7/2}\sqrt{b^2-4ac}\sqrt{b+}}$$

$$+ \frac{\left(2c^3e+c^2\left(-bg+\sqrt{b^2-4ac}g-2aj\right)+b^2\left(-b+\sqrt{b^2-4ac}\right)l+c\left(b^2j-b\sqrt{b^2-4ac}j+3abl-a\sqrt{b^2-\right.\right.}{4c^3\sqrt{b^2-4ac}}$$

$$+ \frac{\left(-2c^3e+c^2\left(bg+\sqrt{b^2-4ac}g+2aj\right)+b^2\left(b+\sqrt{b^2-4ac}\right)l-c\left(b^2j+b\sqrt{b^2-4ac}j+3abl+a\sqrt{b^2-4\right.\right.}{4c^3\sqrt{b^2-4ac}}$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4 + j*x^5 + k*x^6 + l*x^7 + m*x^8
)/(a + b*x^2 + c*x^4),x]
```

output
```
((c^2*h + b^2*m - c*(b*k + a*m))*x)/c^3 + ((c*j - b*l)*x^2)/(2*c^2) + ((c*
k - b*m)*x^3)/(3*c^2) + (l*x^4)/(4*c) + (m*x^5)/(5*c) + ((2*c^4*d + c^3*(-
(b*f) + Sqrt[b^2 - 4*a*c]*f - 2*a*h) + b^3*(b - Sqrt[b^2 - 4*a*c])*m + c^2
*(b^2*h - b*Sqrt[b^2 - 4*a*c]*h + 3*a*b*k - a*Sqrt[b^2 - 4*a*c]*k + 2*a^2*
m) + b*c*(-(b^2*k) + b*Sqrt[b^2 - 4*a*c]*k - 4*a*b*m + 2*a*Sqrt[b^2 - 4*a*
c]*m))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*c
^(7/2)*Sqrt[b^2 - 4*a*c]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) - ((2*c^4*d - c^3*(b
*f + Sqrt[b^2 - 4*a*c]*f + 2*a*h) + b^3*(b + Sqrt[b^2 - 4*a*c])*m + c^2*(b
^2*h + b*Sqrt[b^2 - 4*a*c]*h + 3*a*b*k + a*Sqrt[b^2 - 4*a*c]*k + 2*a^2*m)
- b*c*(b^2*k + b*Sqrt[b^2 - 4*a*c]*k + 4*a*b*m + 2*a*Sqrt[b^2 - 4*a*c]*m))
*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*c^(7/2)
*Sqrt[b^2 - 4*a*c]*Sqrt[b + Sqrt[b^2 - 4*a*c]]) + ((2*c^3*e + c^2*(-(b*g)
+ Sqrt[b^2 - 4*a*c]*g - 2*a*j) + b^2*(-b + Sqrt[b^2 - 4*a*c])*l + c*(b^2*j
 - b*Sqrt[b^2 - 4*a*c]*j + 3*a*b*l - a*Sqrt[b^2 - 4*a*c]*l))*Log[-b + Sqrt
[b^2 - 4*a*c] - 2*c*x^2])/(4*c^3*Sqrt[b^2 - 4*a*c]) + ((-2*c^3*e + c^2*(b*
g + Sqrt[b^2 - 4*a*c]*g + 2*a*j) + b^2*(b + Sqrt[b^2 - 4*a*c])*l - c*(b^2*
j + b*Sqrt[b^2 - 4*a*c]*j + 3*a*b*l + a*Sqrt[b^2 - 4*a*c]*l))*Log[b + Sqrt
[b^2 - 4*a*c] + 2*c*x^2])/(4*c^3*Sqrt[b^2 - 4*a*c])
```

3.25.3 Rubi [A] (verified)

Time = 2.99 (sec) , antiderivative size = 545, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 6, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.109, Rules used = {2202, 2194, 2188, 2009, 2205, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{a+bx^2+cx^4} \, dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{mx^8+kx^6+hx^4+fx^2+d}{cx^4+bx^2+a}dx+\int \frac{x\left(lx^6+jx^4+gx^2+e\right)}{cx^4+bx^2+a}dx$$

$$\Big\downarrow \text{2194}$$

$$\int \frac{mx^8+kx^6+hx^4+fx^2+d}{cx^4+bx^2+a}dx+\frac{1}{2}\int \frac{lx^6+jx^4+gx^2+e}{cx^4+bx^2+a}dx^2$$

$$\Big\downarrow \text{2188}$$

$$\frac{1}{2}\int \left(\frac{lx^2}{c}+\frac{cj-bl}{c^2}+\frac{ec^2-ajc+\left(lb^2+c^2g-c(bj+al)\right)x^2+abl}{c^2\left(cx^4+bx^2+a\right)}\right)dx^2+\int \frac{mx^8+kx^6+hx^4+fx^2+d}{cx^4+bx^2+a}dx$$

$$\Big\downarrow \; 2009$$

$$\int \frac{mx^8+kx^6+hx^4+fx^2+d}{cx^4+bx^2+a}dx+\frac{1}{2}\left(-\frac{\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-c^2(2aj+bg)+bc(3al+bj)+b^3(-l)+2c^3e\right)}{c^3\sqrt{b^2-4ac}}+\frac{\log\left(a+bx^2+cx^4\right)\left(-c(al+bj)+b^2\right.}{2c^3}\right.$$

$$\Big\downarrow \; 2205$$

$$\int \left(\frac{mx^4}{c}+\frac{(ck-bm)x^2}{c^2}+\frac{mb^2+c^2h-c(bk+am)}{c^3}+\frac{dc^3-ahc^2+a(bk+am)c+\left(-mb^3+c(bk+2am)b+c^3\right.}{c^3\left(cx^4+bx^2+a\right)}\right.$$
$$\frac{1}{2}\left(-\frac{\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-c^2(2aj+bg)+bc(3al+bj)+b^3(-l)+2c^3e\right)}{c^3\sqrt{b^2-4ac}}+\frac{\log\left(a+bx^2+cx^4\right)\left(-c(al+bj)+b^2\right.}{2c^3}\right.$$

$$\Big\downarrow \; 2009$$

$$\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{c^2\left(2a^2m+3abk+b^2h\right)-b^2c(4am+bk)-c^3(2ah+bf)+b^4m+2c^4d}{\sqrt{b^2-4ac}}-c^2(ak+bh)+bc(2am+bk)+b^3(-m\right.}{\sqrt{2}c^{7/2}\sqrt{b-\sqrt{b^2-4ac}}}$$
$$\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{c^2\left(2a^2m+3abk+b^2h\right)-b^2c(4am+bk)-c^3(2ah+bf)+b^4m+2c^4d}{\sqrt{b^2-4ac}}-c^2(ak+bh)+bc(2am+bk)+b^3(-\right.}{\sqrt{2}c^{7/2}\sqrt{\sqrt{b^2-4ac}+b}}$$
$$\frac{1}{2}\left(-\frac{\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-c^2(2aj+bg)+bc(3al+bj)+b^3(-l)+2c^3e\right)}{c^3\sqrt{b^2-4ac}}+\frac{\log\left(a+bx^2+cx^4\right)\left(-c(al+bj)+b^2\right.}{2c^3}\right.$$
$$\frac{x\left(-c(am+bk)+b^2m+c^2h\right)}{c^3}+\frac{x^3(ck-bm)}{3c^2}+\frac{mx^5}{5c}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4 + j*x^5 + k*x^6 + l*x^7 + m*x^8)/(a +
 b*x^2 + c*x^4),x]
```

3.25. $\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{a+bx^2+cx^4}\,dx$

output
```
((c^2*h + b^2*m - c*(b*k + a*m))*x)/c^3 + ((c*k - b*m)*x^3)/(3*c^2) + (m*x
^5)/(5*c) + ((c^3*f - c^2*(b*h + a*k) - b^3*m + b*c*(b*k + 2*a*m) + (2*c^4
*d - c^3*(b*f + 2*a*h) + b^4*m - b^2*c*(b*k + 4*a*m) + c^2*(b^2*h + 3*a*b*
k + 2*a^2*m))/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[
b^2 - 4*a*c]]])/(Sqrt[2]*c^(7/2)*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + ((c^3*f -
c^2*(b*h + a*k) - b^3*m + b*c*(b*k + 2*a*m) - (2*c^4*d - c^3*(b*f + 2*a*h)
 + b^4*m - b^2*c*(b*k + 4*a*m) + c^2*(b^2*h + 3*a*b*k + 2*a^2*m))/Sqrt[b^2
 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(Sqrt[
2]*c^(7/2)*Sqrt[b + Sqrt[b^2 - 4*a*c]]) + (((c*j - b*l)*x^2)/c^2 + (l*x^4)
/(2*c) - ((2*c^3*e - c^2*(b*g + 2*a*j) - b^3*l + b*c*(b*j + 3*a*l))*ArcTan
h[(b + 2*c*x^2)/Sqrt[b^2 - 4*a*c]])/(c^3*Sqrt[b^2 - 4*a*c]) + ((c^2*g + b^
2*l - c*(b*j + a*l))*Log[a + b*x^2 + c*x^4])/(2*c^3))/2
```

3.25.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2188
```
Int[(Pq_)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_.), x_Symbol] :> Int[Expand
Integrand[Pq*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c}, x] && PolyQ[Pq
, x] && IGtQ[p, -2]
```

rule 2194
```
Int[(Pq_)*(x_)^(m_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :
> Simp[1/2   Subst[Int[x^((m - 1)/2)*SubstFor[x^2, Pq, x]*(a + b*x + c*x^2)
^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pq, x^2] && IntegerQ
[(m - 1)/2]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2205
```
Int[(Px_)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :> Int[ExpandInte
grand[Px/(a + b*x^2 + c*x^4), x], x] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x^
2] && Expon[Px, x^2] > 1
```

3.25.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.76 (sec) , antiderivative size = 246, normalized size of antiderivative = 0.45

method	result
risch	$\frac{m x^5}{5c}+\frac{l x^4}{4c}-\frac{bm x^3}{3c^2}+\frac{k x^3}{3c}-\frac{bl x^2}{2c^2}+\frac{j x^2}{2c}-\frac{amx}{c^2}+\frac{b^2mx}{c^3}-\frac{bkx}{c^2}+\frac{hx}{c}+\frac{\sum_{\_R=\mathrm{RootOf}\left(c\_Z^4+\_Z^2b+a\right)} \frac{\left(c\left(-acl+b^2\right.\right.}{}}{}$
default	$-\frac{-\frac{1}{5}m x^5c^2-\frac{1}{4}l x^4c^2+\frac{1}{3}bcm x^3-\frac{1}{3}c^2k x^3+\frac{1}{2}bcl x^2-\frac{1}{2}c^2j x^2+acmx-b^2mx+bckx-c^2hx}{c^3}+\frac{-\frac{\sqrt{-4ac+b^2}\left(\frac{\left(-\sqrt{-4ac+b^2}\,a\,c^2l+\sqrt{-\right.}{}\right.}{}}{}$

input
```
int((m*x^8+l*x^7+k*x^6+j*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a),x,me
thod=_RETURNVERBOSE)
```

output
```
1/5*m*x^5/c+1/4*l*x^4/c-1/3/c^2*b*m*x^3+1/3/c*k*x^3-1/2/c^2*b*l*x^2+1/2/c*
j*x^2-1/c^2*a*m*x+1/c^3*b^2*m*x-1/c^2*b*k*x+h*x/c+1/2/c^3*sum((c*(-a*c*l+b
^2*l-b*c*j+c^2*g)*_R^3+_R^2*(2*a*b*c*m-a*c^2*k-b^3*m+b^2*c*k-b*c^2*h+c^3*f
)+c*(a*b*l-a*c*j+c^2*e)*_R+a^2*c*m-a*b^2*m+a*b*c*k-a*c^2*h+c^3*d)/(2*_R^3*
c+_R*b)*ln(x-_R),_R=RootOf(_Z^4*c+_Z^2*b+a))
```

3.25.5 Fricas [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{a+bx^2+cx^4}\,dx=\text{Timed out}$$

input
```
integrate((m*x^8+l*x^7+k*x^6+j*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a
),x, algorithm="fricas")
```

output
```
Timed out
```

3.25.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{a+bx^2+cx^4}\,dx = \text{Timed out}$$

input
```
integrate((m*x**8+l*x**7+k*x**6+j*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(c*x**4
+b*x**2+a),x)
```

output
```
Timed out
```

3.25.7 Maxima [F]

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{a+bx^2+cx^4}\,dx = \int \frac{mx^8+lx^7+kx^6+jx^5+hx^4+gx^3+fx^2+ex+d}{cx^4+bx^2+a}\,dx$$

input
```
integrate((m*x^8+l*x^7+k*x^6+j*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a
),x, algorithm="maxima")
```

output
```
1/60*(12*c^2*m*x^5 + 15*c^2*l*x^4 + 20*(c^2*k - b*c*m)*x^3 + 30*(c^2*j - b
*c*l)*x^2 + 60*(c^2*h - b*c*k + (b^2 - a*c)*m)*x)/c^3 - integrate(-(c^3*d
- a*c^2*h + a*b*c*k + (c^3*g - b*c^2*j + (b^2*c - a*c^2)*l)*x^3 + (c^3*f -
 b*c^2*h + (b^2*c - a*c^2)*k - (b^3 - 2*a*b*c)*m)*x^2 - (a*b^2 - a^2*c)*m
+ (c^3*e - a*c^2*j + a*b*c*l)*x)/(c*x^4 + b*x^2 + a), x)/c^3
```

3.25.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 11830 vs. 2(495) = 990.

Time = 2.10 (sec) , antiderivative size = 11830, normalized size of antiderivative = 21.71

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{a+bx^2+cx^4}\,dx = \text{Too large to display}$$

input
```
integrate((m*x^8+l*x^7+k*x^6+j*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a
),x, algorithm="giac")
```

output
```
1/8*((2*b^4*c^5 - 16*a*b^2*c^6 + 32*a^2*c^7 - sqrt(2)*sqrt(b^2 - 4*a*c)*sq
rt(b*c + sqrt(b^2 - 4*a*c)*c)*b^4*c^3 + 8*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b
*c + sqrt(b^2 - 4*a*c)*c)*a*b^2*c^4 + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c
 + sqrt(b^2 - 4*a*c)*c)*b^3*c^4 - 16*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c +
sqrt(b^2 - 4*a*c)*c)*a^2*c^5 - 8*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt
(b^2 - 4*a*c)*c)*a*b*c^5 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 -
 4*a*c)*c)*b^2*c^5 + 4*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a
*c)*c)*a*c^6 - 2*(b^2 - 4*a*c)*b^2*c^5 + 8*(b^2 - 4*a*c)*a*c^6)*c^2*f - (2
*b^5*c^4 - 16*a*b^3*c^5 + 32*a^2*b*c^6 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*
c + sqrt(b^2 - 4*a*c)*c)*b^5*c^2 + 8*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c +
sqrt(b^2 - 4*a*c)*c)*a*b^3*c^3 + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sq
rt(b^2 - 4*a*c)*c)*b^4*c^3 - 16*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(
b^2 - 4*a*c)*c)*a^2*b*c^4 - 8*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^
2 - 4*a*c)*c)*a*b^2*c^4 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 -
4*a*c)*c)*b^3*c^4 + 4*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*
c)*c)*a*b*c^5 - 2*(b^2 - 4*a*c)*b^3*c^4 + 8*(b^2 - 4*a*c)*a*b*c^5)*c^2*h +
 (2*b^6*c^3 - 18*a*b^4*c^4 + 48*a^2*b^2*c^5 - 32*a^3*c^6 - sqrt(2)*sqrt(b^
2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^6*c + 9*sqrt(2)*sqrt(b^2 - 4*
a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^4*c^2 + 2*sqrt(2)*sqrt(b^2 - 4*a*
c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^5*c^2 - 24*sqrt(2)*sqrt(b^2 - 4*a*...
```

3.25.9 Mupad [B] (verification not implemented)

Time = 10.69 (sec) , antiderivative size = 49150, normalized size of antiderivative = 90.18

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4 + jx^5 + kx^6 + lx^7 + mx^8}{a + bx^2 + cx^4} \, dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4 + j*x^5 + k*x^6 + l*x^7 + m*x^8)/(a +
 b*x^2 + c*x^4),x)
```

output
```
x^2*(j/(2*c) - (b*l)/(2*c^2)) - x*((b*(k/c - (b*m)/c^2))/c - h/c + (a*m)/c
^2) + x^3*(k/(3*c) - (b*m)/(3*c^2)) + symsum(log((c^7*d*e^2 - a*c^6*f^3 -
c^7*d^2*f + b^7*d*m^2 + a^4*c^3*k^3 + a^4*b^3*m^3 + a^2*b*c^4*h^3 + b^2*c^
5*d*g^2 + b^3*c^4*d*h^2 + a^2*c^5*d*j^2 - a^2*c^5*f*h^2 + a^2*c^5*g^2*h +
b^4*c^3*d*j^2 - a^3*c^4*d*l^2 - b^2*c^5*d^2*k + b^5*c^2*d*k^2 + 3*a^2*c^5*
f^2*k - 3*a^3*c^4*f*k^2 + a^2*c^5*e^2*m - a^3*c^4*h*j^2 + b^3*c^4*d^2*m +
a^3*c^4*h^2*k - a^4*c^3*f*m^2 + a^2*b^5*h*m^2 - a^3*c^4*g^2*m + a^4*c^3*h*
l^2 - a^3*b^4*k*m^2 + a^4*c^3*j^2*m + a^5*c^2*k*m^2 - a^5*c^2*l^2*m - a^3*
b^2*c^2*k^3 - a*c^6*d*g^2 + b*c^6*d*f^2 - a*c^6*e^2*h + b*c^6*d^2*h + a*c^
6*d^2*k - 2*a^5*b*c*m^3 + b^6*c*d*l^2 - a*b^6*f*m^2 - 2*a*b*c^5*d*h^2 - a*
b*c^5*f*g^2 + 2*a*b*c^5*f^2*h + a*b*c^5*e^2*k - 2*a*b*c^5*d^2*m - 6*a*b^5*
c*d*m^2 - 2*b^2*c^5*d*f*h - a*b^5*c*f*l^2 + 2*b^2*c^5*d*e*j - 2*b^3*c^4*d*
e*l + 2*b^3*c^4*d*f*k - 2*b^3*c^4*d*g*j - 2*a^2*c^5*d*f*m + 2*a^2*c^5*d*g*
l - 2*a^2*c^5*d*h*k - 2*a^2*c^5*e*f*l - 2*a^2*c^5*e*g*k + 2*a^2*c^5*e*h*j
- 2*a^2*c^5*f*g*j - 2*b^4*c^3*d*f*m + 2*b^4*c^3*d*g*l - 2*b^4*c^3*d*h*k +
2*b^5*c^2*d*h*m + 2*a^3*c^4*f*h*m - 2*a^3*c^4*g*h*l - 2*b^5*c^2*d*j*l + 2*
a^3*c^4*d*k*m - 2*a^3*c^4*e*j*m + 2*a^3*c^4*e*k*l + 2*a^3*c^4*f*j*l + 2*a^
3*c^4*g*j*k + 2*a^4*c^3*g*l*m - 2*a^4*c^3*h*k*m - 2*a^4*c^3*j*k*l - 3*a*b^
2*c^4*d*j^2 - a*b^2*c^4*f*h^2 - 4*a*b^3*c^3*d*k^2 + 3*a^2*b*c^4*d*k^2 - a*
b^3*c^3*f*j^2 - 5*a*b^4*c^2*d*l^2 + 2*a^2*b*c^4*f*j^2 - 2*a*b^2*c^4*f^2...
```

3.25. $\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{a+bx^2+cx^4}\,dx$

3.26 $\int \frac{d+ex}{(4-5x^2+x^4)^2} \, dx$

3.26.1 Optimal result

Integrand size = 18, antiderivative size = 94

$$\int \frac{d+ex}{(4-5x^2+x^4)^2} \, dx = \frac{dx(17-5x^2)}{72(4-5x^2+x^4)} + \frac{e(5-2x^2)}{18(4-5x^2+x^4)} + \frac{19}{432} d \operatorname{arctanh}\left(\frac{x}{2}\right) - \frac{1}{54} d \operatorname{arctanh}(x) + \frac{1}{27} e \log(1-x^2) - \frac{1}{27} e \log(4-x^2)$$

output
```
1/72*d*x*(-5*x^2+17)/(x^4-5*x^2+4)+1/18*e*(-2*x^2+5)/(x^4-5*x^2+4)+19/432*
d*arctanh(1/2*x)-1/54*d*arctanh(x)+1/27*e*ln(-x^2+1)-1/27*e*ln(-x^2+4)
```

3.26.2 Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 90, normalized size of antiderivative = 0.96

$$\int \frac{d+ex}{(4-5x^2+x^4)^2} \, dx = \frac{1}{864} \left(\frac{12(e(20-8x^2)+dx(17-5x^2))}{4-5x^2+x^4} + 8(d+4e)\log(1-x) - (19d+32e)\log(2-x) - 8(d-4e)\log(1+x) + (19d-32e)\log(2+x) \right)$$

input
```
Integrate[(d + e*x)/(4 - 5*x^2 + x^4)^2,x]
```

output
```
((12*(e*(20 - 8*x^2) + d*x*(17 - 5*x^2)))/(4 - 5*x^2 + x^4) + 8*(d + 4*e)*
Log[1 - x] - (19*d + 32*e)*Log[2 - x] - 8*(d - 4*e)*Log[1 + x] + (19*d - 3
2*e)*Log[2 + x])/864
```

3.26.3 Rubi [A] (verified)

Time = 0.32 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.12, number of steps used = 10, number of rules used = 9, $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$, Rules used = {2202, 27, 1405, 25, 1432, 1084, 1480, 220, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex}{\left(x^4-5x^2+4\right)^2}dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{d}{\left(x^4-5x^2+4\right)^2}dx+\int \frac{ex}{\left(x^4-5x^2+4\right)^2}dx$$

$$\Big\downarrow \text{27}$$

$$d\int \frac{1}{\left(x^4-5x^2+4\right)^2}dx+e\int \frac{x}{\left(x^4-5x^2+4\right)^2}dx$$

$$\Big\downarrow \text{1405}$$

$$d\left(\frac{x\left(17-5x^2\right)}{72\left(x^4-5x^2+4\right)}-\frac{1}{72}\int -\frac{1-5x^2}{x^4-5x^2+4}dx\right)+e\int \frac{x}{\left(x^4-5x^2+4\right)^2}dx$$

$$\Big\downarrow \text{25}$$

$$d\left(\frac{1}{72}\int \frac{1-5x^2}{x^4-5x^2+4}dx+\frac{x\left(17-5x^2\right)}{72\left(x^4-5x^2+4\right)}\right)+e\int \frac{x}{\left(x^4-5x^2+4\right)^2}dx$$

$$\Big\downarrow \text{1432}$$

$$d\left(\frac{1}{72}\int \frac{1-5x^2}{x^4-5x^2+4}dx+\frac{x\left(17-5x^2\right)}{72\left(x^4-5x^2+4\right)}\right)+\frac{1}{2}e\int \frac{1}{\left(x^4-5x^2+4\right)^2}dx^2$$

$$\Big\downarrow \text{1084}$$

$$d\left(\frac{1}{72}\int \frac{1-5x^2}{x^4-5x^2+4}dx+\frac{x\left(17-5x^2\right)}{72\left(x^4-5x^2+4\right)}\right)+$$
$$\frac{1}{2}e\int \left(\frac{2}{27\left(4-x^2\right)}+\frac{1}{9\left(4-x^2\right)^2}-\frac{2}{27\left(1-x^2\right)}+\frac{1}{9\left(1-x^2\right)^2}\right)dx^2$$

$$\Big\downarrow \text{1480}$$

$$d\left(\frac{1}{72}\left(\frac{4}{3}\int \frac{1}{x^2-1}dx-\frac{19}{3}\int \frac{1}{x^2-4}dx\right)+\frac{x\left(17-5x^2\right)}{72\left(x^4-5x^2+4\right)}\right)+\frac{1}{2}e\int \left(\frac{2}{27\left(4-x^2\right)}+\frac{1}{9\left(4-x^2\right)^2}-\frac{2}{27\left(1-x^2\right)}+\frac{1}{9\left(1-x^2\right)^2}\right)dx^2$$

$$\Big\downarrow \text{220}$$

$$\frac{1}{2}e\int \left(\frac{2}{27\left(4-x^2\right)}+\frac{1}{9\left(4-x^2\right)^2}-\frac{2}{27\left(1-x^2\right)}+\frac{1}{9\left(1-x^2\right)^2}\right)dx^2+d\left(\frac{1}{72}\left(\frac{19}{6}\text{arctanh}\left(\frac{x}{2}\right)-\frac{4\text{arctanh}(x)}{3}\right)+\frac{x\left(17-5x^2\right)}{72\left(x^4-5x^2+4\right)}\right)$$

$$\Big\downarrow \text{2009}$$

$$d\left(\frac{1}{72}\left(\frac{19}{6}\text{arctanh}\left(\frac{x}{2}\right)-\frac{4\text{arctanh}(x)}{3}\right)+\frac{x\left(17-5x^2\right)}{72\left(x^4-5x^2+4\right)}\right)+\frac{1}{2}e\left(\frac{1}{9\left(1-x^2\right)}+\frac{1}{9\left(4-x^2\right)}+\frac{2}{27}\log\left(1-x^2\right)-\frac{2}{27}\log\left(4-x^2\right)\right)$$

input
```
Int[(d + e*x)/(4 - 5*x^2 + x^4)^2,x]
```

output
```
d*((x*(17 - 5*x^2))/(72*(4 - 5*x^2 + x^4)) + ((19*ArcTanh[x/2])/6 - (4*Arc
Tanh[x])/3)/72) + (e*(1/(9*(1 - x^2)) + 1/(9*(4 - x^2)) + (2*Log[1 - x^2])
/27 - (2*Log[4 - x^2])/27))/2
```

3.26.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 220
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[b, 2])^(-
1))*ArcTanh[Rt[b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] &&
 (LtQ[a, 0] || GtQ[b, 0])
```

rule 1084
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> With[{q = Rt[b^2
- 4*a*c, 2]}, Simp[1/c^p   Int[ExpandIntegrand[(b/2 - q/2 + c*x)^p*(b/2 + q
/2 + c*x)^p, x], x], x] /;  !FractionalPowerFactorQ[q]] /; FreeQ[{a, b, c},
 x] && IntegerQ[p] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 1405
```
Int[((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[(-x)*(b^2
- 2*a*c + b*c*x^2)*((a + b*x^2 + c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))
), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(b^2 - 2*a*c + 2*(p + 1)*(
b^2 - 4*a*c) + b*c*(4*p + 7)*x^2)*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; Fr
eeQ[{a, b, c}, x] && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1] && IntegerQ[2*p]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.26. $\int \frac{d+ex}{(4-5x^2+x^4)^2}\,dx$

3.26.4 Maple [A] (verified)

Time = 0.08 (sec) , antiderivative size = 83, normalized size of antiderivative = 0.88

method	result
norman	$\frac{-\frac{1}{9}e\,x^2+\frac{17}{72}dx-\frac{5}{72}x^3d+\frac{5}{18}e}{x^4-5x^2+4}+\left(-\frac{19d}{864}-\frac{e}{27}\right)\ln\left(x-2\right)+\left(-\frac{d}{108}+\frac{e}{27}\right)\ln\left(x+1\right)+\left(\frac{d}{108}+\frac{e}{27}\right)\ln\left(x-\right.$
risch	$\frac{-\frac{1}{9}e\,x^2+\frac{17}{72}dx-\frac{5}{72}x^3d+\frac{5}{18}e}{x^4-5x^2+4}-\frac{\ln(x+1)d}{108}+\frac{\ln(x+1)e}{27}+\frac{\ln(1-x)d}{108}+\frac{\ln(1-x)e}{27}-\frac{19\ln(2-x)d}{864}-\frac{\ln(2-x)e}{27}+\frac{19\ln(}{8}$
default	$-\frac{\frac{d}{144}-\frac{e}{72}}{x+2}+\left(\frac{19d}{864}-\frac{e}{27}\right)\ln\left(x+2\right)+\left(-\frac{d}{108}+\frac{e}{27}\right)\ln\left(x+1\right)-\frac{\frac{d}{36}-\frac{e}{36}}{x+1}-\frac{\frac{d}{36}+\frac{e}{36}}{x-1}+\left(\frac{d}{108}+\frac{e}{27}\right)\ln\left(\right.$
parallelrisch	$-\frac{-240e-204dx+76\ln(x-2)d+128\ln(x-2)e-32\ln(x-1)d-128\ln(x-1)e+32\ln(x-2)x^4e+96e\,x^2-160\ln(x-2)x^2e+40\ln(x}{}$

input
```
int((e*x+d)/(x^4-5*x^2+4)^2,x,method=_RETURNVERBOSE)
```

output
```
(-1/9*e*x^2+17/72*d*x-5/72*x^3*d+5/18*e)/(x^4-5*x^2+4)+(-19/864*d-1/27*e)*
ln(x-2)+(-1/108*d+1/27*e)*ln(x+1)+(1/108*d+1/27*e)*ln(x-1)+(19/864*d-1/27*
e)*ln(x+2)
```

3.26.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 169 vs. 2(80) = 160.

Time = 0.28 (sec) , antiderivative size = 169, normalized size of antiderivative = 1.80

$$\int \frac{d+ex}{(4-5x^2+x^4)^2}\,dx = -\frac{60\,dx^3+96\,ex^2-204\,dx-((19\,d-32\,e)x^4-5\,(19\,d-32\,e)x^2+76\,d-128\,e)\log\left(x+2\right)+8\,((d-4\,e}{}$$

input
```
integrate((e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="fricas")
```

output
```
-1/864*(60*d*x^3 + 96*e*x^2 - 204*d*x - ((19*d - 32*e)*x^4 - 5*(19*d - 32*
e)*x^2 + 76*d - 128*e)*log(x + 2) + 8*((d - 4*e)*x^4 - 5*(d - 4*e)*x^2 + 4
*d - 16*e)*log(x + 1) - 8*((d + 4*e)*x^4 - 5*(d + 4*e)*x^2 + 4*d + 16*e)*l
og(x - 1) + ((19*d + 32*e)*x^4 - 5*(19*d + 32*e)*x^2 + 76*d + 128*e)*log(x
 - 2) - 240*e)/(x^4 - 5*x^2 + 4)
```

3.26.6 Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 604 vs. 2(78) = 156.

Time = 2.04 (sec) , antiderivative size = 604, normalized size of antiderivative = 6.43

$$\int \frac{d+ex}{(4-5x^2+x^4)^2}\,dx =$$

$$-\frac{(d-4e)\log\left(x+\frac{-6006260d^4e+2341251d^4(d-4e)-18247680d^2e^3+24099840d^2e^2(d-4e)+7387904d^2e(d-4e)^2-665280d^2(d-4e)^3+5}{1675971d^5-66150400d^3e^2+318767104de^4}\right.}{108}$$

$$+\frac{(d+4e)\log\left(x+\frac{-6006260d^4e-2341251d^4(d+4e)-18247680d^2e^3-24099840d^2e^2(d+4e)+7387904d^2e(d+4e)^2+665280d^2(d+4e)^3+58}{1675971d^5-66150400d^3e^2+318767104de^4}\right.}{108}$$

$$+\frac{(19d-32e)\log\left(x+\frac{-6006260d^4e-\frac{2341251d^4\cdot(19d-32e)}{8}-18247680d^2e^3-3012480d^2e^2\cdot(19d-32e)+115436d^2e(19d-32e)^2+\frac{10395d^2(}{}}{1675971d^5-66150400d^3e^2+31}\right.}{864}$$

$$-\frac{(19d+32e)\log\left(x+\frac{-6006260d^4e+\frac{2341251d^4\cdot(19d+32e)}{8}-18247680d^2e^3+3012480d^2e^2\cdot(19d+32e)+115436d^2e(19d+32e)^2-\frac{10395d^2(}{}}{1675971d^5-66150400d^3e^2+31}\right.}{864}$$

$$+\frac{-5dx^3+17dx-8ex^2+20e}{72x^4-360x^2+288}$$

input
```
integrate((e*x+d)/(x**4-5*x**2+4)**2,x)
```

output
```
-(d - 4*e)*log(x + (-6006260*d**4*e + 2341251*d**4*(d - 4*e) - 18247680*d*
*2*e**3 + 24099840*d**2*e**2*(d - 4*e) + 7387904*d**2*e*(d - 4*e)**2 - 665
280*d**2*(d - 4*e)**3 + 587202560*e**5 - 12582912*e**4*(d - 4*e) - 3670016
0*e**3*(d - 4*e)**2 + 786432*e**2*(d - 4*e)**3)/(1675971*d**5 - 66150400*d
**3*e**2 + 318767104*d*e**4))/108 + (d + 4*e)*log(x + (-6006260*d**4*e - 2
341251*d**4*(d + 4*e) - 18247680*d**2*e**3 - 24099840*d**2*e**2*(d + 4*e)
+ 7387904*d**2*e*(d + 4*e)**2 + 665280*d**2*(d + 4*e)**3 + 587202560*e**5
+ 12582912*e**4*(d + 4*e) - 36700160*e**3*(d + 4*e)**2 - 786432*e**2*(d +
4*e)**3)/(1675971*d**5 - 66150400*d**3*e**2 + 318767104*d*e**4))/108 + (19
*d - 32*e)*log(x + (-6006260*d**4*e - 2341251*d**4*(19*d - 32*e)/8 - 18247
680*d**2*e**3 - 3012480*d**2*e**2*(19*d - 32*e) + 115436*d**2*e*(19*d - 32
*e)**2 + 10395*d**2*(19*d - 32*e)**3/8 + 587202560*e**5 + 1572864*e**4*(19
*d - 32*e) - 573440*e**3*(19*d - 32*e)**2 - 1536*e**2*(19*d - 32*e)**3)/(1
675971*d**5 - 66150400*d**3*e**2 + 318767104*d*e**4))/864 - (19*d + 32*e)*
log(x + (-6006260*d**4*e + 2341251*d**4*(19*d + 32*e)/8 - 18247680*d**2*e*
*3 + 3012480*d**2*e**2*(19*d + 32*e) + 115436*d**2*e*(19*d + 32*e)**2 - 10
395*d**2*(19*d + 32*e)**3/8 + 587202560*e**5 - 1572864*e**4*(19*d + 32*e)
- 573440*e**3*(19*d + 32*e)**2 + 1536*e**2*(19*d + 32*e)**3)/(1675971*d**5
 - 66150400*d**3*e**2 + 318767104*d*e**4))/864 + (-5*d*x**3 + 17*d*x - 8*e
*x**2 + 20*e)/(72*x**4 - 360*x**2 + 288)
```

3.26.7 Maxima [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 83, normalized size of antiderivative = 0.88

$$\int \frac{d+ex}{\left(4-5x^2+x^4\right)^2}\,dx = \frac{1}{864}\,(19\,d-32\,e)\log\left(x+2\right)-\frac{1}{108}\,(d-4\,e)\log\left(x+1\right) + \frac{1}{108}\,(d+4\,e)\log\left(x-1\right)-\frac{1}{864}\,(19\,d+32\,e)\log\left(x-2\right) - \frac{5\,dx^3+8\,ex^2-17\,dx-20\,e}{72\,(x^4-5\,x^2+4)}$$

input
```
integrate((e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="maxima")
```

output
```
1/864*(19*d - 32*e)*log(x + 2) - 1/108*(d - 4*e)*log(x + 1) + 1/108*(d + 4
*e)*log(x - 1) - 1/864*(19*d + 32*e)*log(x - 2) - 1/72*(5*d*x^3 + 8*e*x^2
- 17*d*x - 20*e)/(x^4 - 5*x^2 + 4)
```

3.26.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 87, normalized size of antiderivative = 0.93

$$\int \frac{d+ex}{(4-5x^2+x^4)^2}\,dx = \frac{1}{864}\,(19\,d-32\,e)\log(|x+2|) - \frac{1}{108}\,(d-4\,e)\log(|x+1|) + \frac{1}{108}\,(d+4\,e)\log(|x-1|) - \frac{1}{864}\,(19\,d+32\,e)\log(|x-2|) - \frac{5\,dx^3+8\,ex^2-17\,dx-20\,e}{72\,(x^4-5\,x^2+4)}$$

input
```
integrate((e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="giac")
```

output
```
1/864*(19*d - 32*e)*log(abs(x + 2)) - 1/108*(d - 4*e)*log(abs(x + 1)) + 1/
108*(d + 4*e)*log(abs(x - 1)) - 1/864*(19*d + 32*e)*log(abs(x - 2)) - 1/72
*(5*d*x^3 + 8*e*x^2 - 17*d*x - 20*e)/(x^4 - 5*x^2 + 4)
```

3.26.9 Mupad [B] (verification not implemented)

Time = 0.05 (sec) , antiderivative size = 84, normalized size of antiderivative = 0.89

$$\int \frac{d+ex}{(4-5x^2+x^4)^2}\,dx = \ln(x-1)\left(\frac{d}{108}+\frac{e}{27}\right) - \ln(x+1)\left(\frac{d}{108}-\frac{e}{27}\right) - \ln(x-2)\left(\frac{19\,d}{864}+\frac{e}{27}\right) + \ln(x+2)\left(\frac{19\,d}{864}-\frac{e}{27}\right) + \frac{-\frac{5\,d\,x^3}{72}-\frac{e\,x^2}{9}+\frac{17\,d\,x}{72}+\frac{5\,e}{18}}{x^4-5\,x^2+4}$$

input
```
int((d + e*x)/(x^4 - 5*x^2 + 4)^2,x)
```

output
```
log(x - 1)*(d/108 + e/27) - log(x + 1)*(d/108 - e/27) - log(x - 2)*((19*d)
/864 + e/27) + log(x + 2)*((19*d)/864 - e/27) + ((5*e)/18 + (17*d*x)/72 -
(5*d*x^3)/72 - (e*x^2)/9)/(x^4 - 5*x^2 + 4)
```

3.27 $\int \frac{d+ex+fx^2}{(4-5x^2+x^4)^2} dx$

3.27.1 Optimal result

Integrand size = 23, antiderivative size = 115

$$\int \frac{d+ex+fx^2}{(4-5x^2+x^4)^2} dx = \frac{e(5-2x^2)}{18(4-5x^2+x^4)} + \frac{x(17d+20f-(5d+8f)x^2)}{72(4-5x^2+x^4)} + \frac{1}{432}(19d+52f)\text{arctanh}\left(\frac{x}{2}\right) - \frac{1}{54}(d+7f)\text{arctanh}(x) + \frac{1}{27}e\log(1-x^2) - \frac{1}{27}e\log(4-x^2)$$

output
```
1/18*e*(-2*x^2+5)/(x^4-5*x^2+4)+1/72*x*(17*d+20*f-(5*d+8*f)*x^2)/(x^4-5*x^
2+4)+1/432*(19*d+52*f)*arctanh(1/2*x)-1/54*(d+7*f)*arctanh(x)+1/27*e*ln(-x
^2+1)-1/27*e*ln(-x^2+4)
```

3.27.2 Mathematica [A] (verified)

Time = 0.05 (sec) , antiderivative size = 112, normalized size of antiderivative = 0.97

$$\int \frac{d+ex+fx^2}{(4-5x^2+x^4)^2} dx = \frac{1}{864}\left(\frac{12(17dx+20fx-5dx^3-8fx^3+e(20-8x^2))}{4-5x^2+x^4} + 8(d+4e+7f)\log(1-x) - (19d+32e+52f)\log(2-x) - 8(d-4e+7f)\log(1+x) + (19d-32e+52f)\log(2+x)\right)$$

input
```
Integrate[(d + e*x + f*x^2)/(4 - 5*x^2 + x^4)^2,x]
```

output
```
((12*(17*d*x + 20*f*x - 5*d*x^3 - 8*f*x^3 + e*(20 - 8*x^2)))/(4 - 5*x^2 +
x^4) + 8*(d + 4*e + 7*f)*Log[1 - x] - (19*d + 32*e + 52*f)*Log[2 - x] - 8*
(d - 4*e + 7*f)*Log[1 + x] + (19*d - 32*e + 52*f)*Log[2 + x])/864
```

3.27.3 Rubi [A] (verified)

Time = 0.37 (sec) , antiderivative size = 126, normalized size of antiderivative = 1.10, number of steps used = 10, number of rules used = 9, $\frac{\text{number of rules}}{\text{integrand size}} = 0.391$, Rules used = {2202, 27, 1432, 1084, 1492, 25, 1480, 220, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2}{\left(x^4-5x^2+4\right)^2}dx$$

$$\Big\downarrow \text{ 2202}$$

$$\int \frac{fx^2+d}{\left(x^4-5x^2+4\right)^2}dx+\int \frac{ex}{\left(x^4-5x^2+4\right)^2}dx$$

$$\Big\downarrow \text{ 27}$$

$$\int \frac{fx^2+d}{\left(x^4-5x^2+4\right)^2}dx+e\int \frac{x}{\left(x^4-5x^2+4\right)^2}dx$$

$$\Big\downarrow \text{ 1432}$$

$$\int \frac{fx^2+d}{\left(x^4-5x^2+4\right)^2}dx+\frac{1}{2}e\int \frac{1}{\left(x^4-5x^2+4\right)^2}dx^2$$

$$\Big\downarrow \text{ 1084}$$

$$\int \frac{fx^2+d}{\left(x^4-5x^2+4\right)^2}dx+\frac{1}{2}e\int \left(\frac{2}{27\left(4-x^2\right)}+\frac{1}{9\left(4-x^2\right)^2}-\frac{2}{27\left(1-x^2\right)}+\frac{1}{9\left(1-x^2\right)^2}\right)dx^2$$

$$\Big\downarrow \text{ 1492}$$

$$-\frac{1}{72}\int -\frac{-\left(\left(5d+8f\right)x^2\right)+d-20f}{x^4-5x^2+4}dx+\frac{1}{2}e\int \left(\frac{2}{27\left(4-x^2\right)}+\frac{1}{9\left(4-x^2\right)^2}-\frac{2}{27\left(1-x^2\right)}+\frac{1}{9\left(1-x^2\right)^2}\right)dx^2+\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{72\left(x^4-5x^2+4\right)}$$

$$\Big\downarrow \; 25$$

$$\frac{1}{72}\int \frac{-((5d+8f)x^2)+d-20f}{x^4-5x^2+4}dx+\frac{1}{2}e\int \left(\frac{2}{27(4-x^2)}+\frac{1}{9(4-x^2)^2}-\frac{2}{27(1-x^2)}+\frac{1}{9(1-x^2)^2}\right)dx^2+\frac{x(-(x^2(5d+8f))+17d+20f)}{72(x^4-5x^2+4)}$$

$$\Big\downarrow \; 1480$$

$$\frac{1}{72}\left(\frac{4}{3}(d+7f)\int \frac{1}{x^2-1}dx-\frac{1}{3}(19d+52f)\int \frac{1}{x^2-4}dx\right)+\frac{1}{2}e\int \left(\frac{2}{27(4-x^2)}+\frac{1}{9(4-x^2)^2}-\frac{2}{27(1-x^2)}+\frac{1}{9(1-x^2)^2}\right)dx^2+\frac{x(-(x^2(5d+8f))+17d+20f)}{72(x^4-5x^2+4)}$$

$$\Big\downarrow \; 220$$

$$\frac{1}{2}e\int \left(\frac{2}{27(4-x^2)}+\frac{1}{9(4-x^2)^2}-\frac{2}{27(1-x^2)}+\frac{1}{9(1-x^2)^2}\right)dx^2+\frac{1}{72}\left(\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(19d+52f)-\frac{4}{3}\text{arctanh}(x)(d+7f)\right)+\frac{x(-(x^2(5d+8f))+17d+20f)}{72(x^4-5x^2+4)}$$

$$\Big\downarrow \; 2009$$

$$\frac{1}{72}\left(\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(19d+52f)-\frac{4}{3}\text{arctanh}(x)(d+7f)\right)+\frac{x(-(x^2(5d+8f))+17d+20f)}{72(x^4-5x^2+4)}+\frac{1}{2}e\left(\frac{1}{9(1-x^2)}+\frac{1}{9(4-x^2)}+\frac{2}{27}\log(1-x^2)-\frac{2}{27}\log(4-x^2)\right)$$

input
```
Int[(d + e*x + f*x^2)/(4 - 5*x^2 + x^4)^2,x]
```

output
```
(x*(17*d + 20*f - (5*d + 8*f)*x^2))/(72*(4 - 5*x^2 + x^4)) + (((19*d + 52*
f)*ArcTanh[x/2])/6 - (4*(d + 7*f)*ArcTanh[x])/3)/72 + (e*(1/(9*(1 - x^2))
+ 1/(9*(4 - x^2)) + (2*Log[1 - x^2])/27 - (2*Log[4 - x^2])/27))/2
```

3.27. $\int \frac{d+ex+fx^2}{(4-5x^2+x^4)^2}\,dx$

3.27.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 220
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[b, 2])^(-
1))*ArcTanh[Rt[b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] &&
 (LtQ[a, 0] || GtQ[b, 0])
```

rule 1084
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> With[{q = Rt[b^2
- 4*a*c, 2]}, Simp[1/c^p   Int[ExpandIntegrand[(b/2 - q/2 + c*x)^p*(b/2 + q
/2 + c*x)^p, x], x], x] /;  !FractionalPowerFactorQ[q]] /; FreeQ[{a, b, c},
 x] && IntegerQ[p] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.27.4 Maple [A] (verified)

Time = 0.09 (sec) , antiderivative size = 105, normalized size of antiderivative = 0.91

method	result
norman	$\frac{\left(-\frac{5d}{72}-\frac{f}{9}\right)x^3+\left(\frac{17d}{72}+\frac{5f}{18}\right)x-\frac{e\,x^2}{9}+\frac{5e}{18}}{x^4-5x^2+4}+\left(-\frac{19d}{864}-\frac{e}{27}-\frac{13f}{216}\right)\ln\left(x-2\right)+\left(-\frac{d}{108}+\frac{e}{27}-\frac{7f}{108}\right)\ln\left(x+1\right)$
default	$-\frac{\frac{d}{144}-\frac{e}{72}+\frac{f}{36}}{x+2}+\left(\frac{19d}{864}-\frac{e}{27}+\frac{13f}{216}\right)\ln\left(x+2\right)+\left(-\frac{d}{108}+\frac{e}{27}-\frac{7f}{108}\right)\ln\left(x+1\right)-\frac{\frac{d}{36}-\frac{e}{36}+\frac{f}{36}}{x+1}-$
risch	$\frac{\left(-\frac{5d}{72}-\frac{f}{9}\right)x^3+\left(\frac{17d}{72}+\frac{5f}{18}\right)x-\frac{e\,x^2}{9}+\frac{5e}{18}}{x^4-5x^2+4}+\frac{19\ln(x+2)d}{864}-\frac{\ln(x+2)e}{27}+\frac{13\ln(x+2)f}{216}-\frac{\ln(x+1)d}{108}+\frac{\ln(x+1)e}{27}-$
parallelrisch	$-\frac{-240e+96f\,x^3-204dx+76\ln(x-2)d+128\ln(x-2)e-32\ln(x-1)d-128\ln(x-1)e+32\ln(x-2)x^4e-208\ln(x+2)f+224\ln(x}{}$

input
```
int((f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x,method=_RETURNVERBOSE)
```

output
```
((-5/72*d-1/9*f)*x^3+(17/72*d+5/18*f)*x-1/9*e*x^2+5/18*e)/(x^4-5*x^2+4)+(-
19/864*d-1/27*e-13/216*f)*ln(x-2)+(-1/108*d+1/27*e-7/108*f)*ln(x+1)+(1/108
*d+1/27*e+7/108*f)*ln(x-1)+(19/864*d-1/27*e+13/216*f)*ln(x+2)
```

3.27.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 217 vs. $2(100) = 200$.

Time = 0.31 (sec) , antiderivative size = 217, normalized size of antiderivative = 1.89

$$\int \frac{d+ex+fx^2}{\left(4-5x^2+x^4\right)^2}\,dx = -\frac{12\,(5\,d+8\,f)x^3+96\,ex^2-12\,(17\,d+20\,f)x-((19\,d-32\,e+52\,f)x^4-5\,(19\,d-32\,e+52\,f)x^2+76}{}$$

input
```
integrate((f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="fricas")
```

output
```
-1/864*(12*(5*d + 8*f)*x^3 + 96*e*x^2 - 12*(17*d + 20*f)*x - ((19*d - 32*e
 + 52*f)*x^4 - 5*(19*d - 32*e + 52*f)*x^2 + 76*d - 128*e + 208*f)*log(x +
2) + 8*((d - 4*e + 7*f)*x^4 - 5*(d - 4*e + 7*f)*x^2 + 4*d - 16*e + 28*f)*l
og(x + 1) - 8*((d + 4*e + 7*f)*x^4 - 5*(d + 4*e + 7*f)*x^2 + 4*d + 16*e +
28*f)*log(x - 1) + ((19*d + 32*e + 52*f)*x^4 - 5*(19*d + 32*e + 52*f)*x^2
+ 76*d + 128*e + 208*f)*log(x - 2) - 240*e)/(x^4 - 5*x^2 + 4)
```

3.27.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2}{(4-5x^2+x^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((f*x**2+e*x+d)/(x**4-5*x**2+4)**2,x)
```

output
```
Timed out
```

3.27.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 106, normalized size of antiderivative = 0.92

$$\begin{aligned}\int \frac{d+ex+fx^2}{(4-5x^2+x^4)^2}\,dx &= \frac{1}{864}\,(19\,d-32\,e+52\,f)\log\left(x+2\right)\\ &\quad-\frac{1}{108}\,(d-4\,e+7\,f)\log\left(x+1\right)+\frac{1}{108}\,(d+4\,e+7\,f)\log\left(x-1\right)\\ &\quad-\frac{1}{864}\,(19\,d+32\,e+52\,f)\log\left(x-2\right)\\ &\quad-\frac{(5\,d+8\,f)x^3+8\,ex^2-(17\,d+20\,f)x-20\,e}{72\,(x^4-5\,x^2+4)}\end{aligned}$$

input
```
integrate((f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="maxima")
```

output
```
1/864*(19*d - 32*e + 52*f)*log(x + 2) - 1/108*(d - 4*e + 7*f)*log(x + 1) +
 1/108*(d + 4*e + 7*f)*log(x - 1) - 1/864*(19*d + 32*e + 52*f)*log(x - 2)
- 1/72*((5*d + 8*f)*x^3 + 8*e*x^2 - (17*d + 20*f)*x - 20*e)/(x^4 - 5*x^2 +
 4)
```

3.27.8 Giac [A] (verification not implemented)

Time = 0.32 (sec) , antiderivative size = 109, normalized size of antiderivative = 0.95

$$\int \frac{d+ex+fx^2}{(4-5x^2+x^4)^2}\,dx = \frac{1}{864}\,(19\,d-32\,e+52\,f)\log\left(|x+2|\right) - \frac{1}{108}\,(d-4\,e+7\,f)\log\left(|x+1|\right) + \frac{1}{108}\,(d+4\,e+7\,f)\log\left(|x-1|\right) - \frac{1}{864}\,(19\,d+32\,e+52\,f)\log\left(|x-2|\right) - \frac{5\,dx^3+8\,fx^3+8\,ex^2-17\,dx-20\,fx-20\,e}{72\,(x^4-5\,x^2+4)}$$

input
```
integrate((f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="giac")
```

output
```
1/864*(19*d - 32*e + 52*f)*log(abs(x + 2)) - 1/108*(d - 4*e + 7*f)*log(abs
(x + 1)) + 1/108*(d + 4*e + 7*f)*log(abs(x - 1)) - 1/864*(19*d + 32*e + 52
*f)*log(abs(x - 2)) - 1/72*(5*d*x^3 + 8*f*x^3 + 8*e*x^2 - 17*d*x - 20*f*x
- 20*e)/(x^4 - 5*x^2 + 4)
```

3.27.9 Mupad [B] (verification not implemented)

Time = 0.06 (sec) , antiderivative size = 107, normalized size of antiderivative = 0.93

$$\int \frac{d+ex+fx^2}{(4-5x^2+x^4)^2}\,dx = \ln(x-1)\left(\frac{d}{108}+\frac{e}{27}+\frac{7\,f}{108}\right) - \ln(x+1)\left(\frac{d}{108}-\frac{e}{27}+\frac{7\,f}{108}\right) - \ln(x-2)\left(\frac{19\,d}{864}+\frac{e}{27}+\frac{13\,f}{216}\right) + \ln(x+2)\left(\frac{19\,d}{864}-\frac{e}{27}+\frac{13\,f}{216}\right) + \frac{\left(-\frac{5\,d}{72}-\frac{f}{9}\right)\,x^3-\frac{e\,x^2}{9}+\left(\frac{17\,d}{72}+\frac{5\,f}{18}\right)\,x+\frac{5\,e}{18}}{x^4-5\,x^2+4}$$

input
```
int((d + e*x + f*x^2)/(x^4 - 5*x^2 + 4)^2,x)
```

output
```
log(x - 1)*(d/108 + e/27 + (7*f)/108) - log(x + 1)*(d/108 - e/27 + (7*f)/1
08) - log(x - 2)*((19*d)/864 + e/27 + (13*f)/216) + log(x + 2)*((19*d)/864
 - e/27 + (13*f)/216) + ((5*e)/18 - x^3*((5*d)/72 + f/9) - (e*x^2)/9 + x*(
(17*d)/72 + (5*f)/18))/(x^4 - 5*x^2 + 4)
```

3.28 $\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^2}\,dx$

3.28.1 Optimal result

Integrand size = 28, antiderivative size = 138

$$\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^2}\,dx = \frac{x(17d+20f-(5d+8f)x^2)}{72(4-5x^2+x^4)}+\frac{5e+8g-(2e+5g)x^2}{18(4-5x^2+x^4)} + \frac{1}{432}(19d+52f)\operatorname{arctanh}\left(\frac{x}{2}\right)-\frac{1}{54}(d+7f)\operatorname{arctanh}(x) + \frac{1}{54}(2e+5g)\log\left(1-x^2\right)-\frac{1}{54}(2e+5g)\log\left(4-x^2\right)$$

output
```
1/72*x*(17*d+20*f-(5*d+8*f)*x^2)/(x^4-5*x^2+4)+1/18*(5*e+8*g-(2*e+5*g)*x^2
)/(x^4-5*x^2+4)+1/432*(19*d+52*f)*arctanh(1/2*x)-1/54*(d+7*f)*arctanh(x)+1
/54*(2*e+5*g)*ln(-x^2+1)-1/54*(2*e+5*g)*ln(-x^2+4)
```

3.28.2 Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 134, normalized size of antiderivative = 0.97

$$\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^2}\,dx = \frac{1}{864}\left(\frac{12(17dx+20fx-5dx^3-8fx^3+e(20-8x^2)-4g(-8+5x^2))}{4-5x^2+x^4} + 8(d+4e+7f+10g)\log(1-x)-(19d+32e+52f+80g)\log(2-x) - 8(d-4e+7f-10g)\log(1+x)+(19d-32e+52f-80g)\log(2+x)\right)$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3)/(4 - 5*x^2 + x^4)^2,x]
```

output
```
((12*(17*d*x + 20*f*x - 5*d*x^3 - 8*f*x^3 + e*(20 - 8*x^2) - 4*g*(-8 + 5*x
^2)))/(4 - 5*x^2 + x^4) + 8*(d + 4*e + 7*f + 10*g)*Log[1 - x] - (19*d + 32
*e + 52*f + 80*g)*Log[2 - x] - 8*(d - 4*e + 7*f - 10*g)*Log[1 + x] + (19*d
 - 32*e + 52*f - 80*g)*Log[2 + x])/864
```

3.28.3 Rubi [A] (verified)

Time = 0.42 (sec) , antiderivative size = 147, normalized size of antiderivative = 1.07, number of steps used = 9, number of rules used = 8, $\frac{\text{number of rules}}{\text{integrand size}} = 0.286$, Rules used = {2202, 1492, 25, 1480, 220, 1576, 1141, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3}{\left(x^4-5x^2+4\right)^2}dx$$

$$\Big\downarrow \text{ 2202}$$

$$\int \frac{fx^2+d}{\left(x^4-5x^2+4\right)^2}dx+\int \frac{x\left(gx^2+e\right)}{\left(x^4-5x^2+4\right)^2}dx$$

$$\Big\downarrow \text{ 1492}$$

$$-\frac{1}{72}\int -\frac{-\left((5d+8f)x^2\right)+d-20f}{x^4-5x^2+4}dx+\int \frac{x\left(gx^2+e\right)}{\left(x^4-5x^2+4\right)^2}dx+\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{72\left(x^4-5x^2+4\right)}$$

$$\Big\downarrow \text{ 25}$$

$$\frac{1}{72}\int \frac{-\left((5d+8f)x^2\right)+d-20f}{x^4-5x^2+4}dx+\int \frac{x\left(gx^2+e\right)}{\left(x^4-5x^2+4\right)^2}dx+\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{72\left(x^4-5x^2+4\right)}$$

$$\Big\downarrow \text{ 1480}$$

$$\frac{1}{72}\left(\frac{4}{3}(d+7f)\int \frac{1}{x^2-1}dx-\frac{1}{3}(19d+52f)\int \frac{1}{x^2-4}dx\right)+\int \frac{x\left(gx^2+e\right)}{\left(x^4-5x^2+4\right)^2}dx+\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{72\left(x^4-5x^2+4\right)}$$

$$\Big\downarrow \text{ 220}$$

$$\int \frac{x(gx^2+e)}{(x^4-5x^2+4)^2}dx+\frac{1}{72}\left(\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(19d+52f)-\frac{4}{3}\text{arctanh}(x)(d+7f)\right)+\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{72\left(x^4-5x^2+4\right)}$$

$\downarrow$ 1576

$$\frac{1}{2}\int \frac{gx^2+e}{(x^4-5x^2+4)^2}dx^2+\frac{1}{72}\left(\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(19d+52f)-\frac{4}{3}\text{arctanh}(x)(d+7f)\right)+\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{72\left(x^4-5x^2+4\right)}$$

$\downarrow$ 1141

$$\frac{1}{2}\int \left(\frac{e+g}{9(1-x^2)^2}-\frac{2e+5g}{27(1-x^2)}+\frac{2e+5g}{27(4-x^2)}+\frac{e+4g}{9(4-x^2)^2}\right)dx^2+\frac{1}{72}\left(\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(19d+52f)-\frac{4}{3}\text{arctanh}(x)(d+7f)\right)+\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{72\left(x^4-5x^2+4\right)}$$

$\downarrow$ 2009

$$\frac{1}{72}\left(\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(19d+52f)-\frac{4}{3}\text{arctanh}(x)(d+7f)\right)+\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{72\left(x^4-5x^2+4\right)}+\frac{1}{2}\left(\frac{e+g}{9(1-x^2)}+\frac{e+4g}{9(4-x^2)}+\frac{1}{27}(2e+5g)\log\left(1-x^2\right)-\frac{1}{27}(2e+5g)\log\left(4-x^2\right)\right)$$

input
```
Int[(d + e*x + f*x^2 + g*x^3)/(4 - 5*x^2 + x^4)^2,x]
```

output
```
(x*(17*d + 20*f - (5*d + 8*f)*x^2))/(72*(4 - 5*x^2 + x^4)) + (((19*d + 52*
f)*ArcTanh[x/2])/6 - (4*(d + 7*f)*ArcTanh[x])/3)/72 + ((e + g)/(9*(1 - x^2
)) + (e + 4*g)/(9*(4 - x^2)) + ((2*e + 5*g)*Log[1 - x^2])/27 - ((2*e + 5*g
)*Log[4 - x^2])/27)/2
```

3.28.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 220
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[b, 2])^(-
1))*ArcTanh[Rt[b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] &&
 (LtQ[a, 0] || GtQ[b, 0])
```

rule 1141
```
Int[((d_.) + (e_.)*(x_))^(m_.)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_
Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[1/c^p   Int[ExpandIntegrand[
(d + e*x)^m*(b/2 - q/2 + c*x)^p*(b/2 + q/2 + c*x)^p, x], x], x] /; EqQ[p, -
1] ||  !FractionalPowerFactorQ[q]] /; FreeQ[{a, b, c, d, e}, x] && ILtQ[p,
0] && IntegerQ[m] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.28. $\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^2}\,dx$

3.28.4 Maple [A] (verified)

Time = 0.10 (sec) , antiderivative size = 125, normalized size of antiderivative = 0.91

method	result
norman	$\frac{\left(-\frac{5d}{72}-\frac{f}{9}\right)x^3+\left(\frac{17d}{72}+\frac{5f}{18}\right)x+\left(-\frac{e}{9}-\frac{5g}{18}\right)x^2+\frac{5e}{18}+\frac{4g}{9}}{x^4-5x^2+4}+\left(-\frac{19d}{864}-\frac{e}{27}-\frac{13f}{216}-\frac{5g}{54}\right)\ln\left(x-2\right)+\left(-\frac{d}{108}+\frac{e}{27}-\right.$
default	$-\frac{\frac{d}{144}-\frac{e}{72}+\frac{f}{36}-\frac{g}{18}}{x+2}+\left(\frac{19d}{864}-\frac{e}{27}+\frac{13f}{216}-\frac{5g}{54}\right)\ln\left(x+2\right)+\left(-\frac{d}{108}+\frac{e}{27}-\frac{7f}{108}+\frac{5g}{54}\right)\ln\left(x+1\right)-\frac{d}{36}$
risch	$\frac{\left(-\frac{5d}{72}-\frac{f}{9}\right)x^3+\left(\frac{17d}{72}+\frac{5f}{18}\right)x+\left(-\frac{e}{9}-\frac{5g}{18}\right)x^2+\frac{5e}{18}+\frac{4g}{9}}{x^4-5x^2+4}+\frac{19\ln(x+2)d}{864}-\frac{\ln(x+2)e}{27}+\frac{13\ln(x+2)f}{216}-\frac{5\ln(x+2)g}{54}+\frac{\ln(1-}{10}$
parallelrisch	$-\frac{-384g-240e+96f\,x^3+240g\,x^2-204dx+76\ln(x-2)d+128\ln(x-2)e-32\ln(x-1)d-128\ln(x-1)e+32\ln(x-2)x^4e-208\ln(x}{}$

input
```
int((g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x,method=_RETURNVERBOSE)
```

output
```
((-5/72*d-1/9*f)*x^3+(17/72*d+5/18*f)*x+(-1/9*e-5/18*g)*x^2+5/18*e+4/9*g)/
(x^4-5*x^2+4)+(-19/864*d-1/27*e-13/216*f-5/54*g)*ln(x-2)+(-1/108*d+1/27*e-
7/108*f+5/54*g)*ln(x+1)+(1/108*d+1/27*e+7/108*f+5/54*g)*ln(x-1)+(19/864*d-
1/27*e+13/216*f-5/54*g)*ln(x+2)
```

3.28.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 262 vs. 2(122) = 244.

Time = 0.49 (sec) , antiderivative size = 262, normalized size of antiderivative = 1.90

$$\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^2}\,dx =$$
$$-\frac{12\,(5\,d+8\,f)x^3+48\,(2\,e+5\,g)x^2-12\,(17\,d+20\,f)x-((19\,d-32\,e+52\,f-80\,g)x^4-5\,(19\,d-32\,e}{}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="fricas")
```

output
```
-1/864*(12*(5*d + 8*f)*x^3 + 48*(2*e + 5*g)*x^2 - 12*(17*d + 20*f)*x - ((1
9*d - 32*e + 52*f - 80*g)*x^4 - 5*(19*d - 32*e + 52*f - 80*g)*x^2 + 76*d -
 128*e + 208*f - 320*g)*log(x + 2) + 8*((d - 4*e + 7*f - 10*g)*x^4 - 5*(d
- 4*e + 7*f - 10*g)*x^2 + 4*d - 16*e + 28*f - 40*g)*log(x + 1) - 8*((d + 4
*e + 7*f + 10*g)*x^4 - 5*(d + 4*e + 7*f + 10*g)*x^2 + 4*d + 16*e + 28*f +
40*g)*log(x - 1) + ((19*d + 32*e + 52*f + 80*g)*x^4 - 5*(19*d + 32*e + 52*
f + 80*g)*x^2 + 76*d + 128*e + 208*f + 320*g)*log(x - 2) - 240*e - 384*g)/
(x^4 - 5*x^2 + 4)
```

3.28.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4)**2,x)
```

output
```
Timed out
```

3.28.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 127, normalized size of antiderivative = 0.92

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^2}\,dx = {} & \frac{1}{864}\,(19\,d-32\,e+52\,f-80\,g)\log(x+2) \\ & -\frac{1}{108}\,(d-4\,e+7\,f-10\,g)\log(x+1) \\ & +\frac{1}{108}\,(d+4\,e+7\,f+10\,g)\log(x-1) \\ & -\frac{1}{864}\,(19\,d+32\,e+52\,f+80\,g)\log(x-2) \\ & -\frac{(5\,d+8\,f)x^3+4\,(2\,e+5\,g)x^2-(17\,d+20\,f)x-20\,e-32\,g}{72\,(x^4-5\,x^2+4)}\end{aligned}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="maxima")
```

output
```
1/864*(19*d - 32*e + 52*f - 80*g)*log(x + 2) - 1/108*(d - 4*e + 7*f - 10*g
)*log(x + 1) + 1/108*(d + 4*e + 7*f + 10*g)*log(x - 1) - 1/864*(19*d + 32*
e + 52*f + 80*g)*log(x - 2) - 1/72*((5*d + 8*f)*x^3 + 4*(2*e + 5*g)*x^2 -
(17*d + 20*f)*x - 20*e - 32*g)/(x^4 - 5*x^2 + 4)
```

3.28.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 130, normalized size of antiderivative = 0.94

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^2}\,dx &= \frac{1}{864}\,(19\,d-32\,e+52\,f-80\,g)\log\left(|x+2|\right)\\ &\quad-\frac{1}{108}\,(d-4\,e+7\,f-10\,g)\log\left(|x+1|\right)\\ &\quad+\frac{1}{108}\,(d+4\,e+7\,f+10\,g)\log\left(|x-1|\right)\\ &\quad-\frac{1}{864}\,(19\,d+32\,e+52\,f+80\,g)\log\left(|x-2|\right)\\ &\quad-\frac{5\,dx^3+8\,fx^3+8\,ex^2+20\,gx^2-17\,dx-20\,fx-20\,e-32\,g}{72\,(x^4-5\,x^2+4)}\end{aligned}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="giac")
```

output
```
1/864*(19*d - 32*e + 52*f - 80*g)*log(abs(x + 2)) - 1/108*(d - 4*e + 7*f -
 10*g)*log(abs(x + 1)) + 1/108*(d + 4*e + 7*f + 10*g)*log(abs(x - 1)) - 1/
864*(19*d + 32*e + 52*f + 80*g)*log(abs(x - 2)) - 1/72*(5*d*x^3 + 8*f*x^3
+ 8*e*x^2 + 20*g*x^2 - 17*d*x - 20*f*x - 20*e - 32*g)/(x^4 - 5*x^2 + 4)
```

3.28.9 Mupad [B] (verification not implemented)

Time = 7.88 (sec) , antiderivative size = 128, normalized size of antiderivative = 0.93

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^2}\,dx &= \ln\left(x-1\right)\left(\frac{d}{108}+\frac{e}{27}+\frac{7\,f}{108}+\frac{5\,g}{54}\right)\\ &\quad-\ln\left(x+1\right)\left(\frac{d}{108}-\frac{e}{27}+\frac{7\,f}{108}-\frac{5\,g}{54}\right)\\ &\quad-\ln\left(x-2\right)\left(\frac{19\,d}{864}+\frac{e}{27}+\frac{13\,f}{216}+\frac{5\,g}{54}\right)\\ &\quad+\ln\left(x+2\right)\left(\frac{19\,d}{864}-\frac{e}{27}+\frac{13\,f}{216}-\frac{5\,g}{54}\right)\\ &\quad+\frac{\left(-\frac{5\,d}{72}-\frac{f}{9}\right)\,x^3+\left(-\frac{e}{9}-\frac{5\,g}{18}\right)\,x^2+\left(\frac{17\,d}{72}+\frac{5\,f}{18}\right)\,x+\frac{5\,e}{18}+\frac{4\,g}{9}}{x^4-5\,x^2+4}\end{aligned}$$

input
```
int((d + e*x + f*x^2 + g*x^3)/(x^4 - 5*x^2 + 4)^2,x)
```

output
```
log(x - 1)*(d/108 + e/27 + (7*f)/108 + (5*g)/54) - log(x + 1)*(d/108 - e/2
7 + (7*f)/108 - (5*g)/54) - log(x - 2)*((19*d)/864 + e/27 + (13*f)/216 + (
5*g)/54) + log(x + 2)*((19*d)/864 - e/27 + (13*f)/216 - (5*g)/54) + ((5*e)
/18 + (4*g)/9 - x^3*((5*d)/72 + f/9) - x^2*(e/9 + (5*g)/18) + x*((17*d)/72
 + (5*f)/18))/(x^4 - 5*x^2 + 4)
```

3.28. $\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^2} dx$

3.29 $\int \frac{d+ex+fx^2+gx^3+hx^4}{(4-5x^2+x^4)^2}\,dx$

3.29.1 Optimal result

Integrand size = 33, antiderivative size = 150

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3+hx^4}{(4-5x^2+x^4)^2}\,dx &= \frac{5e+8g-(2e+5g)x^2}{18(4-5x^2+x^4)} \\ &+ \frac{x(17d+20f+32h-(5d+8f+20h)x^2)}{72(4-5x^2+x^4)} \\ &+ \frac{1}{432}(19d+52f+112h)\operatorname{arctanh}\left(\frac{x}{2}\right) \\ &- \frac{1}{54}(d+7f+13h)\operatorname{arctanh}(x) \\ &+ \frac{1}{54}(2e+5g)\log\left(1-x^2\right)-\frac{1}{54}(2e+5g)\log\left(4-x^2\right)\end{aligned}$$

output
```
1/18*(5*e+8*g-(2*e+5*g)*x^2)/(x^4-5*x^2+4)+1/72*x*(17*d+20*f+32*h-(5*d+8*f
+20*h)*x^2)/(x^4-5*x^2+4)+1/432*(19*d+52*f+112*h)*arctanh(1/2*x)-1/54*(d+7
*f+13*h)*arctanh(x)+1/54*(2*e+5*g)*ln(-x^2+1)-1/54*(2*e+5*g)*ln(-x^2+4)
```

3.29.2 Mathematica [A] (verified)

Time = 0.05 (sec) , antiderivative size = 159, normalized size of antiderivative = 1.06

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(4-5x^2+x^4)^2}\,dx$$
$$=\frac{1}{864}\left(-\frac{12(4e(-5+2x^2)+4g(-8+5x^2)+x(4f(-5+2x^2)+d(-17+5x^2)+4h(-8+5x^2)))}{4-5x^2+x^4}+8(d+4e+7f+10g+13h)\log(1-x)-(19d+32e+52f+80g+112h)\log(2-x)-8(d-4e+7f-10g+13h)\log(1+x)+(19d-32e+52f-80g+112h)\log(2+x)\right)$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4)/(4 - 5*x^2 + x^4)^2,x]
```

output
```
((-12*(4*e*(-5 + 2*x^2) + 4*g*(-8 + 5*x^2) + x*(4*f*(-5 + 2*x^2) + d*(-17
+ 5*x^2) + 4*h*(-8 + 5*x^2))))/(4 - 5*x^2 + x^4) + 8*(d + 4*e + 7*f + 10*g
 + 13*h)*Log[1 - x] - (19*d + 32*e + 52*f + 80*g + 112*h)*Log[2 - x] - 8*(
d - 4*e + 7*f - 10*g + 13*h)*Log[1 + x] + (19*d - 32*e + 52*f - 80*g + 112
*h)*Log[2 + x])/864
```

3.29.3 Rubi [A] (verified)

Time = 0.49 (sec) , antiderivative size = 159, normalized size of antiderivative = 1.06, number of steps used = 9, number of rules used = 8, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.242, Rules used = {2202, 1576, 1141, 2009, 2206, 25, 1480, 220}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(x^4-5x^2+4)^2}dx$$
$$\Big\downarrow \ 2202$$
$$\int \frac{hx^4+fx^2+d}{(x^4-5x^2+4)^2}dx+\int \frac{x(gx^2+e)}{(x^4-5x^2+4)^2}dx$$
$$\Big\downarrow \ 1576$$
$$\int \frac{hx^4+fx^2+d}{(x^4-5x^2+4)^2}dx+\frac{1}{2}\int \frac{gx^2+e}{(x^4-5x^2+4)^2}dx^2$$
$$\Big\downarrow \ 1141$$

$$\int \frac{hx^4+fx^2+d}{(x^4-5x^2+4)^2}dx+\frac{1}{2}\int \left(\frac{e+g}{9(1-x^2)^2}-\frac{2e+5g}{27(1-x^2)}+\frac{2e+5g}{27(4-x^2)}+\frac{e+4g}{9(4-x^2)^2}\right)dx^2$$

$$\Big\downarrow \text{2009}$$

$$\int \frac{hx^4+fx^2+d}{(x^4-5x^2+4)^2}dx+\frac{1}{2}\left(\frac{e+g}{9(1-x^2)}+\frac{e+4g}{9(4-x^2)}+\frac{1}{27}(2e+5g)\log(1-x^2)-\frac{1}{27}(2e+5g)\log(4-x^2)\right)$$

$$\Big\downarrow \text{2206}$$

$$-\frac{1}{72}\int -\frac{-((5d+8f+20h)x^2)+d-20f-32h}{x^4-5x^2+4}dx+\frac{x(-(x^2(5d+8f+20h))+17d+20f+32h)}{72(x^4-5x^2+4)}+\frac{1}{2}\left(\frac{e+g}{9(1-x^2)}+\frac{e+4g}{9(4-x^2)}+\frac{1}{27}(2e+5g)\log(1-x^2)-\frac{1}{27}(2e+5g)\log(4-x^2)\right)$$

$$\Big\downarrow \text{25}$$

$$\frac{1}{72}\int \frac{-((5d+8f+20h)x^2)+d-20f-32h}{x^4-5x^2+4}dx+\frac{x(-(x^2(5d+8f+20h))+17d+20f+32h)}{72(x^4-5x^2+4)}+\frac{1}{2}\left(\frac{e+g}{9(1-x^2)}+\frac{e+4g}{9(4-x^2)}+\frac{1}{27}(2e+5g)\log(1-x^2)-\frac{1}{27}(2e+5g)\log(4-x^2)\right)$$

$$\Big\downarrow \text{1480}$$

$$\frac{1}{72}\left(\frac{4}{3}(d+7f+13h)\int \frac{1}{x^2-1}dx-\frac{1}{3}(19d+52f+112h)\int \frac{1}{x^2-4}dx\right)+\frac{x(-(x^2(5d+8f+20h))+17d+20f+32h)}{72(x^4-5x^2+4)}+\frac{1}{2}\left(\frac{e+g}{9(1-x^2)}+\frac{e+4g}{9(4-x^2)}+\frac{1}{27}(2e+5g)\log(1-x^2)-\frac{1}{27}(2e+5g)\log(4-x^2)\right)$$

$$\Big\downarrow \text{220}$$

$$\frac{1}{72}\left(\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(19d+52f+112h)-\frac{4}{3}\text{arctanh}(x)(d+7f+13h)\right)+\frac{x(-(x^2(5d+8f+20h))+17d+20f+32h)}{72(x^4-5x^2+4)}+\frac{1}{2}\left(\frac{e+g}{9(1-x^2)}+\frac{e+4g}{9(4-x^2)}+\frac{1}{27}(2e+5g)\log(1-x^2)-\frac{1}{27}(2e+5g)\log(4-x^2)\right)$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4)/(4 - 5*x^2 + x^4)^2,x]
```

3.29. $\int \frac{d+ex+fx^2+gx^3+hx^4}{(4-5x^2+x^4)^2}\,dx$

output
```
(x*(17*d + 20*f + 32*h - (5*d + 8*f + 20*h)*x^2))/(72*(4 - 5*x^2 + x^4)) +
 (((19*d + 52*f + 112*h)*ArcTanh[x/2])/6 - (4*(d + 7*f + 13*h)*ArcTanh[x])
/3)/72 + ((e + g)/(9*(1 - x^2)) + (e + 4*g)/(9*(4 - x^2)) + ((2*e + 5*g)*L
og[1 - x^2])/27 - ((2*e + 5*g)*Log[4 - x^2])/27)/2
```

3.29.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 220
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[b, 2])^(-
1))*ArcTanh[Rt[b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] &&
 (LtQ[a, 0] || GtQ[b, 0])
```

rule 1141
```
Int[((d_.) + (e_.)*(x_))^(m_.)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_
Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[1/c^p   Int[ExpandIntegrand[
(d + e*x)^m*(b/2 - q/2 + c*x)^p*(b/2 + q/2 + c*x)^p, x], x], x] /; EqQ[p, -
1] ||  !FractionalPowerFactorQ[q]] /; FreeQ[{a, b, c, d, e}, x] && ILtQ[p,
0] && IntegerQ[m] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2206
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> With[{d =
 Coeff[PolynomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 0], e = Coeff[Poly
nomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 2]}, Simp[x*(a + b*x^2 + c*x^
4)^(p + 1)*((a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)/(2*a*(p + 1)*(b
^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(a + b*x^2 + c
*x^4)^(p + 1)*ExpandToSum[2*a*(p + 1)*(b^2 - 4*a*c)*PolynomialQuotient[Px,
a + b*x^2 + c*x^4, x] + b^2*d*(2*p + 3) - 2*a*c*d*(4*p + 5) - a*b*e + c*(4*
p + 7)*(b*d - 2*a*e)*x^2, x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x
^2] && Expon[Px, x^2] > 1 && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1]
```

3.29.4 Maple [A] (verified)

Time = 0.10 (sec) , antiderivative size = 143, normalized size of antiderivative = 0.95

method	result
norman	$\frac{\left(-\frac{5d}{72}-\frac{f}{9}-\frac{5h}{18}\right)x^3+\left(\frac{17d}{72}+\frac{5f}{18}+\frac{4h}{9}\right)x+\left(-\frac{e}{9}-\frac{5g}{18}\right)x^2+\frac{5e}{18}+\frac{4g}{9}}{x^4-5x^2+4}+\left(-\frac{19d}{864}-\frac{e}{27}-\frac{13f}{216}-\frac{5g}{54}-\frac{7h}{54}\right)\ln\left(x-2\right)+($
default	$-\frac{\frac{d}{144}-\frac{e}{72}+\frac{f}{36}-\frac{g}{18}+\frac{h}{9}}{x+2}+\left(\frac{19d}{864}-\frac{e}{27}+\frac{13f}{216}-\frac{5g}{54}+\frac{7h}{54}\right)\ln\left(x+2\right)+\left(-\frac{d}{108}+\frac{e}{27}-\frac{7f}{108}+\frac{5g}{54}-\frac{13h}{108}\right)\ln$
risch	$\frac{\left(-\frac{5d}{72}-\frac{f}{9}-\frac{5h}{18}\right)x^3+\left(\frac{17d}{72}+\frac{5f}{18}+\frac{4h}{9}\right)x+\left(-\frac{e}{9}-\frac{5g}{18}\right)x^2+\frac{5e}{18}+\frac{4g}{9}}{x^4-5x^2+4}-\frac{\ln(x+1)d}{108}+\frac{\ln(x+1)e}{27}-\frac{7\ln(x+1)f}{108}+\frac{5\ln(x+1)g}{54}-$
parallelrisch	$-\frac{-384g-240e+96f\,x^3+240g\,x^2+240h\,x^3-204dx+76\ln(x-2)d+128\ln(x-2)e-32\ln(x-1)d-128\ln(x-1)e+32\ln(x-2)x^4e}{}$

input
```
int((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x,method=_RETURNVERBOSE)
```

output
```
((-5/72*d-1/9*f-5/18*h)*x^3+(17/72*d+5/18*f+4/9*h)*x+(-1/9*e-5/18*g)*x^2+5
/18*e+4/9*g)/(x^4-5*x^2+4)+(-19/864*d-1/27*e-13/216*f-5/54*g-7/54*h)*ln(x-
2)+(-1/108*d+1/27*e-7/108*f+5/54*g-13/108*h)*ln(x+1)+(1/108*d+1/27*e+7/108
*f+5/54*g+13/108*h)*ln(x-1)+(19/864*d-1/27*e+13/216*f-5/54*g+7/54*h)*ln(x+
2)
```

3.29.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 304 vs. $2(134) = 268$.

Time = 1.40 (sec) , antiderivative size = 304, normalized size of antiderivative = 2.03

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(4-5x^2+x^4)^2}\,dx = -\frac{12\,(5\,d+8\,f+20\,h)x^3+48\,(2\,e+5\,g)x^2-12\,(17\,d+20\,f+32\,h)x-((19\,d-32\,e+52\,f-80\,g+11$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="fricas")
```

output
```
-1/864*(12*(5*d + 8*f + 20*h)*x^3 + 48*(2*e + 5*g)*x^2 - 12*(17*d + 20*f +
 32*h)*x - ((19*d - 32*e + 52*f - 80*g + 112*h)*x^4 - 5*(19*d - 32*e + 52*
f - 80*g + 112*h)*x^2 + 76*d - 128*e + 208*f - 320*g + 448*h)*log(x + 2) +
 8*((d - 4*e + 7*f - 10*g + 13*h)*x^4 - 5*(d - 4*e + 7*f - 10*g + 13*h)*x^
2 + 4*d - 16*e + 28*f - 40*g + 52*h)*log(x + 1) - 8*((d + 4*e + 7*f + 10*g
 + 13*h)*x^4 - 5*(d + 4*e + 7*f + 10*g + 13*h)*x^2 + 4*d + 16*e + 28*f + 4
0*g + 52*h)*log(x - 1) + ((19*d + 32*e + 52*f + 80*g + 112*h)*x^4 - 5*(19*
d + 32*e + 52*f + 80*g + 112*h)*x^2 + 76*d + 128*e + 208*f + 320*g + 448*h
)*log(x - 2) - 240*e - 384*g)/(x^4 - 5*x^2 + 4)
```

3.29.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(4-5x^2+x^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((h*x**4+g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4)**2,x)
```

output
```
Timed out
```

3.29.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 145, normalized size of antiderivative = 0.97

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(4-5x^2+x^4)^2}\,dx$$
$$= \frac{1}{864}\,(19\,d-32\,e+52\,f-80\,g+112\,h)\log{(x+2)} - \frac{1}{108}\,(d-4\,e+7\,f-10\,g+13\,h)\log{(x+1)} + \frac{1}{108}\,(d+4\,e+7\,f+10\,g+13\,h)\log{(x-1)} - \frac{1}{864}\,(19\,d+32\,e+52\,f+80\,g+112\,h)\log{(x-2)} - \frac{(5\,d+8\,f+20\,h)x^3+4\,(2\,e+5\,g)x^2-(17\,d+20\,f+32\,h)x-20\,e-32\,g}{72\,(x^4-5\,x^2+4)}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="maxima")
```

output
```
1/864*(19*d - 32*e + 52*f - 80*g + 112*h)*log(x + 2) - 1/108*(d - 4*e + 7*
f - 10*g + 13*h)*log(x + 1) + 1/108*(d + 4*e + 7*f + 10*g + 13*h)*log(x -
1) - 1/864*(19*d + 32*e + 52*f + 80*g + 112*h)*log(x - 2) - 1/72*((5*d + 8
*f + 20*h)*x^3 + 4*(2*e + 5*g)*x^2 - (17*d + 20*f + 32*h)*x - 20*e - 32*g)
/(x^4 - 5*x^2 + 4)
```

3.29.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 152, normalized size of antiderivative = 1.01

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(4-5x^2+x^4)^2}\,dx$$
$$= \frac{1}{864}\,(19\,d-32\,e+52\,f-80\,g+112\,h)\log{(|x+2|)} - \frac{1}{108}\,(d-4\,e+7\,f-10\,g+13\,h)\log{(|x+1|)} + \frac{1}{108}\,(d+4\,e+7\,f+10\,g+13\,h)\log{(|x-1|)} - \frac{1}{864}\,(19\,d+32\,e+52\,f+80\,g+112\,h)\log{(|x-2|)} - \frac{5\,dx^3+8\,fx^3+20\,hx^3+8\,ex^2+20\,gx^2-17\,dx-20\,fx-32\,hx-20\,e-32\,g}{72\,(x^4-5\,x^2+4)}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="giac")
```

output
```
1/864*(19*d - 32*e + 52*f - 80*g + 112*h)*log(abs(x + 2)) - 1/108*(d - 4*e
 + 7*f - 10*g + 13*h)*log(abs(x + 1)) + 1/108*(d + 4*e + 7*f + 10*g + 13*h
)*log(abs(x - 1)) - 1/864*(19*d + 32*e + 52*f + 80*g + 112*h)*log(abs(x -
2)) - 1/72*(5*d*x^3 + 8*f*x^3 + 20*h*x^3 + 8*e*x^2 + 20*g*x^2 - 17*d*x - 2
0*f*x - 32*h*x - 20*e - 32*g)/(x^4 - 5*x^2 + 4)
```

3.29.9 Mupad [B] (verification not implemented)

Time = 7.98 (sec) , antiderivative size = 146, normalized size of antiderivative = 0.97

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(4-5x^2+x^4)^2}\,dx = \frac{\left(-\frac{5d}{72}-\frac{f}{9}-\frac{5h}{18}\right)x^3+\left(-\frac{e}{9}-\frac{5g}{18}\right)x^2+\left(\frac{17d}{72}+\frac{5f}{18}+\frac{4h}{9}\right)x+\frac{5e}{18}+\frac{4g}{9}}{x^4-5x^2+4} + \ln(x-1)\left(\frac{d}{108}+\frac{e}{27}+\frac{7f}{108}+\frac{5g}{54}+\frac{13h}{108}\right)-\ln(x+1)\left(\frac{d}{108}-\frac{e}{27}+\frac{7f}{108}-\frac{5g}{54}+\frac{13h}{108}\right) - \ln(x-2)\left(\frac{19d}{864}+\frac{e}{27}+\frac{13f}{216}+\frac{5g}{54}+\frac{7h}{54}\right)+\ln(x+2)\left(\frac{19d}{864}-\frac{e}{27}+\frac{13f}{216}-\frac{5g}{54}+\frac{7h}{54}\right)$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4)/(x^4 - 5*x^2 + 4)^2,x)
```

output
```
((5*e)/18 + (4*g)/9 - x^2*(e/9 + (5*g)/18) + x*((17*d)/72 + (5*f)/18 + (4*
h)/9) - x^3*((5*d)/72 + f/9 + (5*h)/18))/(x^4 - 5*x^2 + 4) + log(x - 1)*(d
/108 + e/27 + (7*f)/108 + (5*g)/54 + (13*h)/108) - log(x + 1)*(d/108 - e/2
7 + (7*f)/108 - (5*g)/54 + (13*h)/108) - log(x - 2)*((19*d)/864 + e/27 + (
13*f)/216 + (5*g)/54 + (7*h)/54) + log(x + 2)*((19*d)/864 - e/27 + (13*f)/
216 - (5*g)/54 + (7*h)/54)
```

3.30 $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(4-5x^2+x^4)^2}\,dx$

3.30.1 Optimal result

Integrand size = 38, antiderivative size = 162

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(4-5x^2+x^4)^2}\,dx &= \frac{x(17d+20f+32h-(5d+8f+20h)x^2)}{72\,(4-5x^2+x^4)} \\ &\quad+\frac{5e+8g+20i-(2e+5g+17i)x^2}{18\,(4-5x^2+x^4)} \\ &\quad+\frac{1}{432}(19d+52f+112h)\operatorname{arctanh}\left(\frac{x}{2}\right) \\ &\quad-\frac{1}{54}(d+7f+13h)\operatorname{arctanh}(x) \\ &\quad+\frac{1}{54}(2e+5g+8i)\log\left(1-x^2\right) \\ &\quad-\frac{1}{54}(2e+5g+8i)\log\left(4-x^2\right)\end{aligned}$$

output
```
1/72*x*(17*d+20*f+32*h-(5*d+8*f+20*h)*x^2)/(x^4-5*x^2+4)+1/18*(5*e+8*g+20*
i-(2*e+5*g+17*i)*x^2)/(x^4-5*x^2+4)+1/432*(19*d+52*f+112*h)*arctanh(1/2*x)
-1/54*(d+7*f+13*h)*arctanh(x)+1/54*(2*e+5*g+8*i)*ln(-x^2+1)-1/54*(2*e+5*g+
8*i)*ln(-x^2+4)
```

3.30.2 Mathematica [A] (verified)

Time = 0.06 (sec) , antiderivative size = 185, normalized size of antiderivative = 1.14

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(4-5x^2+x^4)^2}\,dx$$
$$= \frac{20e+32g+80i+17dx+20fx+32hx-8ex^2-20gx^2-68ix^2-5dx^3-8fx^3-20hx^3}{72(4-5x^2+x^4)}$$
$$+ \frac{1}{108}(d+4e+7f+10g+13h+16i)\log(1-x)$$
$$+ \frac{1}{864}(-19d-32e-52f-80g-112h-128i)\log(2-x)$$
$$+ \frac{1}{108}(-d+4e-7f+10g-13h+16i)\log(1+x)$$
$$+ \frac{1}{864}(19d-32e+52f-80g+112h-128i)\log(2+x)$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(4 - 5*x^2 + x^4)^2,x]
```

output
```
(20*e + 32*g + 80*i + 17*d*x + 20*f*x + 32*h*x - 8*e*x^2 - 20*g*x^2 - 68*i
*x^2 - 5*d*x^3 - 8*f*x^3 - 20*h*x^3)/(72*(4 - 5*x^2 + x^4)) + ((d + 4*e +
7*f + 10*g + 13*h + 16*i)*Log[1 - x])/108 + ((-19*d - 32*e - 52*f - 80*g -
 112*h - 128*i)*Log[2 - x])/864 + ((-d + 4*e - 7*f + 10*g - 13*h + 16*i)*L
og[1 + x])/108 + ((19*d - 32*e + 52*f - 80*g + 112*h - 128*i)*Log[2 + x])/
864
```

3.30.3 Rubi [A] (verified)

Time = 0.56 (sec) , antiderivative size = 167, normalized size of antiderivative = 1.03, number of steps used = 11, number of rules used = 10, $\frac{\text{number of rules}}{\text{integrand size}} = 0.263$, Rules used = {2202, 2194, 2191, 27, 1081, 2009, 2206, 25, 1480, 220}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(x^4-5x^2+4)^2}\,dx$$
$$\Big\downarrow \text{2202}$$
$$\int \frac{hx^4+fx^2+d}{(x^4-5x^2+4)^2}dx + \int \frac{x(ix^4+gx^2+e)}{(x^4-5x^2+4)^2}dx$$

$$\Big\downarrow \text{2194}$$

$$\int \frac{hx^4+fx^2+d}{\left(x^4-5x^2+4\right)^2}dx+\frac{1}{2}\int \frac{ix^4+gx^2+e}{\left(x^4-5x^2+4\right)^2}dx^2$$

$$\Big\downarrow \text{2191}$$

$$\int \frac{hx^4+fx^2+d}{\left(x^4-5x^2+4\right)^2}dx+\frac{1}{2}\left(\frac{-\left(x^2(2e+5g+17i)\right)+5e+8g+20i}{9\left(x^4-5x^2+4\right)}-\frac{1}{9}\int \frac{2e+5g+8i}{x^4-5x^2+4}dx^2\right)$$

$$\Big\downarrow \text{27}$$

$$\int \frac{hx^4+fx^2+d}{\left(x^4-5x^2+4\right)^2}dx+ \frac{1}{2}\left(\frac{-\left(x^2(2e+5g+17i)\right)+5e+8g+20i}{9\left(x^4-5x^2+4\right)}-\frac{1}{9}(2e+5g+8i)\int \frac{1}{x^4-5x^2+4}dx^2\right)$$

$$\Big\downarrow \text{1081}$$

$$\int \frac{hx^4+fx^2+d}{\left(x^4-5x^2+4\right)^2}dx+ \frac{1}{2}\left(\frac{-\left(x^2(2e+5g+17i)\right)+5e+8g+20i}{9\left(x^4-5x^2+4\right)}-\frac{1}{9}(2e+5g+8i)\int \left(\frac{1}{3\left(1-x^2\right)}-\frac{1}{3\left(4-x^2\right)}\right)dx^2\right)$$

$$\Big\downarrow \text{2009}$$

$$\int \frac{hx^4+fx^2+d}{\left(x^4-5x^2+4\right)^2}dx+ \frac{1}{2}\left(\frac{-\left(x^2(2e+5g+17i)\right)+5e+8g+20i}{9\left(x^4-5x^2+4\right)}-\frac{1}{9}\left(\frac{1}{3}\log\left(4-x^2\right)-\frac{1}{3}\log\left(1-x^2\right)\right)(2e+5g+8i)\right)$$

$$\Big\downarrow \text{2206}$$

$$-\frac{1}{72}\int -\frac{-\left((5d+8f+20h)x^2\right)+d-20f-32h}{x^4-5x^2+4}dx+ \frac{x\left(-\left(x^2(5d+8f+20h)\right)+17d+20f+32h\right)}{72\left(x^4-5x^2+4\right)}+ \frac{1}{2}\left(\frac{-\left(x^2(2e+5g+17i)\right)+5e+8g+20i}{9\left(x^4-5x^2+4\right)}-\frac{1}{9}\left(\frac{1}{3}\log\left(4-x^2\right)-\frac{1}{3}\log\left(1-x^2\right)\right)(2e+5g+8i)\right)$$

$$\Big\downarrow \text{25}$$

$$\frac{1}{72}\int \frac{-\left((5d+8f+20h)x^2\right)+d-20f-32h}{x^4-5x^2+4}dx+\frac{x\left(-\left(x^2(5d+8f+20h)\right)+17d+20f+32h\right)}{72\left(x^4-5x^2+4\right)}+ \frac{1}{2}\left(\frac{-\left(x^2(2e+5g+17i)\right)+5e+8g+20i}{9\left(x^4-5x^2+4\right)}-\frac{1}{9}\left(\frac{1}{3}\log\left(4-x^2\right)-\frac{1}{3}\log\left(1-x^2\right)\right)(2e+5g+8i)\right)$$

$$\Big\downarrow \text{1480}$$

3.30. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{\left(4-5x^2+x^4\right)^2}\,dx$

$$\frac{1}{72}\left(\frac{4}{3}(d+7f+13h)\int\frac{1}{x^2-1}dx-\frac{1}{3}(19d+52f+112h)\int\frac{1}{x^2-4}dx\right)+\frac{x\left(-\left(x^2(5d+8f+20h)\right)+17d+20f+32h\right)}{72\left(x^4-5x^2+4\right)}+\frac{1}{2}\left(\frac{-\left(x^2(2e+5g+17i)\right)+5e+8g+20i}{9\left(x^4-5x^2+4\right)}-\frac{1}{9}\left(\frac{1}{3}\log\left(4-x^2\right)-\frac{1}{3}\log\left(1-x^2\right)\right)(2e+5g+8i)\right)$$

$$\Big\downarrow \ 220$$

$$\frac{1}{72}\left(\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(19d+52f+112h)-\frac{4}{3}\text{arctanh}(x)(d+7f+13h)\right)+\frac{x\left(-\left(x^2(5d+8f+20h)\right)+17d+20f+32h\right)}{72\left(x^4-5x^2+4\right)}+\frac{1}{2}\left(\frac{-\left(x^2(2e+5g+17i)\right)+5e+8g+20i}{9\left(x^4-5x^2+4\right)}-\frac{1}{9}\left(\frac{1}{3}\log\left(4-x^2\right)-\frac{1}{3}\log\left(1-x^2\right)\right)(2e+5g+8i)\right)$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(4 - 5*x^2 + x^4)^2,x]
```

output
```
(x*(17*d + 20*f + 32*h - (5*d + 8*f + 20*h)*x^2))/(72*(4 - 5*x^2 + x^4)) +
 (((19*d + 52*f + 112*h)*ArcTanh[x/2])/6 - (4*(d + 7*f + 13*h)*ArcTanh[x])
/3)/72 + ((5*e + 8*g + 20*i - (2*e + 5*g + 17*i)*x^2)/(9*(4 - 5*x^2 + x^4)
) - ((2*e + 5*g + 8*i)*(-1/3*Log[1 - x^2] + Log[4 - x^2]/3))/9)/2
```

3.30.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 220
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[b, 2])^(-
1))*ArcTanh[Rt[b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] &&
 (LtQ[a, 0] || GtQ[b, 0])
```

rule 1081
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> With[{q = Rt[b^2
- 4*a*c, 2]}, Simp[c   Int[ExpandIntegrand[1/((b/2 - q/2 + c*x)*(b/2 + q/2
+ c*x)), x], x], x]] /; FreeQ[{a, b, c}, x] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2191
```
Int[(Pq_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> With[{Q =
PolynomialQuotient[Pq, a + b*x + c*x^2, x], f = Coeff[PolynomialRemainder[P
q, a + b*x + c*x^2, x], x, 0], g = Coeff[PolynomialRemainder[Pq, a + b*x +
c*x^2, x], x, 1]}, Simp[(b*f - 2*a*g + (2*c*f - b*g)*x)*((a + b*x + c*x^2)^
(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] + Simp[1/((p + 1)*(b^2 - 4*a*c))   Int
[(a + b*x + c*x^2)^(p + 1)*ExpandToSum[(p + 1)*(b^2 - 4*a*c)*Q - (2*p + 3)*
(2*c*f - b*g), x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Pq, x] && NeQ[b^
2 - 4*a*c, 0] && LtQ[p, -1]
```

rule 2194
```
Int[(Pq_)*(x_)^(m_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :
> Simp[1/2   Subst[Int[x^((m - 1)/2)*SubstFor[x^2, Pq, x]*(a + b*x + c*x^2)
^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pq, x^2] && IntegerQ
[(m - 1)/2]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2206
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> With[{d =
 Coeff[PolynomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 0], e = Coeff[Poly
nomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 2]}, Simp[x*(a + b*x^2 + c*x^
4)^(p + 1)*((a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)/(2*a*(p + 1)*(b
^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(a + b*x^2 + c
*x^4)^(p + 1)*ExpandToSum[2*a*(p + 1)*(b^2 - 4*a*c)*PolynomialQuotient[Px,
a + b*x^2 + c*x^4, x] + b^2*d*(2*p + 3) - 2*a*c*d*(4*p + 5) - a*b*e + c*(4*
p + 7)*(b*d - 2*a*e)*x^2, x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x
^2] && Expon[Px, x^2] > 1 && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1]
```

3.30. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(4-5x^2+x^4)^2}\,dx$

3.30.4 Maple [A] (verified)

Time = 0.12 (sec) , antiderivative size = 161, normalized size of antiderivative = 0.99

method	result
norman	$\frac{\left(-\frac{5d}{72}-\frac{f}{9}-\frac{5h}{18}\right)x^3+\left(\frac{17d}{72}+\frac{5f}{18}+\frac{4h}{9}\right)x+\left(-\frac{5g}{18}-\frac{e}{9}-\frac{17i}{18}\right)x^2+\frac{4g}{9}+\frac{5e}{18}+\frac{10i}{9}}{x^4-5x^2+4}+\left(-\frac{19d}{864}-\frac{e}{27}-\frac{13f}{216}-\frac{5g}{54}-\frac{7h}{54}-\frac{4i}{27}\right)\ln$
default	$-\frac{\frac{d}{144}-\frac{e}{72}+\frac{f}{36}-\frac{g}{18}+\frac{h}{9}-\frac{2i}{9}}{x+2}+\left(\frac{19d}{864}-\frac{e}{27}+\frac{13f}{216}-\frac{5g}{54}+\frac{7h}{54}-\frac{4i}{27}\right)\ln\left(x+2\right)+\left(-\frac{d}{108}+\frac{e}{27}-\frac{7f}{108}+\frac{5g}{54}\right.$
risch	$\frac{4\ln(x+1)i}{27}+\frac{\left(-\frac{5d}{72}-\frac{f}{9}-\frac{5h}{18}\right)x^3+\left(\frac{17d}{72}+\frac{5f}{18}+\frac{4h}{9}\right)x+\left(-\frac{5g}{18}-\frac{e}{9}-\frac{17i}{18}\right)x^2+\frac{4g}{9}+\frac{5e}{18}+\frac{10i}{9}}{x^4-5x^2+4}+\frac{13\ln(x+2)f}{216}+\frac{7\ln(1-x)f}{108}-$
parallelrisch	$-\frac{-960i-384g-240e+96f x^3+240g x^2+240h x^3-204dx+816i x^2+76\ln(x-2)d+128\ln(x-2)e-32\ln(x-1)d-128\ln(x-1)e-}{}$

input
```
int((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x,method=_RETURNVERBOS
E)
```

output
```
((-5/72*d-1/9*f-5/18*h)*x^3+(17/72*d+5/18*f+4/9*h)*x+(-5/18*g-1/9*e-17/18*
i)*x^2+4/9*g+5/18*e+10/9*i)/(x^4-5*x^2+4)+(-19/864*d-1/27*e-13/216*f-5/54*
g-7/54*h-4/27*i)*ln(x-2)+(-1/108*d+1/27*e-7/108*f+5/54*g-13/108*h+4/27*i)*
ln(x+1)+(1/108*d+1/27*e+7/108*f+5/54*g+13/108*h+4/27*i)*ln(x-1)+(19/864*d-
1/27*e+13/216*f-5/54*g+7/54*h-4/27*i)*ln(x+2)
```

3.30.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 346 vs. 2(146) = 292.

Time = 6.24 (sec) , antiderivative size = 346, normalized size of antiderivative = 2.14

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{\left(4-5x^2+x^4\right)^2}\,dx = -\frac{12\,(5\,d+8\,f+20\,h)x^3+48\,(2\,e+5\,g+17\,i)x^2-12\,(17\,d+20\,f+32\,h)x-((19\,d-32\,e+52\,f-80}{}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="fr
icas")
```

output
```
-1/864*(12*(5*d + 8*f + 20*h)*x^3 + 48*(2*e + 5*g + 17*i)*x^2 - 12*(17*d +
 20*f + 32*h)*x - ((19*d - 32*e + 52*f - 80*g + 112*h - 128*i)*x^4 - 5*(19
*d - 32*e + 52*f - 80*g + 112*h - 128*i)*x^2 + 76*d - 128*e + 208*f - 320*
g + 448*h - 512*i)*log(x + 2) + 8*((d - 4*e + 7*f - 10*g + 13*h - 16*i)*x^
4 - 5*(d - 4*e + 7*f - 10*g + 13*h - 16*i)*x^2 + 4*d - 16*e + 28*f - 40*g
+ 52*h - 64*i)*log(x + 1) - 8*((d + 4*e + 7*f + 10*g + 13*h + 16*i)*x^4 -
5*(d + 4*e + 7*f + 10*g + 13*h + 16*i)*x^2 + 4*d + 16*e + 28*f + 40*g + 52
*h + 64*i)*log(x - 1) + ((19*d + 32*e + 52*f + 80*g + 112*h + 128*i)*x^4 -
 5*(19*d + 32*e + 52*f + 80*g + 112*h + 128*i)*x^2 + 76*d + 128*e + 208*f
+ 320*g + 448*h + 512*i)*log(x - 2) - 240*e - 384*g - 960*i)/(x^4 - 5*x^2
+ 4)
```

3.30.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(4-5x^2+x^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((i*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4)**2,x)
```

output
```
Timed out
```

3.30.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 163, normalized size of antiderivative = 1.01

$$\begin{aligned}
&\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(4-5x^2+x^4)^2}\,dx \\
&= \frac{1}{864}\,(19\,d-32\,e+52\,f-80\,g+112\,h-128\,i)\log{(x+2)} \\
&\quad - \frac{1}{108}\,(d-4\,e+7\,f-10\,g+13\,h-16\,i)\log{(x+1)} \\
&\quad + \frac{1}{108}\,(d+4\,e+7\,f+10\,g+13\,h+16\,i)\log{(x-1)} \\
&\quad - \frac{1}{864}\,(19\,d+32\,e+52\,f+80\,g+112\,h+128\,i)\log{(x-2)} \\
&\quad - \frac{(5\,d+8\,f+20\,h)x^3+4\,(2\,e+5\,g+17\,i)x^2-(17\,d+20\,f+32\,h)x-20\,e-32\,g-80\,i}{72\,(x^4-5\,x^2+4)}
\end{aligned}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="ma
xima")
```

output
```
1/864*(19*d - 32*e + 52*f - 80*g + 112*h - 128*i)*log(x + 2) - 1/108*(d -
4*e + 7*f - 10*g + 13*h - 16*i)*log(x + 1) + 1/108*(d + 4*e + 7*f + 10*g +
 13*h + 16*i)*log(x - 1) - 1/864*(19*d + 32*e + 52*f + 80*g + 112*h + 128*
i)*log(x - 2) - 1/72*((5*d + 8*f + 20*h)*x^3 + 4*(2*e + 5*g + 17*i)*x^2 -
(17*d + 20*f + 32*h)*x - 20*e - 32*g - 80*i)/(x^4 - 5*x^2 + 4)
```

3.30.8 Giac [A] (verification not implemented)

Time = 0.31 (sec) , antiderivative size = 173, normalized size of antiderivative = 1.07

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(4-5x^2+x^4)^2}\,dx &= \frac{1}{864}\left(19\,d-32\,e+52\,f-80\,g+112\,h-128\,i\right)\log\left(|x+2|\right)\\ &\quad-\frac{1}{108}\left(d-4\,e+7\,f-10\,g+13\,h-16\,i\right)\log\left(|x+1|\right)\\ &\quad+\frac{1}{108}\left(d+4\,e+7\,f+10\,g+13\,h+16\,i\right)\log\left(|x-1|\right)\\ &\quad-\frac{1}{864}\left(19\,d+32\,e+52\,f+80\,g+112\,h+128\,i\right)\log\left(|x-2|\right)\\ &\quad-\frac{5\,dx^3+8\,fx^3+20\,hx^3+8\,ex^2+20\,gx^2+68\,ix^2-17\,dx-20\,fx-32\,hx-20\,e-32\,g-80\,i}{72\left(x^4-5\,x^2+4\right)}\end{aligned}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="gi
ac")
```

output
```
1/864*(19*d - 32*e + 52*f - 80*g + 112*h - 128*i)*log(abs(x + 2)) - 1/108*
(d - 4*e + 7*f - 10*g + 13*h - 16*i)*log(abs(x + 1)) + 1/108*(d + 4*e + 7*
f + 10*g + 13*h + 16*i)*log(abs(x - 1)) - 1/864*(19*d + 32*e + 52*f + 80*g
 + 112*h + 128*i)*log(abs(x - 2)) - 1/72*(5*d*x^3 + 8*f*x^3 + 20*h*x^3 + 8
*e*x^2 + 20*g*x^2 + 68*i*x^2 - 17*d*x - 20*f*x - 32*h*x - 20*e - 32*g - 80
*i)/(x^4 - 5*x^2 + 4)
```

3.30.9 Mupad [B] (verification not implemented)

Time = 0.35 (sec) , antiderivative size = 164, normalized size of antiderivative = 1.01

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(4-5x^2+x^4)^2}\,dx$$
$$= \frac{\left(-\frac{5d}{72}-\frac{f}{9}-\frac{5h}{18}\right)x^3+\left(-\frac{e}{9}-\frac{5g}{18}-\frac{17i}{18}\right)x^2+\left(\frac{17d}{72}+\frac{5f}{18}+\frac{4h}{9}\right)x+\frac{5e}{18}+\frac{4g}{9}+\frac{10i}{9}}{x^4-5x^2+4}$$
$$+\ln(x-1)\left(\frac{d}{108}+\frac{e}{27}+\frac{7f}{108}+\frac{5g}{54}+\frac{13h}{108}+\frac{4i}{27}\right)$$
$$-\ln(x+1)\left(\frac{d}{108}-\frac{e}{27}+\frac{7f}{108}-\frac{5g}{54}+\frac{13h}{108}-\frac{4i}{27}\right)$$
$$-\ln(x-2)\left(\frac{19d}{864}+\frac{e}{27}+\frac{13f}{216}+\frac{5g}{54}+\frac{7h}{54}+\frac{4i}{27}\right)$$
$$+\ln(x+2)\left(\frac{19d}{864}-\frac{e}{27}+\frac{13f}{216}-\frac{5g}{54}+\frac{7h}{54}-\frac{4i}{27}\right)$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(x^4 - 5*x^2 + 4)^2,x)
```

output
```
((5*e)/18 + (4*g)/9 + (10*i)/9 + x*((17*d)/72 + (5*f)/18 + (4*h)/9) - x^3*
((5*d)/72 + f/9 + (5*h)/18) - x^2*(e/9 + (5*g)/18 + (17*i)/18))/(x^4 - 5*x
^2 + 4) + log(x - 1)*(d/108 + e/27 + (7*f)/108 + (5*g)/54 + (13*h)/108 + (
4*i)/27) - log(x + 1)*(d/108 - e/27 + (7*f)/108 - (5*g)/54 + (13*h)/108 -
(4*i)/27) - log(x - 2)*((19*d)/864 + e/27 + (13*f)/216 + (5*g)/54 + (7*h)/
54 + (4*i)/27) + log(x + 2)*((19*d)/864 - e/27 + (13*f)/216 - (5*g)/54 + (
7*h)/54 - (4*i)/27)
```

3.31 $\int \frac{d+ex}{(1+x^2+x^4)^2}\,dx$

3.31.1 Optimal result

Integrand size = 16, antiderivative size = 140

$$\int \frac{d+ex}{(1+x^2+x^4)^2}\,dx = \frac{dx(1-x^2)}{6(1+x^2+x^4)}+\frac{e(1+2x^2)}{6(1+x^2+x^4)}-\frac{d\arctan\left(\frac{1-2x}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{d\arctan\left(\frac{1+2x}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2e\arctan\left(\frac{1+2x^2}{\sqrt{3}}\right)}{3\sqrt{3}}-\frac{1}{4}d\log\left(1-x+x^2\right)+\frac{1}{4}d\log\left(1+x+x^2\right)$$

output
```
1/6*d*x*(-x^2+1)/(x^4+x^2+1)+1/6*e*(2*x^2+1)/(x^4+x^2+1)-1/4*d*ln(x^2-x+1)
+1/4*d*ln(x^2+x+1)-1/9*d*arctan(1/3*(1-2*x)*3^(1/2))*3^(1/2)+1/9*d*arctan(
1/3*(1+2*x)*3^(1/2))*3^(1/2)+2/9*e*arctan(1/3*(2*x^2+1)*3^(1/2))*3^(1/2)
```

3.31.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.35 (sec) , antiderivative size = 146, normalized size of antiderivative = 1.04

$$\int \frac{d+ex}{(1+x^2+x^4)^2}\,dx = \frac{e+dx+2ex^2-dx^3}{6(1+x^2+x^4)}-\frac{\left(-11i+\sqrt{3}\right)d\arctan\left(\frac{1}{2}\left(-i+\sqrt{3}\right)x\right)}{6\sqrt{6+6i\sqrt{3}}}-\frac{\left(11i+\sqrt{3}\right)d\arctan\left(\frac{1}{2}\left(i+\sqrt{3}\right)x\right)}{6\sqrt{6-6i\sqrt{3}}}-\frac{2e\arctan\left(\frac{\sqrt{3}}{1+2x^2}\right)}{3\sqrt{3}}$$

input
```
Integrate[(d + e*x)/(1 + x^2 + x^4)^2,x]
```

output
```
(e + d*x + 2*e*x^2 - d*x^3)/(6*(1 + x^2 + x^4)) - ((-11*I + Sqrt[3])*d*Arc
Tan[((-I + Sqrt[3])*x)/2])/(6*Sqrt[6 + (6*I)*Sqrt[3]]) - ((11*I + Sqrt[3])
*d*ArcTan[((I + Sqrt[3])*x)/2])/(6*Sqrt[6 - (6*I)*Sqrt[3]]) - (2*e*ArcTan[
Sqrt[3]/(1 + 2*x^2)])/(3*Sqrt[3])
```

3.31.3 Rubi [A] (verified)

Time = 0.39 (sec) , antiderivative size = 149, normalized size of antiderivative = 1.06, number of steps used = 14, number of rules used = 13, $\frac{\text{number of rules}}{\text{integrand size}} = 0.812$, Rules used = {2202, 27, 1405, 1432, 1086, 1083, 217, 1483, 1142, 25, 1083, 217, 1103}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex}{(x^4+x^2+1)^2}dx$$

$$\Big\downarrow \text{ 2202}$$

$$\int \frac{d}{(x^4+x^2+1)^2}dx+\int \frac{ex}{(x^4+x^2+1)^2}dx$$

$$\Big\downarrow \text{ 27}$$

$$d\int \frac{1}{(x^4+x^2+1)^2}dx+e\int \frac{x}{(x^4+x^2+1)^2}dx$$

$$\Big\downarrow \text{ 1405}$$

$$d\left(\frac{1}{6}\int \frac{5-x^2}{x^4+x^2+1}dx+\frac{x(1-x^2)}{6(x^4+x^2+1)}\right)+e\int \frac{x}{(x^4+x^2+1)^2}dx$$

$$\Big\downarrow \text{ 1432}$$

$$d\left(\frac{1}{6}\int \frac{5-x^2}{x^4+x^2+1}dx+\frac{x(1-x^2)}{6(x^4+x^2+1)}\right)+\frac{1}{2}e\int \frac{1}{(x^4+x^2+1)^2}dx^2$$

$$\Big\downarrow \text{ 1086}$$

$$d\left(\frac{1}{6}\int \frac{5-x^2}{x^4+x^2+1}dx+\frac{x(1-x^2)}{6(x^4+x^2+1)}\right)+\frac{1}{2}e\left(\frac{2}{3}\int \frac{1}{x^4+x^2+1}dx^2+\frac{2x^2+1}{3(x^4+x^2+1)}\right)$$

$$\Big\downarrow \text{ 1083}$$

$$d\left(\frac{1}{6}\int \frac{5-x^2}{x^4+x^2+1}dx+\frac{x(1-x^2)}{6(x^4+x^2+1)}\right)+\frac{1}{2}e\left(\frac{2x^2+1}{3(x^4+x^2+1)}-\frac{4}{3}\int \frac{1}{-x^4-3}d(2x^2+1)\right)$$

$$\Big\downarrow \; 217$$

$$d\left(\frac{1}{6}\int \frac{5-x^2}{x^4+x^2+1}dx+\frac{x(1-x^2)}{6(x^4+x^2+1)}\right)+\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3(x^4+x^2+1)}\right)$$

$$\Big\downarrow \; 1483$$

$$d\left(\frac{1}{6}\left(\frac{1}{2}\int \frac{5-6x}{x^2-x+1}dx+\frac{1}{2}\int \frac{6x+5}{x^2+x+1}dx\right)+\frac{x(1-x^2)}{6(x^4+x^2+1)}\right)+$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3(x^4+x^2+1)}\right)$$

$$\Big\downarrow \; 1142$$

$$d\left(\frac{1}{6}\left(\frac{1}{2}\left(2\int \frac{1}{x^2-x+1}dx-3\int -\frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(2\int \frac{1}{x^2+x+1}dx+3\int \frac{2x+1}{x^2+x+1}dx\right)\right)+\frac{x(1-}{6(x^4+x^}\right.$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3(x^4+x^2+1)}\right)$$

$$\Big\downarrow \; 25$$

$$d\left(\frac{1}{6}\left(\frac{1}{2}\left(2\int \frac{1}{x^2-x+1}dx+3\int \frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(2\int \frac{1}{x^2+x+1}dx+3\int \frac{2x+1}{x^2+x+1}dx\right)\right)+\frac{x(1-x^2}{6(x^4+x^2}\right.$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3(x^4+x^2+1)}\right)$$

$$\Big\downarrow \; 1083$$

$$d\left(\frac{1}{6}\left(\frac{1}{2}\left(3\int \frac{1-2x}{x^2-x+1}dx-4\int \frac{1}{-(2x-1)^2-3}d(2x-1)\right)+\frac{1}{2}\left(3\int \frac{2x+1}{x^2+x+1}dx-4\int \frac{1}{-(2x+1)^2-3}d(2x\right.\right.\right.$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3(x^4+x^2+1)}\right)$$

$$\Big\downarrow \; 217$$

3.31. $\int \frac{d+ex}{(1+x^2+x^4)^2}\,dx$

$$d\left(\frac{1}{6}\left(\frac{1}{2}\left(3\int \frac{1-2x}{x^2-x+1}dx+\frac{4\arctan\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}\right)+\frac{1}{2}\left(3\int \frac{2x+1}{x^2+x+1}dx+\frac{4\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}\right)\right)+\frac{x(1-x^2)}{6(x^4+x^2+\right.$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3(x^4+x^2+1)}\right)$$

$\big\downarrow$ 1103

$$d\left(\frac{1}{6}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}-3\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}+3\log\left(x^2+x+1\right)\right)\right)+\frac{x(1-x^2)}{6(x^4+x^2+\right.$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3(x^4+x^2+1)}\right)$$

input
```
Int[(d + e*x)/(1 + x^2 + x^4)^2,x]
```

output
```
(e*((1 + 2*x^2)/(3*(1 + x^2 + x^4)) + (4*ArcTan[(1 + 2*x^2)/Sqrt[3]])/(3*S
qrt[3])))/2 + d*((x*(1 - x^2))/(6*(1 + x^2 + x^4)) + (((4*ArcTan[(-1 + 2*x
)/Sqrt[3]])/Sqrt[3] - 3*Log[1 - x + x^2])/2 + ((4*ArcTan[(1 + 2*x)/Sqrt[3]
])/Sqrt[3] + 3*Log[1 + x + x^2])/2)/6)
```

3.31.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 217
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1086
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] - Simp[2*c*((2*p +
 3)/((p + 1)*(b^2 - 4*a*c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; Fre
eQ[{a, b, c}, x] && ILtQ[p, -1]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1405
```
Int[((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[(-x)*(b^2
- 2*a*c + b*c*x^2)*((a + b*x^2 + c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))
), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(b^2 - 2*a*c + 2*(p + 1)*(
b^2 - 4*a*c) + b*c*(4*p + 7)*x^2)*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; Fr
eeQ[{a, b, c}, x] && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1] && IntegerQ[2*p]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 1483
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[a/c, 2]}, With[{r = Rt[2*q - b/c, 2]}, Simp[1/(2*c*q*r)   In
t[(d*r - (d - e*q)*x)/(q - r*x + x^2), x], x] + Simp[1/(2*c*q*r)   Int[(d*r
 + (d - e*q)*x)/(q + r*x + x^2), x], x]]] /; FreeQ[{a, b, c, d, e}, x] && N
eQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NegQ[b^2 - 4*a*c]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.31. $\int \frac{d+ex}{(1+x^2+x^4)^2} \, dx$

3.31.4 Maple [A] (verified)

Time = 0.15 (sec) , antiderivative size = 124, normalized size of antiderivative = 0.89

method	result
default	$-\frac{\left(\frac{d}{3}-\frac{e}{3}\right)x-\frac{2d}{3}-\frac{e}{3}}{4(x^2-x+1)}-\frac{d\ln(x^2-x+1)}{4}-\frac{(-2d-4e)\sqrt{3}\,\arctan\left(\frac{(2x-1)\sqrt{3}}{3}\right)}{18}+\frac{\left(-\frac{d}{3}-\frac{e}{3}\right)x-\frac{2d}{3}+\frac{e}{3}}{4x^2+4x+4}+\frac{d\ln(x^2+x+1)}{4}+\frac{(2d-4e}{}$
risch	$\frac{d\ln(10044d^2x^2+5952e^2x^2+10044d^2x+5952e^2x+10044d^2+5952e^2)}{4}+\frac{\sqrt{3}\,d\arctan\left(\frac{18d^2x\sqrt{3}}{27d^2+16e^2}+\frac{32e^2x\sqrt{3}}{3(27d^2+16e^2)}+\frac{9\sqrt{3}\,d^2}{27d^2+16e^2}+\frac{1}{3(27}\right.}{9}$

input
```
int((e*x+d)/(x^4+x^2+1)^2,x,method=_RETURNVERBOSE)
```

output
```
-1/4*((1/3*d-1/3*e)*x-2/3*d-1/3*e)/(x^2-x+1)-1/4*d*ln(x^2-x+1)-1/18*(-2*d-
4*e)*3^(1/2)*arctan(1/3*(2*x-1)*3^(1/2))+1/4*((-1/3*d-1/3*e)*x-2/3*d+1/3*e
)/(x^2+x+1)+1/4*d*ln(x^2+x+1)+1/18*(2*d-4*e)*arctan(1/3*(1+2*x)*3^(1/2))*3
^(1/2)
```

3.31.5 Fricas [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 154, normalized size of antiderivative = 1.10

$$\int \frac{d+ex}{(1+x^2+x^4)^2}\,dx =$$
$$-\frac{6\,dx^3-12\,ex^2-4\,\sqrt{3}((d-2\,e)x^4+(d-2\,e)x^2+d-2\,e)\arctan\left(\frac{1}{3}\,\sqrt{3}(2\,x+1)\right)-4\,\sqrt{3}((d+2\,e)x^4}{}$$

input
```
integrate((e*x+d)/(x^4+x^2+1)^2,x, algorithm="fricas")
```

output
```
-1/36*(6*d*x^3 - 12*e*x^2 - 4*sqrt(3)*((d - 2*e)*x^4 + (d - 2*e)*x^2 + d -
 2*e)*arctan(1/3*sqrt(3)*(2*x + 1)) - 4*sqrt(3)*((d + 2*e)*x^4 + (d + 2*e)
*x^2 + d + 2*e)*arctan(1/3*sqrt(3)*(2*x - 1)) - 6*d*x - 9*(d*x^4 + d*x^2 +
 d)*log(x^2 + x + 1) + 9*(d*x^4 + d*x^2 + d)*log(x^2 - x + 1) - 6*e)/(x^4
+ x^2 + 1)
```

3.31.6 Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 2.01 (sec) , antiderivative size = 952, normalized size of antiderivative = 6.80

$$\int \frac{d+ex}{\left(1+x^2+x^4\right)^2}\,dx = \text{Too large to display}$$

input
```
integrate((e*x+d)/(x**4+x**2+1)**2,x)
```

output
```
(-d/4 - sqrt(3)*I*(d + 2*e)/18)*log(x + (-10309*d**4*e + 1026*d**4*(-d/4 -
 sqrt(3)*I*(d + 2*e)/18) - 7200*d**2*e**3 - 31536*d**2*e**2*(-d/4 - sqrt(3
)*I*(d + 2*e)/18) + 108432*d**2*e*(-d/4 - sqrt(3)*I*(d + 2*e)/18)**2 + 163
296*d**2*(-d/4 - sqrt(3)*I*(d + 2*e)/18)**3 + 1792*e**5 + 11520*e**4*(-d/4
 - sqrt(3)*I*(d + 2*e)/18) + 48384*e**3*(-d/4 - sqrt(3)*I*(d + 2*e)/18)**2
 + 311040*e**2*(-d/4 - sqrt(3)*I*(d + 2*e)/18)**3)/(3348*d**5 - 11408*d**3
*e**2 - 7936*d*e**4)) + (-d/4 + sqrt(3)*I*(d + 2*e)/18)*log(x + (-10309*d*
*4*e + 1026*d**4*(-d/4 + sqrt(3)*I*(d + 2*e)/18) - 7200*d**2*e**3 - 31536*
d**2*e**2*(-d/4 + sqrt(3)*I*(d + 2*e)/18) + 108432*d**2*e*(-d/4 + sqrt(3)*
I*(d + 2*e)/18)**2 + 163296*d**2*(-d/4 + sqrt(3)*I*(d + 2*e)/18)**3 + 1792
*e**5 + 11520*e**4*(-d/4 + sqrt(3)*I*(d + 2*e)/18) + 48384*e**3*(-d/4 + sq
rt(3)*I*(d + 2*e)/18)**2 + 311040*e**2*(-d/4 + sqrt(3)*I*(d + 2*e)/18)**3)
/(3348*d**5 - 11408*d**3*e**2 - 7936*d*e**4)) + (d/4 - sqrt(3)*I*(d - 2*e)
/18)*log(x + (-10309*d**4*e + 1026*d**4*(d/4 - sqrt(3)*I*(d - 2*e)/18) - 7
200*d**2*e**3 - 31536*d**2*e**2*(d/4 - sqrt(3)*I*(d - 2*e)/18) + 108432*d*
*2*e*(d/4 - sqrt(3)*I*(d - 2*e)/18)**2 + 163296*d**2*(d/4 - sqrt(3)*I*(d -
 2*e)/18)**3 + 1792*e**5 + 11520*e**4*(d/4 - sqrt(3)*I*(d - 2*e)/18) + 483
84*e**3*(d/4 - sqrt(3)*I*(d - 2*e)/18)**2 + 311040*e**2*(d/4 - sqrt(3)*I*(
d - 2*e)/18)**3)/(3348*d**5 - 11408*d**3*e**2 - 7936*d*e**4)) + (d/4 + sqr
t(3)*I*(d - 2*e)/18)*log(x + (-10309*d**4*e + 1026*d**4*(d/4 + sqrt(3)*...
```

3.31.7 Maxima [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 96, normalized size of antiderivative = 0.69

$$\begin{aligned}\int \frac{d+ex}{\left(1+x^2+x^4\right)^2}\,dx &= \frac{1}{9}\sqrt{3}(d-2e)\arctan\left(\frac{1}{3}\sqrt{3}(2x+1)\right) \\ &+ \frac{1}{9}\sqrt{3}(d+2e)\arctan\left(\frac{1}{3}\sqrt{3}(2x-1)\right) + \frac{1}{4}d\log\left(x^2+x+1\right) \\ &- \frac{1}{4}d\log\left(x^2-x+1\right) - \frac{dx^3-2ex^2-dx-e}{6\left(x^4+x^2+1\right)}\end{aligned}$$

input
```
integrate((e*x+d)/(x^4+x^2+1)^2,x, algorithm="maxima")
```

output
```
1/9*sqrt(3)*(d - 2*e)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/9*sqrt(3)*(d + 2*e
)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/4*d*log(x^2 + x + 1) - 1/4*d*log(x^2 -
 x + 1) - 1/6*(d*x^3 - 2*e*x^2 - d*x - e)/(x^4 + x^2 + 1)
```

3.31.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 96, normalized size of antiderivative = 0.69

$$\int \frac{d+ex}{(1+x^2+x^4)^2}\,dx = \frac{1}{9}\sqrt{3}(d-2e)\arctan\left(\frac{1}{3}\sqrt{3}(2x+1)\right) + \frac{1}{9}\sqrt{3}(d+2e)\arctan\left(\frac{1}{3}\sqrt{3}(2x-1)\right) + \frac{1}{4}d\log\left(x^2+x+1\right) - \frac{1}{4}d\log\left(x^2-x+1\right) - \frac{dx^3-2ex^2-dx-e}{6(x^4+x^2+1)}$$

input
```
integrate((e*x+d)/(x^4+x^2+1)^2,x, algorithm="giac")
```

output
```
1/9*sqrt(3)*(d - 2*e)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/9*sqrt(3)*(d + 2*e
)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/4*d*log(x^2 + x + 1) - 1/4*d*log(x^2 -
 x + 1) - 1/6*(d*x^3 - 2*e*x^2 - d*x - e)/(x^4 + x^2 + 1)
```

3.31.9 Mupad [B] (verification not implemented)

Time = 0.14 (sec) , antiderivative size = 149, normalized size of antiderivative = 1.06

$$\int \frac{d+ex}{(1+x^2+x^4)^2}\,dx = \frac{-\frac{d\,x^3}{6}+\frac{e\,x^2}{3}+\frac{d\,x}{6}+\frac{e}{6}}{x^4+x^2+1} - \ln\left(x-\frac{1}{2}-\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{d}{4}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{18}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}\right) + \ln\left(x+\frac{1}{2}-\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{d}{4}-\frac{\sqrt{3}\,d\,1\mathrm{i}}{18}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}\right) + \ln\left(x-\frac{1}{2}+\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(-\frac{d}{4}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{18}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}\right) + \ln\left(x+\frac{1}{2}+\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{d}{4}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{18}-\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}\right)$$

input
```
int((d + e*x)/(x^2 + x^4 + 1)^2,x)
```

output
```
(e/6 + (d*x)/6 - (d*x^3)/6 + (e*x^2)/3)/(x^2 + x^4 + 1) - log(x - (3^(1/2)
*1i)/2 - 1/2)*(d/4 + (3^(1/2)*d*1i)/18 + (3^(1/2)*e*1i)/9) + log(x - (3^(1
/2)*1i)/2 + 1/2)*(d/4 - (3^(1/2)*d*1i)/18 + (3^(1/2)*e*1i)/9) + log(x + (3
^(1/2)*1i)/2 - 1/2)*((3^(1/2)*d*1i)/18 - d/4 + (3^(1/2)*e*1i)/9) + log(x +
 (3^(1/2)*1i)/2 + 1/2)*(d/4 + (3^(1/2)*d*1i)/18 - (3^(1/2)*e*1i)/9)
```

3.31. $\int \frac{d+ex}{(1+x^2+x^4)^2} \, dx$

3.32 $\int \frac{d+ex+fx^2}{(1+x^2+x^4)^2} \, dx$

3.32.1 Optimal result

Integrand size = 21, antiderivative size = 165

$$\begin{aligned}\int \frac{d+ex+fx^2}{(1+x^2+x^4)^2} \, dx = {} & \frac{e(1+2x^2)}{6(1+x^2+x^4)} + \frac{x(d+f-(d-2f)x^2)}{6(1+x^2+x^4)} - \frac{(4d+f)\arctan\left(\frac{1-2x}{\sqrt{3}}\right)}{12\sqrt{3}} \\ & + \frac{(4d+f)\arctan\left(\frac{1+2x}{\sqrt{3}}\right)}{12\sqrt{3}} + \frac{2e\arctan\left(\frac{1+2x^2}{\sqrt{3}}\right)}{3\sqrt{3}} \\ & - \frac{1}{8}(2d-f)\log\left(1-x+x^2\right) + \frac{1}{8}(2d-f)\log\left(1+x+x^2\right)\end{aligned}$$

output
```
1/6*e*(2*x^2+1)/(x^4+x^2+1)+1/6*x*(d+f-(d-2*f)*x^2)/(x^4+x^2+1)-1/8*(2*d-f
)*ln(x^2-x+1)+1/8*(2*d-f)*ln(x^2+x+1)-1/36*(4*d+f)*arctan(1/3*(1-2*x)*3^(1
/2))*3^(1/2)+1/36*(4*d+f)*arctan(1/3*(1+2*x)*3^(1/2))*3^(1/2)+2/9*e*arctan
(1/3*(2*x^2+1)*3^(1/2))*3^(1/2)
```

3.32.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.28 (sec) , antiderivative size = 186, normalized size of antiderivative = 1.13

$$\int \frac{d+ex+fx^2}{(1+x^2+x^4)^2}\,dx = \frac{1}{36}\left(\frac{6(e+2ex^2+x(d+f-dx^2+2fx^2))}{1+x^2+x^4} - \frac{\left(\left(-11i+\sqrt{3}\right)d-2\left(-2i+\sqrt{3}\right)f\right)\arctan\left(\frac{1}{2}\left(-i+\sqrt{3}\right)x\right)}{\sqrt{\frac{1}{6}\left(1+i\sqrt{3}\right)}} - \frac{\left(\left(11i+\sqrt{3}\right)d-2\left(2i+\sqrt{3}\right)f\right)\arctan\left(\frac{1}{2}\left(i+\sqrt{3}\right)x\right)}{\sqrt{\frac{1}{6}\left(1-i\sqrt{3}\right)}} - 8\sqrt{3}e\arctan\left(\frac{\sqrt{3}}{1+2x^2}\right)\right)$$

input
```
Integrate[(d + e*x + f*x^2)/(1 + x^2 + x^4)^2,x]
```

output
```
((6*(e + 2*e*x^2 + x*(d + f - d*x^2 + 2*f*x^2)))/(1 + x^2 + x^4) - (((-11*
I + Sqrt[3])*d - 2*(-2*I + Sqrt[3])*f)*ArcTan[((-I + Sqrt[3])*x)/2])/Sqrt[
(1 + I*Sqrt[3])/6] - (((11*I + Sqrt[3])*d - 2*(2*I + Sqrt[3])*f)*ArcTan[((
I + Sqrt[3])*x)/2])/Sqrt[(1 - I*Sqrt[3])/6] - 8*Sqrt[3]*e*ArcTan[Sqrt[3]/(
1 + 2*x^2)])/36
```

3.32.3 Rubi [A] (verified)

Time = 0.45 (sec) , antiderivative size = 178, normalized size of antiderivative = 1.08, number of steps used = 14, number of rules used = 13, $\frac{\text{number of rules}}{\text{integrand size}} = 0.619$, Rules used = {2202, 27, 1432, 1086, 1083, 217, 1492, 1483, 1142, 25, 1083, 217, 1103}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2}{(x^4+x^2+1)^2}dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{fx^2+d}{(x^4+x^2+1)^2}dx+\int \frac{ex}{(x^4+x^2+1)^2}dx$$

$$\Big\downarrow \text{27}$$

$$\int \frac{fx^2+d}{(x^4+x^2+1)^2}dx+e\int \frac{x}{(x^4+x^2+1)^2}dx$$

$$\Big\downarrow \text{ 1432}$$

$$\int \frac{fx^2+d}{(x^4+x^2+1)^2}dx+\frac{1}{2}e\int \frac{1}{(x^4+x^2+1)^2}dx^2$$

$$\Big\downarrow \text{ 1086}$$

$$\int \frac{fx^2+d}{(x^4+x^2+1)^2}dx+\frac{1}{2}e\left(\frac{2}{3}\int \frac{1}{x^4+x^2+1}dx^2+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)$$

$$\Big\downarrow \text{ 1083}$$

$$\int \frac{fx^2+d}{(x^4+x^2+1)^2}dx+\frac{1}{2}e\left(\frac{2x^2+1}{3\left(x^4+x^2+1\right)}-\frac{4}{3}\int \frac{1}{-x^4-3}d(2x^2+1)\right)$$

$$\Big\downarrow \text{ 217}$$

$$\int \frac{fx^2+d}{(x^4+x^2+1)^2}dx+\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)$$

$$\Big\downarrow \text{ 1492}$$

$$\frac{1}{6}\int \frac{-\left((d-2f)x^2\right)+5d-f}{x^4+x^2+1}dx+\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \text{ 1483}$$

$$\frac{1}{6}\left(\frac{1}{2}\int \frac{5d-f-3(2d-f)x}{x^2-x+1}dx+\frac{1}{2}\int \frac{5d-f+3(2d-f)x}{x^2+x+1}dx\right)+\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \text{ 1142}$$

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{1}{2}(4d+f)\int \frac{1}{x^2-x+1}dx-\frac{3}{2}(2d-f)\int -\frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(4d+f)\int \frac{1}{x^2+x+1}dx+\frac{3}{2}(2d-f\right.\right.$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \text{ 25}$$

3.32. $\int \frac{d+ex+fx^2}{(1+x^2+x^4)^2}\,dx$

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{1}{2}(4d+f)\int\frac{1}{x^2-x+1}dx+\frac{3}{2}(2d-f)\int\frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(4d+f)\int\frac{1}{x^2+x+1}dx+\frac{3}{2}(2d-f)\int\right.\right.$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \text{ 1083}$$

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{3}{2}(2d-f)\int\frac{1-2x}{x^2-x+1}dx-(4d+f)\int\frac{1}{-(2x-1)^2-3}d(2x-1)\right)+\frac{1}{2}\left(\frac{3}{2}(2d-f)\int\frac{2x+1}{x^2+x+1}dx-(4\right.\right.$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \text{ 217}$$

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{3}{2}(2d-f)\int\frac{1-2x}{x^2-x+1}dx+\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(4d+f)}{\sqrt{3}}\right)+\frac{1}{2}\left(\frac{3}{2}(2d-f)\int\frac{2x+1}{x^2+x+1}dx+\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right.}{\sqrt{3}}\right.\right.$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \text{ 1103}$$

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(4d+f)}{\sqrt{3}}-\frac{3}{2}(2d-f)\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(4d+f)}{\sqrt{3}}+\frac{3}{2}(2d-f)\log\left(x^2\right.\right.\right.$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{6\left(x^4+x^2+1\right)}$$

input
```
Int[(d + e*x + f*x^2)/(1 + x^2 + x^4)^2,x]
```

output
```
(x*(d + f - (d - 2*f)*x^2))/(6*(1 + x^2 + x^4)) + (e*((1 + 2*x^2)/(3*(1 +
x^2 + x^4)) + (4*ArcTan[(1 + 2*x^2)/Sqrt[3]])/(3*Sqrt[3])))/2 + ((((4*d +
f)*ArcTan[(-1 + 2*x)/Sqrt[3]])/Sqrt[3] - (3*(2*d - f)*Log[1 - x + x^2])/2)
/2 + (((4*d + f)*ArcTan[(1 + 2*x)/Sqrt[3]])/Sqrt[3] + (3*(2*d - f)*Log[1 +
 x + x^2])/2)/2)/6
```

3.32. $\int \frac{d+ex+fx^2}{(1+x^2+x^4)^2}\,dx$

3.32.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 217
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1086
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] - Simp[2*c*((2*p +
 3)/((p + 1)*(b^2 - 4*a*c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; Fre
eQ[{a, b, c}, x] && ILtQ[p, -1]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 1483
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[a/c, 2]}, With[{r = Rt[2*q - b/c, 2]}, Simp[1/(2*c*q*r)   In
t[(d*r - (d - e*q)*x)/(q - r*x + x^2), x], x] + Simp[1/(2*c*q*r)   Int[(d*r
 + (d - e*q)*x)/(q + r*x + x^2), x], x]]] /; FreeQ[{a, b, c, d, e}, x] && N
eQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NegQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.32.4 Maple [A] (verified)

Time = 0.27 (sec) , antiderivative size = 154, normalized size of antiderivative = 0.93

method	result
default	$-\frac{\left(\frac{d}{3}-\frac{e}{3}-\frac{2f}{3}\right)x-\frac{2d}{3}-\frac{e}{3}+\frac{f}{3}}{4(x^2-x+1)}-\frac{(6d-3f)\ln\left(x^2-x+1\right)}{24}-\frac{\left(-2d-4e-\frac{f}{2}\right)\sqrt{3}\arctan\left(\frac{(2x-1)\sqrt{3}}{3}\right)}{18}+\frac{\left(-\frac{d}{3}-\frac{e}{3}+\frac{2f}{3}\right)x-\frac{2d}{3}+\frac{e}{3}+\frac{f}{3}}{4x^2+4x+4}$
risch	Expression too large to display

input
```
int((f*x^2+e*x+d)/(x^4+x^2+1)^2,x,method=_RETURNVERBOSE)
```

output
```
-1/4*((1/3*d-1/3*e-2/3*f)*x-2/3*d-1/3*e+1/3*f)/(x^2-x+1)-1/24*(6*d-3*f)*ln
(x^2-x+1)-1/18*(-2*d-4*e-1/2*f)*3^(1/2)*arctan(1/3*(2*x-1)*3^(1/2))+1/4*((
-1/3*d-1/3*e+2/3*f)*x-2/3*d+1/3*e+1/3*f)/(x^2+x+1)+1/24*(6*d-3*f)*ln(x^2+x
+1)+1/18*(2*d-4*e+1/2*f)*arctan(1/3*(1+2*x)*3^(1/2))*3^(1/2)
```

3.32.5 Fricas [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 212, normalized size of antiderivative = 1.28

$$\int \frac{d+ex+fx^2}{(1+x^2+x^4)^2}\,dx =$$
$$-\frac{12\,(d-2\,f)x^3-24\,ex^2-2\,\sqrt{3}((4\,d-8\,e+f)x^4+(4\,d-8\,e+f)x^2+4\,d-8\,e+f)\arctan\left(\frac{1}{3}\,\sqrt{3}(2\,x$$

input
```
integrate((f*x^2+e*x+d)/(x^4+x^2+1)^2,x, algorithm="fricas")
```

output
```
-1/72*(12*(d - 2*f)*x^3 - 24*e*x^2 - 2*sqrt(3)*((4*d - 8*e + f)*x^4 + (4*d
 - 8*e + f)*x^2 + 4*d - 8*e + f)*arctan(1/3*sqrt(3)*(2*x + 1)) - 2*sqrt(3)
*((4*d + 8*e + f)*x^4 + (4*d + 8*e + f)*x^2 + 4*d + 8*e + f)*arctan(1/3*sq
rt(3)*(2*x - 1)) - 12*(d + f)*x - 9*((2*d - f)*x^4 + (2*d - f)*x^2 + 2*d -
 f)*log(x^2 + x + 1) + 9*((2*d - f)*x^4 + (2*d - f)*x^2 + 2*d - f)*log(x^2
 - x + 1) - 12*e)/(x^4 + x^2 + 1)
```

3.32.6 Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 81.03 (sec) , antiderivative size = 4106, normalized size of antiderivative = 24.88

$$\int \frac{d+ex+fx^2}{(1+x^2+x^4)^2}\,dx = \text{Too large to display}$$

input
```
integrate((f*x**2+e*x+d)/(x**4+x**2+1)**2,x)
```

output
```
(-d/4 + f/8 - sqrt(3)*I*(4*d + 8*e + f)/72)*log(x + (-164944*d**5*e + 1641
6*d**5*(-d/4 + f/8 - sqrt(3)*I*(4*d + 8*e + f)/72) + 336520*d**4*e*f + 200
664*d**4*f*(-d/4 + f/8 - sqrt(3)*I*(4*d + 8*e + f)/72) - 115200*d**3*e**3
- 504576*d**3*e**2*(-d/4 + f/8 - sqrt(3)*I*(4*d + 8*e + f)/72) - 272380*d*
*3*e*f**2 + 1734912*d**3*e*(-d/4 + f/8 - sqrt(3)*I*(4*d + 8*e + f)/72)**2
- 229500*d**3*f**2*(-d/4 + f/8 - sqrt(3)*I*(4*d + 8*e + f)/72) + 2612736*d
**3*(-d/4 + f/8 - sqrt(3)*I*(4*d + 8*e + f)/72)**3 + 51840*d**2*e**3*f + 8
81280*d**2*e**2*f*(-d/4 + f/8 - sqrt(3)*I*(4*d + 8*e + f)/72) + 119420*d**
2*e*f**3 - 2477952*d**2*e*f*(-d/4 + f/8 - sqrt(3)*I*(4*d + 8*e + f)/72)**2
 + 50436*d**2*f**3*(-d/4 + f/8 - sqrt(3)*I*(4*d + 8*e + f)/72) - 2519424*d
**2*f*(-d/4 + f/8 - sqrt(3)*I*(4*d + 8*e + f)/72)**3 + 28672*d*e**5 + 1843
20*d*e**4*(-d/4 + f/8 - sqrt(3)*I*(4*d + 8*e + f)/72) + 8640*d*e**3*f**2 +
 774144*d*e**3*(-d/4 + f/8 - sqrt(3)*I*(4*d + 8*e + f)/72)**2 - 409536*d*e
**2*f**2*(-d/4 + f/8 - sqrt(3)*I*(4*d + 8*e + f)/72) + 4976640*d*e**2*(-d/
4 + f/8 - sqrt(3)*I*(4*d + 8*e + f)/72)**3 - 31040*d*e*f**4 + 1270080*d*e*
f**2*(-d/4 + f/8 - sqrt(3)*I*(4*d + 8*e + f)/72)**2 + 14040*d*f**4*(-d/4 +
 f/8 - sqrt(3)*I*(4*d + 8*e + f)/72) + 139968*d*f**2*(-d/4 + f/8 - sqrt(3)
*I*(4*d + 8*e + f)/72)**3 - 20480*e**5*f - 36864*e**4*f*(-d/4 + f/8 - sqrt
(3)*I*(4*d + 8*e + f)/72) - 2880*e**3*f**3 - 552960*e**3*f*(-d/4 + f/8 - s
qrt(3)*I*(4*d + 8*e + f)/72)**2 + 70848*e**2*f**3*(-d/4 + f/8 - sqrt(3)...
```

3.32.7 Maxima [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 120, normalized size of antiderivative = 0.73

$$\begin{aligned}\int \frac{d+ex+fx^2}{(1+x^2+x^4)^2}\,dx &= \frac{1}{36}\sqrt{3}(4\,d-8\,e+f)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x+1)\right)\\&\quad+\frac{1}{36}\sqrt{3}(4\,d+8\,e+f)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x-1)\right)\\&\quad+\frac{1}{8}\,(2\,d-f)\log\left(x^2+x+1\right)-\frac{1}{8}\,(2\,d-f)\log\left(x^2-x+1\right)\\&\quad-\frac{(d-2\,f)x^3-2\,ex^2-(d+f)x-e}{6\,(x^4+x^2+1)}\end{aligned}$$

input
```
integrate((f*x^2+e*x+d)/(x^4+x^2+1)^2,x, algorithm="maxima")
```

output
```
1/36*sqrt(3)*(4*d - 8*e + f)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/36*sqrt(3)*
(4*d + 8*e + f)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/8*(2*d - f)*log(x^2 + x
+ 1) - 1/8*(2*d - f)*log(x^2 - x + 1) - 1/6*((d - 2*f)*x^3 - 2*e*x^2 - (d
+ f)*x - e)/(x^4 + x^2 + 1)
```

3.32.8 Giac [A] (verification not implemented)

Time = 0.32 (sec) , antiderivative size = 124, normalized size of antiderivative = 0.75

$$\int \frac{d+ex+fx^2}{(1+x^2+x^4)^2}\,dx = \frac{1}{36}\sqrt{3}(4\,d-8\,e+f)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x+1)\right) + \frac{1}{36}\sqrt{3}(4\,d+8\,e+f)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x-1)\right) + \frac{1}{8}(2\,d-f)\log\left(x^2+x+1\right) - \frac{1}{8}(2\,d-f)\log\left(x^2-x+1\right) - \frac{dx^3-2\,fx^3-2\,ex^2-dx-fx-e}{6\,(x^4+x^2+1)}$$

input
```
integrate((f*x^2+e*x+d)/(x^4+x^2+1)^2,x, algorithm="giac")
```

output
```
1/36*sqrt(3)*(4*d - 8*e + f)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/36*sqrt(3)*
(4*d + 8*e + f)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/8*(2*d - f)*log(x^2 + x
+ 1) - 1/8*(2*d - f)*log(x^2 - x + 1) - 1/6*(d*x^3 - 2*f*x^3 - 2*e*x^2 - d
*x - f*x - e)/(x^4 + x^2 + 1)
```

3.32.9 Mupad [B] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 201, normalized size of antiderivative = 1.22

$$\int \frac{d+ex+fx^2}{(1+x^2+x^4)^2}\,dx = \frac{\left(\frac{f}{3}-\frac{d}{6}\right)x^3+\frac{e\,x^2}{3}+\left(\frac{d}{6}+\frac{f}{6}\right)x+\frac{e}{6}}{x^4+x^2+1} - \ln\left(x-\frac{1}{2}-\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{d}{4}-\frac{f}{8}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{18}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{72}\right) - \ln\left(x+\frac{1}{2}-\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{f}{8}-\frac{d}{4}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{18}-\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{72}\right) + \ln\left(x-\frac{1}{2}+\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{f}{8}-\frac{d}{4}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{18}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{72}\right) + \ln\left(x+\frac{1}{2}+\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{d}{4}-\frac{f}{8}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{18}-\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{72}\right)$$

input
```
int((d + e*x + f*x^2)/(x^2 + x^4 + 1)^2,x)
```

output
```
(e/6 - x^3*(d/6 - f/3) + (e*x^2)/3 + x*(d/6 + f/6))/(x^2 + x^4 + 1) - log(
x - (3^(1/2)*1i)/2 - 1/2)*(d/4 - f/8 + (3^(1/2)*d*1i)/18 + (3^(1/2)*e*1i)/
9 + (3^(1/2)*f*1i)/72) - log(x - (3^(1/2)*1i)/2 + 1/2)*(f/8 - d/4 + (3^(1/
2)*d*1i)/18 - (3^(1/2)*e*1i)/9 + (3^(1/2)*f*1i)/72) + log(x + (3^(1/2)*1i)
/2 - 1/2)*(f/8 - d/4 + (3^(1/2)*d*1i)/18 + (3^(1/2)*e*1i)/9 + (3^(1/2)*f*1
i)/72) + log(x + (3^(1/2)*1i)/2 + 1/2)*(d/4 - f/8 + (3^(1/2)*d*1i)/18 - (3
^(1/2)*e*1i)/9 + (3^(1/2)*f*1i)/72)
```

3.32. $\int \frac{d+ex+fx^2}{(1+x^2+x^4)^2}\,dx$

3.33 $\int \frac{d+ex+fx^2+gx^3}{(1+x^2+x^4)^2}\,dx$

3.33.1 Optimal result

Integrand size = 26, antiderivative size = 179

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3}{(1+x^2+x^4)^2}\,dx &= \frac{x(d+f-(d-2f)x^2)}{6(1+x^2+x^4)}+\frac{e-2g+(2e-g)x^2}{6(1+x^2+x^4)}\\ &\quad -\frac{(4d+f)\arctan\left(\frac{1-2x}{\sqrt{3}}\right)}{12\sqrt{3}}+\frac{(4d+f)\arctan\left(\frac{1+2x}{\sqrt{3}}\right)}{12\sqrt{3}}\\ &\quad +\frac{(2e-g)\arctan\left(\frac{1+2x^2}{\sqrt{3}}\right)}{3\sqrt{3}}-\frac{1}{8}(2d-f)\log\left(1-x+x^2\right)\\ &\quad +\frac{1}{8}(2d-f)\log\left(1+x+x^2\right)\end{aligned}$$

output
```
1/6*x*(d+f-(d-2*f)*x^2)/(x^4+x^2+1)+1/6*(e-2*g+(2*e-g)*x^2)/(x^4+x^2+1)-1/
8*(2*d-f)*ln(x^2-x+1)+1/8*(2*d-f)*ln(x^2+x+1)-1/36*(4*d+f)*arctan(1/3*(1-2
*x)*3^(1/2))*3^(1/2)+1/36*(4*d+f)*arctan(1/3*(1+2*x)*3^(1/2))*3^(1/2)+1/9*
(2*e-g)*arctan(1/3*(2*x^2+1)*3^(1/2))*3^(1/2)
```

3.33.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.27 (sec) , antiderivative size = 200, normalized size of antiderivative = 1.12

$$\int \frac{d+ex+fx^2+gx^3}{(1+x^2+x^4)^2}\,dx = \frac{1}{36}\left(\frac{6(e+2ex^2-g(2+x^2)+x(d+f-dx^2+2fx^2))}{1+x^2+x^4} - \frac{\left(\left(-11i+\sqrt{3}\right)d-2\left(-2i+\sqrt{3}\right)f\right)\arctan\left(\frac{1}{2}\left(-i+\sqrt{3}\right)x\right)}{\sqrt{\frac{1}{6}\left(1+i\sqrt{3}\right)}} - \frac{\left(\left(11i+\sqrt{3}\right)d-2\left(2i+\sqrt{3}\right)f\right)\arctan\left(\frac{1}{2}\left(i+\sqrt{3}\right)x\right)}{\sqrt{\frac{1}{6}\left(1-i\sqrt{3}\right)}} - 4\sqrt{3}(2e-g)\arctan\left(\frac{\sqrt{3}}{1+2x^2}\right)\right)$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3)/(1 + x^2 + x^4)^2,x]
```

output
```
((6*(e + 2*e*x^2 - g*(2 + x^2) + x*(d + f - d*x^2 + 2*f*x^2)))/(1 + x^2 +
x^4) - (((-11*I + Sqrt[3])*d - 2*(-2*I + Sqrt[3])*f)*ArcTan[((-I + Sqrt[3]
)*x)/2])/Sqrt[(1 + I*Sqrt[3])/6] - (((11*I + Sqrt[3])*d - 2*(2*I + Sqrt[3]
)*f)*ArcTan[((I + Sqrt[3])*x)/2])/Sqrt[(1 - I*Sqrt[3])/6] - 4*Sqrt[3]*(2*e
 - g)*ArcTan[Sqrt[3]/(1 + 2*x^2)])/36
```

3.33.3 Rubi [A] (verified)

Time = 0.49 (sec) , antiderivative size = 193, normalized size of antiderivative = 1.08, number of steps used = 13, number of rules used = 12, $\frac{\text{number of rules}}{\text{integrand size}} = 0.462$, Rules used = {2202, 1492, 1483, 1142, 25, 1083, 217, 1103, 1576, 1159, 1083, 217}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3}{(x^4+x^2+1)^2}\,dx$$

$$\Big\downarrow \; 2202$$

$$\int \frac{fx^2+d}{\left(x^4+x^2+1\right)^2}dx+\int \frac{x\left(gx^2+e\right)}{\left(x^4+x^2+1\right)^2}dx$$

$$\Big\downarrow \text{ 1492}$$

$$\frac{1}{6}\int \frac{-\left((d-2f)x^2\right)+5d-f}{x^4+x^2+1}dx+\int \frac{x\left(gx^2+e\right)}{\left(x^4+x^2+1\right)^2}dx+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \text{ 1483}$$

$$\frac{1}{6}\left(\frac{1}{2}\int \frac{5d-f-3(2d-f)x}{x^2-x+1}dx+\frac{1}{2}\int \frac{5d-f+3(2d-f)x}{x^2+x+1}dx\right)+\int \frac{x\left(gx^2+e\right)}{\left(x^4+x^2+1\right)^2}dx+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \text{ 1142}$$

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{1}{2}(4d+f)\int \frac{1}{x^2-x+1}dx-\frac{3}{2}(2d-f)\int -\frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(4d+f)\int \frac{1}{x^2+x+1}dx+\frac{3}{2}(2d-f)\right.\right.$$
$$\int \frac{x\left(gx^2+e\right)}{\left(x^4+x^2+1\right)^2}dx+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \text{ 25}$$

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{1}{2}(4d+f)\int \frac{1}{x^2-x+1}dx+\frac{3}{2}(2d-f)\int \frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(4d+f)\int \frac{1}{x^2+x+1}dx+\frac{3}{2}(2d-f)\int\right.\right.$$
$$\int \frac{x\left(gx^2+e\right)}{\left(x^4+x^2+1\right)^2}dx+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \text{ 1083}$$

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{3}{2}(2d-f)\int \frac{1-2x}{x^2-x+1}dx-(4d+f)\int \frac{1}{-(2x-1)^2-3}d(2x-1)\right)+\frac{1}{2}\left(\frac{3}{2}(2d-f)\int \frac{2x+1}{x^2+x+1}dx-(4\right.\right.$$
$$\int \frac{x\left(gx^2+e\right)}{\left(x^4+x^2+1\right)^2}dx+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \text{ 217}$$

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{3}{2}(2d-f)\int \frac{1-2x}{x^2-x+1}dx+\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(4d+f)}{\sqrt{3}}\right)+\frac{1}{2}\left(\frac{3}{2}(2d-f)\int \frac{2x+1}{x^2+x+1}dx+\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right.}{\sqrt{3}}\right.\right.$$
$$\int \frac{x\left(gx^2+e\right)}{\left(x^4+x^2+1\right)^2}dx+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \text{ 1103}$$

3.33. $\int \frac{d+ex+fx^2+gx^3}{\left(1+x^2+x^4\right)^2}\,dx$

$$\int \frac{x(gx^2+e)}{(x^4+x^2+1)^2}dx+$$
$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(4d+f)}{\sqrt{3}}-\frac{3}{2}(2d-f)\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(4d+f)}{\sqrt{3}}+\frac{3}{2}(2d-f)\log\left(x^2\right.\right.\right.$$
$$\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \; 1576$$

$$\frac{1}{2}\int \frac{gx^2+e}{(x^4+x^2+1)^2}dx^2+$$
$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(4d+f)}{\sqrt{3}}-\frac{3}{2}(2d-f)\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(4d+f)}{\sqrt{3}}+\frac{3}{2}(2d-f)\log\left(x^2\right.\right.\right.$$
$$\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \; 1159$$

$$\frac{1}{2}\left(\frac{1}{3}(2e-g)\int \frac{1}{x^4+x^2+1}dx^2+\frac{x^2(2e-g)+e-2g}{3\left(x^4+x^2+1\right)}\right)+$$
$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(4d+f)}{\sqrt{3}}-\frac{3}{2}(2d-f)\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(4d+f)}{\sqrt{3}}+\frac{3}{2}(2d-f)\log\left(x^2\right.\right.\right.$$
$$\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \; 1083$$

$$\frac{1}{2}\left(\frac{x^2(2e-g)+e-2g}{3\left(x^4+x^2+1\right)}-\frac{2}{3}(2e-g)\int \frac{1}{-x^4-3}d\left(2x^2+1\right)\right)+$$
$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(4d+f)}{\sqrt{3}}-\frac{3}{2}(2d-f)\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(4d+f)}{\sqrt{3}}+\frac{3}{2}(2d-f)\log\left(x^2\right.\right.\right.$$
$$\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \; 217$$

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(4d+f)}{\sqrt{3}}-\frac{3}{2}(2d-f)\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(4d+f)}{\sqrt{3}}+\frac{3}{2}(2d-f)\log\left(x^2\right.\right.\right.$$
$$\frac{1}{2}\left(\frac{2\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g)}{3\sqrt{3}}+\frac{x^2(2e-g)+e-2g}{3\left(x^4+x^2+1\right)}\right)+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{6\left(x^4+x^2+1\right)}$$

3.33. $\int \frac{d+ex+fx^2+gx^3}{(1+x^2+x^4)^2}\,dx$

input
```
Int[(d + e*x + f*x^2 + g*x^3)/(1 + x^2 + x^4)^2,x]
```

output
```
(x*(d + f - (d - 2*f)*x^2))/(6*(1 + x^2 + x^4)) + ((e - 2*g + (2*e - g)*x^
2)/(3*(1 + x^2 + x^4)) + (2*(2*e - g)*ArcTan[(1 + 2*x^2)/Sqrt[3]])/(3*Sqrt
[3]))/2 + ((((4*d + f)*ArcTan[(-1 + 2*x)/Sqrt[3]])/Sqrt[3] - (3*(2*d - f)*
Log[1 - x + x^2])/2)/2 + (((4*d + f)*ArcTan[(1 + 2*x)/Sqrt[3]])/Sqrt[3] +
(3*(2*d - f)*Log[1 + x + x^2])/2)/2)/6
```

3.33.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 217
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1159
```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[((b*d - 2*a*e + (2*c*d - b*e)*x)/((p + 1)*(b^2 - 4*a*c)))*(a + b*
x + c*x^2)^(p + 1), x] - Simp[(2*p + 3)*((2*c*d - b*e)/((p + 1)*(b^2 - 4*a*
c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; FreeQ[{a, b, c, d, e}, x] &
& LtQ[p, -1] && NeQ[p, -3/2]
```

rule 1483
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[a/c, 2]}, With[{r = Rt[2*q - b/c, 2]}, Simp[1/(2*c*q*r)   In
t[(d*r - (d - e*q)*x)/(q - r*x + x^2), x], x] + Simp[1/(2*c*q*r)   Int[(d*r
 + (d - e*q)*x)/(q + r*x + x^2), x], x]]] /; FreeQ[{a, b, c, d, e}, x] && N
eQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NegQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.33.4 Maple [A] (verified)

Time = 0.28 (sec) , antiderivative size = 172, normalized size of antiderivative = 0.96

method	result
default	$-\frac{\left(\frac{d}{3}-\frac{e}{3}-\frac{g}{3}-\frac{2f}{3}\right)x-\frac{2d}{3}-\frac{e}{3}+\frac{2g}{3}+\frac{f}{3}}{4(x^2-x+1)}-\frac{(6d-3f)\ln\left(x^2-x+1\right)}{24}-\frac{\left(-2d-4e-\frac{f}{2}+2g\right)\sqrt{3}\,\arctan\left(\frac{(2x-1)\sqrt{3}}{3}\right)}{18}+\frac{\left(-\frac{d}{3}-\frac{e}{3}-\frac{g}{3}+\frac{2f}{3}\right.}{4x^2+}$
risch	Expression too large to display

input
```
int((g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^2,x,method=_RETURNVERBOSE)
```

output
```
-1/4*((1/3*d-1/3*e-1/3*g-2/3*f)*x-2/3*d-1/3*e+2/3*g+1/3*f)/(x^2-x+1)-1/24*
(6*d-3*f)*ln(x^2-x+1)-1/18*(-2*d-4*e-1/2*f+2*g)*3^(1/2)*arctan(1/3*(2*x-1)
*3^(1/2))+1/4*((-1/3*d-1/3*e-1/3*g+2/3*f)*x-2/3*d+1/3*e-2/3*g+1/3*f)/(x^2+
x+1)+1/24*(6*d-3*f)*ln(x^2+x+1)+1/18*(2*d-4*e+1/2*f+2*g)*arctan(1/3*(1+2*x
)*3^(1/2))*3^(1/2)
```

3.33.5 Fricas [A] (verification not implemented)

Time = 0.46 (sec) , antiderivative size = 239, normalized size of antiderivative = 1.34

$$\int \frac{d+ex+fx^2+gx^3}{(1+x^2+x^4)^2}\,dx = -\frac{12\,(d-2\,f)x^3-12\,(2\,e-g)x^2-2\,\sqrt{3}((4\,d-8\,e+f+4\,g)x^4+(4\,d-8\,e+f+4\,g)x^2+4\,d-8\,e+f}{}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^2,x, algorithm="fricas")
```

output
```
-1/72*(12*(d - 2*f)*x^3 - 12*(2*e - g)*x^2 - 2*sqrt(3)*((4*d - 8*e + f + 4
*g)*x^4 + (4*d - 8*e + f + 4*g)*x^2 + 4*d - 8*e + f + 4*g)*arctan(1/3*sqrt
(3)*(2*x + 1)) - 2*sqrt(3)*((4*d + 8*e + f - 4*g)*x^4 + (4*d + 8*e + f - 4
*g)*x^2 + 4*d + 8*e + f - 4*g)*arctan(1/3*sqrt(3)*(2*x - 1)) - 12*(d + f)*
x - 9*((2*d - f)*x^4 + (2*d - f)*x^2 + 2*d - f)*log(x^2 + x + 1) + 9*((2*d
 - f)*x^4 + (2*d - f)*x^2 + 2*d - f)*log(x^2 - x + 1) - 12*e + 24*g)/(x^4
+ x^2 + 1)
```

3.33.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3}{(1+x^2+x^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((g*x**3+f*x**2+e*x+d)/(x**4+x**2+1)**2,x)
```

output
```
Timed out
```

3.33.7 Maxima [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 135, normalized size of antiderivative = 0.75

$$\int \frac{d+ex+fx^2+gx^3}{(1+x^2+x^4)^2}\,dx = \frac{1}{36}\sqrt{3}(4d-8e+f+4g)\arctan\left(\frac{1}{3}\sqrt{3}(2x+1)\right) + \frac{1}{36}\sqrt{3}(4d+8e+f-4g)\arctan\left(\frac{1}{3}\sqrt{3}(2x-1)\right) + \frac{1}{8}(2d-f)\log\left(x^2+x+1\right) - \frac{1}{8}(2d-f)\log\left(x^2-x+1\right) - \frac{(d-2f)x^3-(2e-g)x^2-(d+f)x-e+2g}{6(x^4+x^2+1)}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^2,x, algorithm="maxima")
```

output
```
1/36*sqrt(3)*(4*d - 8*e + f + 4*g)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/36*sq
rt(3)*(4*d + 8*e + f - 4*g)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/8*(2*d - f)*
log(x^2 + x + 1) - 1/8*(2*d - f)*log(x^2 - x + 1) - 1/6*((d - 2*f)*x^3 - (
2*e - g)*x^2 - (d + f)*x - e + 2*g)/(x^4 + x^2 + 1)
```

3.33.8 Giac [A] (verification not implemented)

Time = 0.33 (sec) , antiderivative size = 138, normalized size of antiderivative = 0.77

$$\int \frac{d+ex+fx^2+gx^3}{(1+x^2+x^4)^2}\,dx = \frac{1}{36}\sqrt{3}(4d-8e+f+4g)\arctan\left(\frac{1}{3}\sqrt{3}(2x+1)\right) + \frac{1}{36}\sqrt{3}(4d+8e+f-4g)\arctan\left(\frac{1}{3}\sqrt{3}(2x-1)\right) + \frac{1}{8}(2d-f)\log\left(x^2+x+1\right) - \frac{1}{8}(2d-f)\log\left(x^2-x+1\right) - \frac{dx^3-2fx^3-2ex^2+gx^2-dx-fx-e+2g}{6(x^4+x^2+1)}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^2,x, algorithm="giac")
```

output
```
1/36*sqrt(3)*(4*d - 8*e + f + 4*g)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/36*sq
rt(3)*(4*d + 8*e + f - 4*g)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/8*(2*d - f)*
log(x^2 + x + 1) - 1/8*(2*d - f)*log(x^2 - x + 1) - 1/6*(d*x^3 - 2*f*x^3 -
 2*e*x^2 + g*x^2 - d*x - f*x - e + 2*g)/(x^4 + x^2 + 1)
```

3.33.9 Mupad [B] (verification not implemented)

Time = 8.14 (sec) , antiderivative size = 237, normalized size of antiderivative = 1.32

$$\int \frac{d+ex+fx^2+gx^3}{(1+x^2+x^4)^2}\,dx = \frac{\left(\frac{f}{3}-\frac{d}{6}\right)x^3+\left(\frac{e}{3}-\frac{g}{6}\right)x^2+\left(\frac{d}{6}+\frac{f}{6}\right)x+\frac{e}{6}-\frac{g}{3}}{x^4+x^2+1} - \ln\left(x-\frac{1}{2}-\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{d}{4}-\frac{f}{8}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{18}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{72}-\frac{\sqrt{3}\,g\,1\mathrm{i}}{18}\right) - \ln\left(x+\frac{1}{2}-\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{f}{8}-\frac{d}{4}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{18}-\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{72}+\frac{\sqrt{3}\,g\,1\mathrm{i}}{18}\right) + \ln\left(x-\frac{1}{2}+\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{f}{8}-\frac{d}{4}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{18}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{72}-\frac{\sqrt{3}\,g\,1\mathrm{i}}{18}\right) + \ln\left(x+\frac{1}{2}+\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{d}{4}-\frac{f}{8}+\frac{\sqrt{3}\,d\,1\mathrm{i}}{18}-\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{72}+\frac{\sqrt{3}\,g\,1\mathrm{i}}{18}\right)$$

input
```
int((d + e*x + f*x^2 + g*x^3)/(x^2 + x^4 + 1)^2,x)
```

output
```
(e/6 - g/3 - x^3*(d/6 - f/3) + x^2*(e/3 - g/6) + x*(d/6 + f/6))/(x^2 + x^4
 + 1) - log(x - (3^(1/2)*1i)/2 - 1/2)*(d/4 - f/8 + (3^(1/2)*d*1i)/18 + (3^
(1/2)*e*1i)/9 + (3^(1/2)*f*1i)/72 - (3^(1/2)*g*1i)/18) - log(x - (3^(1/2)*
1i)/2 + 1/2)*(f/8 - d/4 + (3^(1/2)*d*1i)/18 - (3^(1/2)*e*1i)/9 + (3^(1/2)*
f*1i)/72 + (3^(1/2)*g*1i)/18) + log(x + (3^(1/2)*1i)/2 - 1/2)*(f/8 - d/4 +
 (3^(1/2)*d*1i)/18 + (3^(1/2)*e*1i)/9 + (3^(1/2)*f*1i)/72 - (3^(1/2)*g*1i)
/18) + log(x + (3^(1/2)*1i)/2 + 1/2)*(d/4 - f/8 + (3^(1/2)*d*1i)/18 - (3^(
1/2)*e*1i)/9 + (3^(1/2)*f*1i)/72 + (3^(1/2)*g*1i)/18)
```

3.34 $\int \frac{d+ex+fx^2+gx^3+hx^4}{(1+x^2+x^4)^2} dx$

3.34.1 Optimal result

Integrand size = 31, antiderivative size = 187

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(1+x^2+x^4)^2} dx = \frac{e-2g+(2e-g)x^2}{6(1+x^2+x^4)} + \frac{x(d+f-2h-(d-2f+h)x^2)}{6(1+x^2+x^4)} - \frac{(4d+f+h)\arctan\left(\frac{1-2x}{\sqrt{3}}\right)}{12\sqrt{3}} + \frac{(4d+f+h)\arctan\left(\frac{1+2x}{\sqrt{3}}\right)}{12\sqrt{3}} + \frac{(2e-g)\arctan\left(\frac{1+2x^2}{\sqrt{3}}\right)}{3\sqrt{3}} - \frac{1}{8}(2d-f+h)\log\left(1-x+x^2\right) + \frac{1}{8}(2d-f+h)\log\left(1+x+x^2\right)$$

output
```
1/6*(e-2*g+(2*e-g)*x^2)/(x^4+x^2+1)+1/6*x*(d+f-2*h-(d-2*f+h)*x^2)/(x^4+x^2
+1)-1/8*(2*d-f+h)*ln(x^2-x+1)+1/8*(2*d-f+h)*ln(x^2+x+1)-1/36*(4*d+f+h)*arc
tan(1/3*(1-2*x)*3^(1/2))*3^(1/2)+1/36*(4*d+f+h)*arctan(1/3*(1+2*x)*3^(1/2)
)*3^(1/2)+1/9*(2*e-g)*arctan(1/3*(2*x^2+1)*3^(1/2))*3^(1/2)
```

3.34.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.39 (sec) , antiderivative size = 234, normalized size of antiderivative = 1.25

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(1+x^2+x^4)^2}\,dx$$
$$= \frac{1}{36}\left(-\frac{6(g(2+x^2)-e(1+2x^2)+x(d(-1+x^2)+h(2+x^2)-f(1+2x^2)))}{1+x^2+x^4}\right.$$
$$-\frac{\left(\left(-11i+\sqrt{3}\right)d-2\left(-2i+\sqrt{3}\right)f+\left(-5i+\sqrt{3}\right)h\right)\arctan\left(\frac{1}{2}\left(-i+\sqrt{3}\right)x\right)}{\sqrt{\frac{1}{6}\left(1+i\sqrt{3}\right)}}$$
$$-\frac{\left(\left(11i+\sqrt{3}\right)d-2\left(2i+\sqrt{3}\right)f+\left(5i+\sqrt{3}\right)h\right)\arctan\left(\frac{1}{2}\left(i+\sqrt{3}\right)x\right)}{\sqrt{\frac{1}{6}\left(1-i\sqrt{3}\right)}}$$
$$\left.-4\sqrt{3}(2e-g)\arctan\left(\frac{\sqrt{3}}{1+2x^2}\right)\right)$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4)/(1 + x^2 + x^4)^2,x]
```

output
```
((-6*(g*(2 + x^2) - e*(1 + 2*x^2) + x*(d*(-1 + x^2) + h*(2 + x^2) - f*(1 +
 2*x^2))))/(1 + x^2 + x^4) - (((-11*I + Sqrt[3])*d - 2*(-2*I + Sqrt[3])*f
+ (-5*I + Sqrt[3])*h)*ArcTan[((-I + Sqrt[3])*x)/2])/Sqrt[(1 + I*Sqrt[3])/6
] - (((11*I + Sqrt[3])*d - 2*(2*I + Sqrt[3])*f + (5*I + Sqrt[3])*h)*ArcTan
[((I + Sqrt[3])*x)/2])/Sqrt[(1 - I*Sqrt[3])/6] - 4*Sqrt[3]*(2*e - g)*ArcTa
n[Sqrt[3]/(1 + 2*x^2)])/36
```

3.34.3 Rubi [A] (verified)

Time = 0.51 (sec) , antiderivative size = 201, normalized size of antiderivative = 1.07, number of steps used = 13, number of rules used = 12, $\frac{\text{number of rules}}{\text{integrand size}} = 0.387$, Rules used = {2202, 1576, 1159, 1083, 217, 2206, 1483, 1142, 25, 1083, 217, 1103}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(x^4+x^2+1)^2}\,dx$$

$$\Big\downarrow \; 2202$$

$$\int \frac{hx^4+fx^2+d}{(x^4+x^2+1)^2}dx+\int \frac{x(gx^2+e)}{(x^4+x^2+1)^2}dx$$

$$\Big\downarrow \; 1576$$

$$\int \frac{hx^4+fx^2+d}{(x^4+x^2+1)^2}dx+\frac{1}{2}\int \frac{gx^2+e}{(x^4+x^2+1)^2}dx^2$$

$$\Big\downarrow \; 1159$$

$$\int \frac{hx^4+fx^2+d}{(x^4+x^2+1)^2}dx+\frac{1}{2}\left(\frac{1}{3}(2e-g)\int \frac{1}{x^4+x^2+1}dx^2+\frac{x^2(2e-g)+e-2g}{3(x^4+x^2+1)}\right)$$

$$\Big\downarrow \; 1083$$

$$\int \frac{hx^4+fx^2+d}{(x^4+x^2+1)^2}dx+\frac{1}{2}\left(\frac{x^2(2e-g)+e-2g}{3(x^4+x^2+1)}-\frac{2}{3}(2e-g)\int \frac{1}{-x^4-3}d(2x^2+1)\right)$$

$$\Big\downarrow \; 217$$

$$\int \frac{hx^4+fx^2+d}{(x^4+x^2+1)^2}dx+\frac{1}{2}\left(\frac{2\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g)}{3\sqrt{3}}+\frac{x^2(2e-g)+e-2g}{3(x^4+x^2+1)}\right)$$

$$\Big\downarrow \; 2206$$

$$\frac{1}{6}\int \frac{-\left((d-2f+h)x^2\right)+5d-f+2h}{x^4+x^2+1}dx+$$
$$\frac{1}{2}\left(\frac{2\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g)}{3\sqrt{3}}+\frac{x^2(2e-g)+e-2g}{3(x^4+x^2+1)}\right)+\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{6(x^4+x^2+1)}$$

$$\Big\downarrow \; 1483$$

$$\frac{1}{6}\left(\frac{1}{2}\int \frac{5d-f+2h-3(2d-f+h)x}{x^2-x+1}dx+\frac{1}{2}\int \frac{5d-f+2h+3(2d-f+h)x}{x^2+x+1}dx\right)+$$
$$\frac{1}{2}\left(\frac{2\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g)}{3\sqrt{3}}+\frac{x^2(2e-g)+e-2g}{3(x^4+x^2+1)}\right)+\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{6(x^4+x^2+1)}$$

$$\Big\downarrow \; 1142$$

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{1}{2}(4d+f+h)\int \frac{1}{x^2-x+1}dx-\frac{3}{2}(2d-f+h)\int -\frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(4d+f+h)\int \frac{1}{x^2+x+1}dx\right.\right.$$
$$\frac{1}{2}\left(\frac{2\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g)}{3\sqrt{3}}+\frac{x^2(2e-g)+e-2g}{3(x^4+x^2+1)}\right)+\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{6(x^4+x^2+1)}$$

3.34. $\int \frac{d+ex+fx^2+gx^3+hx^4}{(1+x^2+x^4)^2}\,dx$

$\downarrow$ 25

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{1}{2}(4d+f+h)\int\frac{1}{x^2-x+1}dx+\frac{3}{2}(2d-f+h)\int\frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(4d+f+h)\int\frac{1}{x^2+x+1}dx+\right.\right.$$
$$\frac{1}{2}\left(\frac{2\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g)}{3\sqrt{3}}+\frac{x^2(2e-g)+e-2g}{3\left(x^4+x^2+1\right)}\right)+\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{6\left(x^4+x^2+1\right)}$$

$\downarrow$ 1083

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{3}{2}(2d-f+h)\int\frac{1-2x}{x^2-x+1}dx-(4d+f+h)\int\frac{1}{-(2x-1)^2-3}d(2x-1)\right)+\frac{1}{2}\left(\frac{3}{2}(2d-f+h)\int\frac{2x}{x^2+}\right.\right.$$
$$\frac{1}{2}\left(\frac{2\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g)}{3\sqrt{3}}+\frac{x^2(2e-g)+e-2g}{3\left(x^4+x^2+1\right)}\right)+\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{6\left(x^4+x^2+1\right)}$$

$\downarrow$ 217

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{3}{2}(2d-f+h)\int\frac{1-2x}{x^2-x+1}dx+\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(4d+f+h)}{\sqrt{3}}\right)+\frac{1}{2}\left(\frac{3}{2}(2d-f+h)\int\frac{2x+1}{x^2+x+1}dx+\right.\right.$$
$$\frac{1}{2}\left(\frac{2\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g)}{3\sqrt{3}}+\frac{x^2(2e-g)+e-2g}{3\left(x^4+x^2+1\right)}\right)+\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{6\left(x^4+x^2+1\right)}$$

$\downarrow$ 1103

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(4d+f+h)}{\sqrt{3}}-\frac{3}{2}\log\left(x^2-x+1\right)(2d-f+h)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(4d+f+h)}{\sqrt{3}}+\frac{3}{2}\text{lo}\right.\right.$$
$$\frac{1}{2}\left(\frac{2\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g)}{3\sqrt{3}}+\frac{x^2(2e-g)+e-2g}{3\left(x^4+x^2+1\right)}\right)+\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{6\left(x^4+x^2+1\right)}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4)/(1 + x^2 + x^4)^2,x]
```

output
```
(x*(d + f - 2*h - (d - 2*f + h)*x^2))/(6*(1 + x^2 + x^4)) + ((e - 2*g + (2
*e - g)*x^2)/(3*(1 + x^2 + x^4)) + (2*(2*e - g)*ArcTan[(1 + 2*x^2)/Sqrt[3]
])/(3*Sqrt[3]))/2 + ((((4*d + f + h)*ArcTan[(-1 + 2*x)/Sqrt[3]])/Sqrt[3] -
 (3*(2*d - f + h)*Log[1 - x + x^2])/2)/2 + (((4*d + f + h)*ArcTan[(1 + 2*x
)/Sqrt[3]])/Sqrt[3] + (3*(2*d - f + h)*Log[1 + x + x^2])/2)/2)/6
```

3.34. $\int \frac{d+ex+fx^2+gx^3+hx^4}{(1+x^2+x^4)^2}\,dx$

3.34.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 217
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1159
```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[((b*d - 2*a*e + (2*c*d - b*e)*x)/((p + 1)*(b^2 - 4*a*c)))*(a + b*
x + c*x^2)^(p + 1), x] - Simp[(2*p + 3)*((2*c*d - b*e)/((p + 1)*(b^2 - 4*a*
c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; FreeQ[{a, b, c, d, e}, x] &
& LtQ[p, -1] && NeQ[p, -3/2]
```

rule 1483
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[a/c, 2]}, With[{r = Rt[2*q - b/c, 2]}, Simp[1/(2*c*q*r)   In
t[(d*r - (d - e*q)*x)/(q - r*x + x^2), x], x] + Simp[1/(2*c*q*r)   Int[(d*r
 + (d - e*q)*x)/(q + r*x + x^2), x], x]]] /; FreeQ[{a, b, c, d, e}, x] && N
eQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NegQ[b^2 - 4*a*c]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

3.34. $\int \frac{d+ex+fx^2+gx^3+hx^4}{(1+x^2+x^4)^2} dx$

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2206
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> With[{d =
 Coeff[PolynomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 0], e = Coeff[Poly
nomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 2]}, Simp[x*(a + b*x^2 + c*x^
4)^(p + 1)*((a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)/(2*a*(p + 1)*(b
^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(a + b*x^2 + c
*x^4)^(p + 1)*ExpandToSum[2*a*(p + 1)*(b^2 - 4*a*c)*PolynomialQuotient[Px,
a + b*x^2 + c*x^4, x] + b^2*d*(2*p + 3) - 2*a*c*d*(4*p + 5) - a*b*e + c*(4*
p + 7)*(b*d - 2*a*e)*x^2, x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x
^2] && Expon[Px, x^2] > 1 && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1]
```

3.34.4 Maple [A] (verified)

Time = 0.41 (sec) , antiderivative size = 196, normalized size of antiderivative = 1.05

method	result
default	$-\frac{\left(\frac{d}{3}-\frac{2f}{3}-\frac{g}{3}-\frac{e}{3}+\frac{h}{3}\right)x-\frac{2d}{3}+\frac{f}{3}+\frac{2g}{3}-\frac{e}{3}+\frac{h}{3}}{4(x^2-x+1)}-\frac{(6d-3f+3h)\ln(x^2-x+1)}{24}-\frac{\left(-2d-4e-\frac{f}{2}+2g-\frac{h}{2}\right)\sqrt{3}\,\arctan\left(\frac{(2x-1)\sqrt{3}}{3}\right)}{18}+$
risch	Expression too large to display

input
```
int((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^2,x,method=_RETURNVERBOSE)
```

output
```
-1/4*((1/3*d-2/3*f-1/3*g-1/3*e+1/3*h)*x-2/3*d+1/3*f+2/3*g-1/3*e+1/3*h)/(x^
2-x+1)-1/24*(6*d-3*f+3*h)*ln(x^2-x+1)-1/18*(-2*d-4*e-1/2*f+2*g-1/2*h)*3^(1
/2)*arctan(1/3*(2*x-1)*3^(1/2))+1/4*((-1/3*d+2/3*f-1/3*g-1/3*e-1/3*h)*x-2/
3*d+1/3*f-2/3*g+1/3*e+1/3*h)/(x^2+x+1)+1/24*(6*d-3*f+3*h)*ln(x^2+x+1)+1/18
*(2*d-4*e+1/2*f+2*g+1/2*h)*arctan(1/3*(1+2*x)*3^(1/2))*3^(1/2)
```

3.34.5 Fricas [A] (verification not implemented)

Time = 1.10 (sec) , antiderivative size = 255, normalized size of antiderivative = 1.36

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(1+x^2+x^4)^2}\,dx =$$
$$-\frac{12\,(d-2\,f+h)x^3-12\,(2\,e-g)x^2-2\,\sqrt{3}((4\,d-8\,e+f+4\,g+h)x^4+(4\,d-8\,e+f+4\,g+h)x^2+$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^2,x, algorithm="fricas")
```

output
```
-1/72*(12*(d - 2*f + h)*x^3 - 12*(2*e - g)*x^2 - 2*sqrt(3)*((4*d - 8*e + f
 + 4*g + h)*x^4 + (4*d - 8*e + f + 4*g + h)*x^2 + 4*d - 8*e + f + 4*g + h)
*arctan(1/3*sqrt(3)*(2*x + 1)) - 2*sqrt(3)*((4*d + 8*e + f - 4*g + h)*x^4
+ (4*d + 8*e + f - 4*g + h)*x^2 + 4*d + 8*e + f - 4*g + h)*arctan(1/3*sqrt
(3)*(2*x - 1)) - 12*(d + f - 2*h)*x - 9*((2*d - f + h)*x^4 + (2*d - f + h)
*x^2 + 2*d - f + h)*log(x^2 + x + 1) + 9*((2*d - f + h)*x^4 + (2*d - f + h
)*x^2 + 2*d - f + h)*log(x^2 - x + 1) - 12*e + 24*g)/(x^4 + x^2 + 1)
```

3.34.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(1+x^2+x^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((h*x**4+g*x**3+f*x**2+e*x+d)/(x**4+x**2+1)**2,x)
```

output
```
Timed out
```

3.34.7 Maxima [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 143, normalized size of antiderivative = 0.76

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(1+x^2+x^4)^2}\,dx$$
$$= \frac{1}{36}\sqrt{3}(4\,d-8\,e+f+4\,g+h)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x+1)\right) + \frac{1}{36}\sqrt{3}(4\,d+8\,e+f-4\,g+h)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x-1)\right) + \frac{1}{8}(2\,d-f+h)\log\left(x^2+x+1\right) - \frac{1}{8}(2\,d-f+h)\log\left(x^2-x+1\right) - \frac{(d-2\,f+h)x^3-(2\,e-g)x^2-(d+f-2\,h)x-e+2\,g}{6\,(x^4+x^2+1)}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^2,x, algorithm="maxima")
```

output
```
1/36*sqrt(3)*(4*d - 8*e + f + 4*g + h)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/3
6*sqrt(3)*(4*d + 8*e + f - 4*g + h)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/8*(2
*d - f + h)*log(x^2 + x + 1) - 1/8*(2*d - f + h)*log(x^2 - x + 1) - 1/6*((
d - 2*f + h)*x^3 - (2*e - g)*x^2 - (d + f - 2*h)*x - e + 2*g)/(x^4 + x^2 +
 1)
```

3.34.8 Giac [A] (verification not implemented)

Time = 0.42 (sec) , antiderivative size = 151, normalized size of antiderivative = 0.81

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(1+x^2+x^4)^2}\,dx$$
$$= \frac{1}{36}\sqrt{3}(4\,d-8\,e+f+4\,g+h)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x+1)\right) + \frac{1}{36}\sqrt{3}(4\,d+8\,e+f-4\,g+h)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x-1)\right) + \frac{1}{8}(2\,d-f+h)\log\left(x^2+x+1\right) - \frac{1}{8}(2\,d-f+h)\log\left(x^2-x+1\right) - \frac{dx^3-2\,fx^3+hx^3-2\,ex^2+gx^2-dx-fx+2\,hx-e+2\,g}{6\,(x^4+x^2+1)}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^2,x, algorithm="giac")
```

output
```
1/36*sqrt(3)*(4*d - 8*e + f + 4*g + h)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/3
6*sqrt(3)*(4*d + 8*e + f - 4*g + h)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/8*(2
*d - f + h)*log(x^2 + x + 1) - 1/8*(2*d - f + h)*log(x^2 - x + 1) - 1/6*(d
*x^3 - 2*f*x^3 + h*x^3 - 2*e*x^2 + g*x^2 - d*x - f*x + 2*h*x - e + 2*g)/(x
^4 + x^2 + 1)
```

3.34.9 Mupad [B] (verification not implemented)

Time = 3.31 (sec) , antiderivative size = 1547, normalized size of antiderivative = 8.27

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4}{(1 + x^2 + x^4)^2} dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4)/(x^2 + x^4 + 1)^2,x)
```

output
```
(e/6 - g/3 + x^2*(e/3 - g/6) + x*(d/6 + f/6 - h/3) - x^3*(d/6 - f/3 + h/6)
)/(x^2 + x^4 + 1) - log(60*d*g - 153*d*f - 120*d*e + 24*e*f + 135*d*h - 48
*e*h - 12*f*g - 81*f*h + 24*g*h + 3^(1/2)*d^2*90i + 3^(1/2)*f^2*9i + 3^(1/
2)*h^2*18i - 198*d^2*x - 36*f^2*x - 45*h^2*x + 126*d^2 + 45*f^2 + 36*h^2 +
 3^(1/2)*d*e*56i - 3^(1/2)*d*f*63i - 3^(1/2)*d*g*28i - 3^(1/2)*e*f*40i + 3
^(1/2)*d*h*81i + 3^(1/2)*e*h*32i + 3^(1/2)*f*g*20i - 3^(1/2)*f*h*27i - 3^(
1/2)*g*h*16i - 24*d*e*x + 171*d*f*x + 12*d*g*x + 48*e*f*x - 189*d*h*x - 24
*e*h*x - 24*f*g*x + 81*f*h*x + 12*g*h*x + 3^(1/2)*d^2*x*18i + 3^(1/2)*f^2*
x*18i + 3^(1/2)*h^2*x*9i - 3^(1/2)*d*f*x*45i + 3^(1/2)*d*g*x*44i + 3^(1/2)
*e*f*x*32i + 3^(1/2)*d*h*x*27i - 3^(1/2)*e*h*x*40i - 3^(1/2)*f*g*x*16i - 3
^(1/2)*f*h*x*27i + 3^(1/2)*g*h*x*20i - 3^(1/2)*d*e*x*88i)*(d/4 - f/8 + h/8
 + (3^(1/2)*d*1i)/18 + (3^(1/2)*e*1i)/9 + (3^(1/2)*f*1i)/72 - (3^(1/2)*g*1
i)/18 + (3^(1/2)*h*1i)/72) - log(120*d*e - 153*d*f - 60*d*g - 24*e*f + 135
*d*h + 48*e*h + 12*f*g - 81*f*h - 24*g*h - 3^(1/2)*d^2*90i - 3^(1/2)*f^2*9
i - 3^(1/2)*h^2*18i + 198*d^2*x + 36*f^2*x + 45*h^2*x + 126*d^2 + 45*f^2 +
 36*h^2 + 3^(1/2)*d*e*56i + 3^(1/2)*d*f*63i - 3^(1/2)*d*g*28i - 3^(1/2)*e*
f*40i - 3^(1/2)*d*h*81i + 3^(1/2)*e*h*32i + 3^(1/2)*f*g*20i + 3^(1/2)*f*h*
27i - 3^(1/2)*g*h*16i - 24*d*e*x - 171*d*f*x + 12*d*g*x + 48*e*f*x + 189*d
*h*x - 24*e*h*x - 24*f*g*x - 81*f*h*x + 12*g*h*x + 3^(1/2)*d^2*x*18i + 3^(
1/2)*f^2*x*18i + 3^(1/2)*h^2*x*9i - 3^(1/2)*d*f*x*45i - 3^(1/2)*d*g*x*4...
```

3.35 $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^2} dx$

3.35.1 Optimal result

Integrand size = 36, antiderivative size = 194

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^2} dx &= \frac{x(d+f-2h-(d-2f+h)x^2)}{6(1+x^2+x^4)} \\ &+ \frac{e-2g+i+(2e-g-i)x^2}{6(1+x^2+x^4)} \\ &- \frac{(4d+f+h)\arctan\left(\frac{1-2x}{\sqrt{3}}\right)}{12\sqrt{3}} \\ &+ \frac{(4d+f+h)\arctan\left(\frac{1+2x}{\sqrt{3}}\right)}{12\sqrt{3}} \\ &+ \frac{(2e-g+2i)\arctan\left(\frac{1+2x^2}{\sqrt{3}}\right)}{3\sqrt{3}} \\ &- \frac{1}{8}(2d-f+h)\log\left(1-x+x^2\right) \\ &+ \frac{1}{8}(2d-f+h)\log\left(1+x+x^2\right)\end{aligned}$$

output
```
1/6*x*(d+f-2*h-(d-2*f+h)*x^2)/(x^4+x^2+1)+1/6*(e-2*g+i+(2*e-g-i)*x^2)/(x^4
+x^2+1)-1/8*(2*d-f+h)*ln(x^2-x+1)+1/8*(2*d-f+h)*ln(x^2+x+1)-1/36*(4*d+f+h)
*arctan(1/3*(1-2*x)*3^(1/2))*3^(1/2)+1/36*(4*d+f+h)*arctan(1/3*(1+2*x)*3^(
1/2))*3^(1/2)+1/9*(2*e-g+2*i)*arctan(1/3*(2*x^2+1)*3^(1/2))*3^(1/2)
```

3.35.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.39 (sec) , antiderivative size = 243, normalized size of antiderivative = 1.25

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^2}\,dx = \frac{1}{36}\left(\frac{6(e+i+dx+fx-2hx+2ex^2-ix^2-dx^3+2fx^3-hx^3-g(2+x^2))}{1+x^2+x^4} - \frac{\left(\left(-11i+\sqrt{3}\right)d-2\left(-2i+\sqrt{3}\right)f+\left(-5i+\sqrt{3}\right)h\right)\arctan\left(\frac{1}{2}\left(-i+\sqrt{3}\right)x\right)}{\sqrt{\frac{1}{6}\left(1+i\sqrt{3}\right)}} - \frac{\left(\left(11i+\sqrt{3}\right)d-2\left(2i+\sqrt{3}\right)f+\left(5i+\sqrt{3}\right)h\right)\arctan\left(\frac{1}{2}\left(i+\sqrt{3}\right)x\right)}{\sqrt{\frac{1}{6}\left(1-i\sqrt{3}\right)}} - 4\sqrt{3}(2e-g+2i)\arctan\left(\frac{\sqrt{3}}{1+2x^2}\right)\right)$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(1 + x^2 + x^4)^2,x]
```

output
```
((6*(e + i + d*x + f*x - 2*h*x + 2*e*x^2 - i*x^2 - d*x^3 + 2*f*x^3 - h*x^3
 - g*(2 + x^2)))/(1 + x^2 + x^4) - (((-11*I + Sqrt[3])*d - 2*(-2*I + Sqrt[
3])*f + (-5*I + Sqrt[3])*h)*ArcTan[((-I + Sqrt[3])*x)/2])/Sqrt[(1 + I*Sqrt
[3])/6] - (((11*I + Sqrt[3])*d - 2*(2*I + Sqrt[3])*f + (5*I + Sqrt[3])*h)*
ArcTan[((I + Sqrt[3])*x)/2])/Sqrt[(1 - I*Sqrt[3])/6] - 4*Sqrt[3]*(2*e - g
+ 2*i)*ArcTan[Sqrt[3]/(1 + 2*x^2)])/36
```

3.35.3 Rubi [A] (verified)

Time = 0.58 (sec) , antiderivative size = 208, normalized size of antiderivative = 1.07, number of steps used = 14, number of rules used = 13, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.361, Rules used = {2202, 2194, 2191, 27, 1083, 217, 2206, 1483, 1142, 25, 1083, 217, 1103}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(x^4+x^2+1)^2}\,dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{hx^4+fx^2+d}{(x^4+x^2+1)^2}dx+\int \frac{x(ix^4+gx^2+e)}{(x^4+x^2+1)^2}dx$$

$$\Big\downarrow \text{2194}$$

$$\int \frac{hx^4+fx^2+d}{(x^4+x^2+1)^2}dx+\frac{1}{2}\int \frac{ix^4+gx^2+e}{(x^4+x^2+1)^2}dx^2$$

$$\Big\downarrow \text{2191}$$

$$\int \frac{hx^4+fx^2+d}{(x^4+x^2+1)^2}dx+\frac{1}{2}\left(\frac{1}{3}\int \frac{2e-g+2i}{x^4+x^2+1}dx^2+\frac{x^2(2e-g-i)+e-2g+i}{3\,(x^4+x^2+1)}\right)$$

$$\Big\downarrow \text{27}$$

$$\int \frac{hx^4+fx^2+d}{(x^4+x^2+1)^2}dx+\frac{1}{2}\left(\frac{1}{3}(2e-g+2i)\int \frac{1}{x^4+x^2+1}dx^2+\frac{x^2(2e-g-i)+e-2g+i}{3\,(x^4+x^2+1)}\right)$$

$$\Big\downarrow \text{1083}$$

$$\int \frac{hx^4+fx^2+d}{(x^4+x^2+1)^2}dx+\frac{1}{2}\left(\frac{x^2(2e-g-i)+e-2g+i}{3\,(x^4+x^2+1)}-\frac{2}{3}(2e-g+2i)\int \frac{1}{-x^4-3}d(2x^2+1)\right)$$

$$\Big\downarrow \text{217}$$

$$\int \frac{hx^4+fx^2+d}{(x^4+x^2+1)^2}dx+\frac{1}{2}\left(\frac{2\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g+2i)}{3\sqrt{3}}+\frac{x^2(2e-g-i)+e-2g+i}{3\,(x^4+x^2+1)}\right)$$

$$\Big\downarrow \text{2206}$$

$$\frac{1}{6}\int \frac{-\left((d-2f+h)x^2\right)+5d-f+2h}{x^4+x^2+1}dx+\frac{1}{2}\left(\frac{2\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g+2i)}{3\sqrt{3}}+\frac{x^2(2e-g-i)+e-2g+i}{3\,(x^4+x^2+1)}\right)+\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{6\,(x^4+x^2+1)}$$

$$\Big\downarrow \text{1483}$$

$$\frac{1}{6}\left(\frac{1}{2}\int \frac{5d-f+2h-3(2d-f+h)x}{x^2-x+1}dx+\frac{1}{2}\int \frac{5d-f+2h+3(2d-f+h)x}{x^2+x+1}dx\right)+\frac{1}{2}\left(\frac{2\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g+2i)}{3\sqrt{3}}+\frac{x^2(2e-g-i)+e-2g+i}{3\,(x^4+x^2+1)}\right)+\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{6\,(x^4+x^2+1)}$$

3.35. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^2}\,dx$

$$\Big\downarrow \text{1142}$$

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{1}{2}(4d+f+h)\int \frac{1}{x^2-x+1}dx-\frac{3}{2}(2d-f+h)\int -\frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(4d+f+h)\int \frac{1}{x^2+x+1}dx\right.\right.$$
$$\frac{1}{2}\left(\frac{2\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g+2i)}{3\sqrt{3}}+\frac{x^2(2e-g-i)+e-2g+i}{3\left(x^4+x^2+1\right)}\right)+$$
$$\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \text{25}$$

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{1}{2}(4d+f+h)\int \frac{1}{x^2-x+1}dx+\frac{3}{2}(2d-f+h)\int \frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(4d+f+h)\int \frac{1}{x^2+x+1}dx+\right.\right.$$
$$\frac{1}{2}\left(\frac{2\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g+2i)}{3\sqrt{3}}+\frac{x^2(2e-g-i)+e-2g+i}{3\left(x^4+x^2+1\right)}\right)+$$
$$\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \text{1083}$$

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{3}{2}(2d-f+h)\int \frac{1-2x}{x^2-x+1}dx-(4d+f+h)\int \frac{1}{-(2x-1)^2-3}d(2x-1)\right)+\frac{1}{2}\left(\frac{3}{2}(2d-f+h)\int \frac{2x}{x^2+}\right.\right.$$
$$\frac{1}{2}\left(\frac{2\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g+2i)}{3\sqrt{3}}+\frac{x^2(2e-g-i)+e-2g+i}{3\left(x^4+x^2+1\right)}\right)+$$
$$\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \text{217}$$

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{3}{2}(2d-f+h)\int \frac{1-2x}{x^2-x+1}dx+\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(4d+f+h)}{\sqrt{3}}\right)+\frac{1}{2}\left(\frac{3}{2}(2d-f+h)\int \frac{2x+1}{x^2+x+1}dx+\right.\right.$$
$$\frac{1}{2}\left(\frac{2\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g+2i)}{3\sqrt{3}}+\frac{x^2(2e-g-i)+e-2g+i}{3\left(x^4+x^2+1\right)}\right)+$$
$$\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{6\left(x^4+x^2+1\right)}$$

$$\Big\downarrow \text{1103}$$

3.35. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{\left(1+x^2+x^4\right)^2}\,dx$

$$\frac{1}{6}\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(4d+f+h)}{\sqrt{3}}-\frac{3}{2}\log\left(x^2-x+1\right)(2d-f+h)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(4d+f+h)}{\sqrt{3}}+\frac{3}{2}\text{lo}\right.\right.$$
$$\frac{1}{2}\left(\frac{2\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)(2e-g+2i)}{3\sqrt{3}}+\frac{x^2(2e-g-i)+e-2g+i}{3\left(x^4+x^2+1\right)}\right)+$$
$$\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{6\left(x^4+x^2+1\right)}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(1 + x^2 + x^4)^2,x]
```

output
```
(x*(d + f - 2*h - (d - 2*f + h)*x^2))/(6*(1 + x^2 + x^4)) + ((e - 2*g + i
+ (2*e - g - i)*x^2)/(3*(1 + x^2 + x^4)) + (2*(2*e - g + 2*i)*ArcTan[(1 +
2*x^2)/Sqrt[3]])/(3*Sqrt[3]))/2 + ((((4*d + f + h)*ArcTan[(-1 + 2*x)/Sqrt[
3]])/Sqrt[3] - (3*(2*d - f + h)*Log[1 - x + x^2])/2)/2 + (((4*d + f + h)*A
rcTan[(1 + 2*x)/Sqrt[3]])/Sqrt[3] + (3*(2*d - f + h)*Log[1 + x + x^2])/2)/
2)/6
```

3.35.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 217
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1483
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[a/c, 2]}, With[{r = Rt[2*q - b/c, 2]}, Simp[1/(2*c*q*r)   In
t[(d*r - (d - e*q)*x)/(q - r*x + x^2), x], x] + Simp[1/(2*c*q*r)   Int[(d*r
 + (d - e*q)*x)/(q + r*x + x^2), x], x]]] /; FreeQ[{a, b, c, d, e}, x] && N
eQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NegQ[b^2 - 4*a*c]
```

rule 2191
```
Int[(Pq_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> With[{Q =
PolynomialQuotient[Pq, a + b*x + c*x^2, x], f = Coeff[PolynomialRemainder[P
q, a + b*x + c*x^2, x], x, 0], g = Coeff[PolynomialRemainder[Pq, a + b*x +
c*x^2, x], x, 1]}, Simp[(b*f - 2*a*g + (2*c*f - b*g)*x)*((a + b*x + c*x^2)^
(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] + Simp[1/((p + 1)*(b^2 - 4*a*c))   Int
[(a + b*x + c*x^2)^(p + 1)*ExpandToSum[(p + 1)*(b^2 - 4*a*c)*Q - (2*p + 3)*
(2*c*f - b*g), x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Pq, x] && NeQ[b^
2 - 4*a*c, 0] && LtQ[p, -1]
```

rule 2194
```
Int[(Pq_)*(x_)^(m_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :
> Simp[1/2   Subst[Int[x^((m - 1)/2)*SubstFor[x^2, Pq, x]*(a + b*x + c*x^2)
^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pq, x^2] && IntegerQ
[(m - 1)/2]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.35. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^2}\,dx$

rule 2206
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> With[{d =
 Coeff[PolynomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 0], e = Coeff[Poly
nomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 2]}, Simp[x*(a + b*x^2 + c*x^
4)^(p + 1)*((a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)/(2*a*(p + 1)*(b
^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(a + b*x^2 + c
*x^4)^(p + 1)*ExpandToSum[2*a*(p + 1)*(b^2 - 4*a*c)*PolynomialQuotient[Px,
a + b*x^2 + c*x^4, x] + b^2*d*(2*p + 3) - 2*a*c*d*(4*p + 5) - a*b*e + c*(4*
p + 7)*(b*d - 2*a*e)*x^2, x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x
^2] && Expon[Px, x^2] > 1 && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1]
```

3.35.4 Maple [A] (verified)

Time = 0.51 (sec) , antiderivative size = 214, normalized size of antiderivative = 1.10

method	result
default	$-\frac{\left(\frac{d}{3}-\frac{e}{3}-\frac{g}{3}+\frac{h}{3}-\frac{2f}{3}+\frac{2i}{3}\right)x-\frac{2d}{3}-\frac{e}{3}+\frac{2g}{3}+\frac{h}{3}+\frac{f}{3}-\frac{i}{3}}{4(x^2-x+1)}-\frac{(6d-3f+3h)\ln(x^2-x+1)}{24}-\frac{\left(-2d-4e-\frac{f}{2}+2g-\frac{h}{2}-4i\right)\sqrt{3}\arctan\left(\frac{(2x-1}{3}\right.}{18}$
risch	Expression too large to display

input
```
int((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^2,x,method=_RETURNVERBOSE)
```

output
```
-1/4*((1/3*d-1/3*e-1/3*g+1/3*h-2/3*f+2/3*i)*x-2/3*d-1/3*e+2/3*g+1/3*h+1/3*
f-1/3*i)/(x^2-x+1)-1/24*(6*d-3*f+3*h)*ln(x^2-x+1)-1/18*(-2*d-4*e-1/2*f+2*g
-1/2*h-4*i)*3^(1/2)*arctan(1/3*(2*x-1)*3^(1/2))+1/4*((-1/3*d-1/3*e-1/3*g-1
/3*h+2/3*f+2/3*i)*x-2/3*d+1/3*e-2/3*g+1/3*h+1/3*f+1/3*i)/(x^2+x+1)+1/24*(6
*d-3*f+3*h)*ln(x^2+x+1)+1/18*(2*d-4*e+1/2*f+2*g+1/2*h-4*i)*arctan(1/3*(1+2
*x)*3^(1/2))*3^(1/2)
```

3.35.5 Fricas [A] (verification not implemented)

Time = 4.87 (sec) , antiderivative size = 279, normalized size of antiderivative = 1.44

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^2}\,dx =$$
$$-\frac{12\,(d-2\,f+h)x^3-12\,(2\,e-g-i)x^2-2\,\sqrt{3}((4\,d-8\,e+f+4\,g+h-8\,i)x^4+(4\,d-8\,e+f+4\,g}{}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^2,x, algorithm="fric
as")
```

output
```
-1/72*(12*(d - 2*f + h)*x^3 - 12*(2*e - g - i)*x^2 - 2*sqrt(3)*((4*d - 8*e
 + f + 4*g + h - 8*i)*x^4 + (4*d - 8*e + f + 4*g + h - 8*i)*x^2 + 4*d - 8*
e + f + 4*g + h - 8*i)*arctan(1/3*sqrt(3)*(2*x + 1)) - 2*sqrt(3)*((4*d + 8
*e + f - 4*g + h + 8*i)*x^4 + (4*d + 8*e + f - 4*g + h + 8*i)*x^2 + 4*d +
8*e + f - 4*g + h + 8*i)*arctan(1/3*sqrt(3)*(2*x - 1)) - 12*(d + f - 2*h)*
x - 9*((2*d - f + h)*x^4 + (2*d - f + h)*x^2 + 2*d - f + h)*log(x^2 + x +
1) + 9*((2*d - f + h)*x^4 + (2*d - f + h)*x^2 + 2*d - f + h)*log(x^2 - x +
 1) - 12*e + 24*g - 12*i)/(x^4 + x^2 + 1)
```

3.35.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((i*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(x**4+x**2+1)**2,x)
```

output
```
Timed out
```

3.35.7 Maxima [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 155, normalized size of antiderivative = 0.80

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^2}\,dx &= \frac{1}{36}\sqrt{3}(4\,d-8\,e+f+4\,g+h-8\,i)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x+1)\right)\\ &\quad+\frac{1}{36}\sqrt{3}(4\,d+8\,e+f-4\,g+h+8\,i)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x-1)\right)\\ &\quad+\frac{1}{8}\,(2\,d-f+h)\log\left(x^2+x+1\right)-\frac{1}{8}\,(2\,d-f+h)\log\left(x^2-x+1\right)\\ &\quad-\frac{(d-2\,f+h)x^3-(2\,e-g-i)x^2-(d+f-2\,h)x-e+2\,g-i}{6\,(x^4+x^2+1)}\end{aligned}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^2,x, algorithm="maxi
ma")
```

output
```
1/36*sqrt(3)*(4*d - 8*e + f + 4*g + h - 8*i)*arctan(1/3*sqrt(3)*(2*x + 1))
 + 1/36*sqrt(3)*(4*d + 8*e + f - 4*g + h + 8*i)*arctan(1/3*sqrt(3)*(2*x -
1)) + 1/8*(2*d - f + h)*log(x^2 + x + 1) - 1/8*(2*d - f + h)*log(x^2 - x +
 1) - 1/6*((d - 2*f + h)*x^3 - (2*e - g - i)*x^2 - (d + f - 2*h)*x - e + 2
*g - i)/(x^4 + x^2 + 1)
```

3.35.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 165, normalized size of antiderivative = 0.85

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^2}\,dx = \frac{1}{36}\sqrt{3}(4\,d-8\,e+f+4\,g+h-8\,i)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x+1)\right) + \frac{1}{36}\sqrt{3}(4\,d+8\,e+f-4\,g+h+8\,i)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x-1)\right) + \frac{1}{8}\,(2\,d-f+h)\log\left(x^2+x+1\right) - \frac{1}{8}\,(2\,d-f+h)\log\left(x^2-x+1\right) - \frac{dx^3-2\,fx^3+hx^3-2\,ex^2+gx^2+ix^2-dx-fx+2\,hx-e+2\,g-i}{6\left(x^4+x^2+1\right)}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^2,x, algorithm="giac
")
```

output
```
1/36*sqrt(3)*(4*d - 8*e + f + 4*g + h - 8*i)*arctan(1/3*sqrt(3)*(2*x + 1))
 + 1/36*sqrt(3)*(4*d + 8*e + f - 4*g + h + 8*i)*arctan(1/3*sqrt(3)*(2*x -
1)) + 1/8*(2*d - f + h)*log(x^2 + x + 1) - 1/8*(2*d - f + h)*log(x^2 - x +
 1) - 1/6*(d*x^3 - 2*f*x^3 + h*x^3 - 2*e*x^2 + g*x^2 + i*x^2 - d*x - f*x +
 2*h*x - e + 2*g - i)/(x^4 + x^2 + 1)
```

3.35.9 Mupad [B] (verification not implemented)

Time = 13.24 (sec) , antiderivative size = 1894, normalized size of antiderivative = 9.76

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^2}\,dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(x^2 + x^4 + 1)^2,x)
```

output
```
(e/6 - g/3 + i/6 + x*(d/6 + f/6 - h/3) - x^3*(d/6 - f/3 + h/6) - x^2*(g/6
- e/3 + i/6))/(x^2 + x^4 + 1) - log(60*d*g - 153*d*f - 120*d*e + 24*e*f +
135*d*h - 120*d*i - 48*e*h - 12*f*g - 81*f*h + 24*f*i + 24*g*h - 48*h*i +
3^(1/2)*d^2*90i + 3^(1/2)*f^2*9i + 3^(1/2)*h^2*18i - 198*d^2*x - 36*f^2*x
- 45*h^2*x + 126*d^2 + 45*f^2 + 36*h^2 + 3^(1/2)*d*e*56i - 3^(1/2)*d*f*63i
 - 3^(1/2)*d*g*28i - 3^(1/2)*e*f*40i + 3^(1/2)*d*h*81i + 3^(1/2)*d*i*56i +
 3^(1/2)*e*h*32i + 3^(1/2)*f*g*20i - 3^(1/2)*f*h*27i - 3^(1/2)*f*i*40i - 3
^(1/2)*g*h*16i + 3^(1/2)*h*i*32i - 24*d*e*x + 171*d*f*x + 12*d*g*x + 48*e*
f*x - 189*d*h*x - 24*d*i*x - 24*e*h*x - 24*f*g*x + 81*f*h*x + 48*f*i*x + 1
2*g*h*x - 24*h*i*x + 3^(1/2)*d^2*x*18i + 3^(1/2)*f^2*x*18i + 3^(1/2)*h^2*x
*9i - 3^(1/2)*d*f*x*45i + 3^(1/2)*d*g*x*44i + 3^(1/2)*e*f*x*32i + 3^(1/2)*
d*h*x*27i - 3^(1/2)*d*i*x*88i - 3^(1/2)*e*h*x*40i - 3^(1/2)*f*g*x*16i - 3^
(1/2)*f*h*x*27i + 3^(1/2)*f*i*x*32i + 3^(1/2)*g*h*x*20i - 3^(1/2)*h*i*x*40
i - 3^(1/2)*d*e*x*88i)*(d/4 - f/8 + h/8 + (3^(1/2)*d*1i)/18 + (3^(1/2)*e*1
i)/9 + (3^(1/2)*f*1i)/72 - (3^(1/2)*g*1i)/18 + (3^(1/2)*h*1i)/72 + (3^(1/2
)*i*1i)/9) - log(120*d*e - 153*d*f - 60*d*g - 24*e*f + 135*d*h + 120*d*i +
 48*e*h + 12*f*g - 81*f*h - 24*f*i - 24*g*h + 48*h*i - 3^(1/2)*d^2*90i - 3
^(1/2)*f^2*9i - 3^(1/2)*h^2*18i + 198*d^2*x + 36*f^2*x + 45*h^2*x + 126*d^
2 + 45*f^2 + 36*h^2 + 3^(1/2)*d*e*56i + 3^(1/2)*d*f*63i - 3^(1/2)*d*g*28i
- 3^(1/2)*e*f*40i - 3^(1/2)*d*h*81i + 3^(1/2)*d*i*56i + 3^(1/2)*e*h*32i...
```

3.35. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^2}\,dx$

3.36 $\int \frac{d+ex}{(a+bx^2+cx^4)^2} dx$

3.36.1 Optimal result

Integrand size = 20, antiderivative size = 330

$$\begin{aligned}\int \frac{d+ex}{(a+bx^2+cx^4)^2} dx =& -\frac{e(b+2cx^2)}{2(b^2-4ac)(a+bx^2+cx^4)} + \frac{dx(b^2-2ac+bcx^2)}{2a(b^2-4ac)(a+bx^2+cx^4)} \\ &+ \frac{\sqrt{c}\left(b^2-12ac+b\sqrt{b^2-4ac}\right) d\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{2\sqrt{2}a(b^2-4ac)^{3/2}\sqrt{b-\sqrt{b^2-4ac}}} \\ &- \frac{\sqrt{c}\left(b^2-12ac-b\sqrt{b^2-4ac}\right) d\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{2\sqrt{2}a(b^2-4ac)^{3/2}\sqrt{b+\sqrt{b^2-4ac}}} \\ &+ \frac{2ce\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}\end{aligned}$$

output
```
-1/2*e*(2*c*x^2+b)/(-4*a*c+b^2)/(c*x^4+b*x^2+a)+1/2*d*x*(b*c*x^2-2*a*c+b^2
)/a/(-4*a*c+b^2)/(c*x^4+b*x^2+a)+2*c*e*arctanh((2*c*x^2+b)/(-4*a*c+b^2)^(1
/2))/(-4*a*c+b^2)^(3/2)+1/4*d*arctan(x*2^(1/2)*c^(1/2)/(b-(-4*a*c+b^2)^(1/
2))^(1/2))*c^(1/2)*(b^2-12*a*c+b*(-4*a*c+b^2)^(1/2))/a/(-4*a*c+b^2)^(3/2)*
2^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2)-1/4*d*arctan(x*2^(1/2)*c^(1/2)/(b+(-4
*a*c+b^2)^(1/2))^(1/2))*c^(1/2)*(b^2-12*a*c-b*(-4*a*c+b^2)^(1/2))/a/(-4*a*
c+b^2)^(3/2)*2^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2)
```

3.36.2 Mathematica [A] (verified)

Time = 0.50 (sec) , antiderivative size = 341, normalized size of antiderivative = 1.03

$$\int \frac{d+ex}{(a+bx^2+cx^4)^2}\,dx = \frac{1}{4}\left(\frac{2abe+4acx(d+ex)-2bdx(b+cx^2)}{a(-b^2+4ac)(a+bx^2+cx^4)} + \frac{\sqrt{2}\sqrt{c}\left(b^2-12ac+b\sqrt{b^2-4ac}\right)d\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{a(b^2-4ac)^{3/2}\sqrt{b-\sqrt{b^2-4ac}}} + \frac{\sqrt{2}\sqrt{c}\left(-b^2+12ac+b\sqrt{b^2-4ac}\right)d\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{a(b^2-4ac)^{3/2}\sqrt{b+\sqrt{b^2-4ac}}} - \frac{4ce\log\left(-b+\sqrt{b^2-4ac}-2cx^2\right)}{(b^2-4ac)^{3/2}} + \frac{4ce\log\left(b+\sqrt{b^2-4ac}+2cx^2\right)}{(b^2-4ac)^{3/2}}\right)$$

input
```
Integrate[(d + e*x)/(a + b*x^2 + c*x^4)^2,x]
```

output
```
((2*a*b*e + 4*a*c*x*(d + e*x) - 2*b*d*x*(b + c*x^2))/(a*(-b^2 + 4*a*c)*(a
+ b*x^2 + c*x^4)) + (Sqrt[2]*Sqrt[c]*(b^2 - 12*a*c + b*Sqrt[b^2 - 4*a*c])*
d*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]])/(a*(b^2 - 4*a*c
)^(3/2)*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + (Sqrt[2]*Sqrt[c]*(-b^2 + 12*a*c + b
*Sqrt[b^2 - 4*a*c])*d*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqrt[b^2 - 4*a*c
]]])/(a*(b^2 - 4*a*c)^(3/2)*Sqrt[b + Sqrt[b^2 - 4*a*c]]) - (4*c*e*Log[-b +
 Sqrt[b^2 - 4*a*c] - 2*c*x^2])/(b^2 - 4*a*c)^(3/2) + (4*c*e*Log[b + Sqrt[b
^2 - 4*a*c] + 2*c*x^2])/(b^2 - 4*a*c)^(3/2))/4
```

3.36.3 Rubi [A] (verified)

Time = 0.62 (sec) , antiderivative size = 316, normalized size of antiderivative = 0.96, number of steps used = 11, number of rules used = 10, $\frac{\text{number of rules}}{\text{integrand size}} = 0.500$, Rules used = {2202, 27, 1405, 25, 1432, 1086, 1083, 219, 1480, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex}{(a+bx^2+cx^4)^2}\,dx$$

$$\Big\downarrow \; 2202$$

$$\int \frac{d}{(cx^4+bx^2+a)^2}dx+\int \frac{ex}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \; 27$$

$$d\int \frac{1}{(cx^4+bx^2+a)^2}dx+e\int \frac{x}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \; 1405$$

$$d\left(\frac{x(-2ac+b^2+bcx^2)}{2a(b^2-4ac)(a+bx^2+cx^4)}-\frac{\int -\frac{b^2+cx^2b-6ac}{cx^4+bx^2+a}dx}{2a(b^2-4ac)}\right)+e\int \frac{x}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \; 25$$

$$d\left(\frac{\int \frac{b^2+cx^2b-6ac}{cx^4+bx^2+a}dx}{2a(b^2-4ac)}+\frac{x(-2ac+b^2+bcx^2)}{2a(b^2-4ac)(a+bx^2+cx^4)}\right)+e\int \frac{x}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \; 1432$$

$$d\left(\frac{\int \frac{b^2+cx^2b-6ac}{cx^4+bx^2+a}dx}{2a(b^2-4ac)}+\frac{x(-2ac+b^2+bcx^2)}{2a(b^2-4ac)(a+bx^2+cx^4)}\right)+\frac{1}{2}e\int \frac{1}{(cx^4+bx^2+a)^2}dx^2$$

$$\Big\downarrow \; 1086$$

$$d\left(\frac{\int \frac{b^2+cx^2b-6ac}{cx^4+bx^2+a}dx}{2a(b^2-4ac)}+\frac{x(-2ac+b^2+bcx^2)}{2a(b^2-4ac)(a+bx^2+cx^4)}\right)+$$
$$\frac{1}{2}e\left(-\frac{2c\int \frac{1}{cx^4+bx^2+a}dx^2}{b^2-4ac}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)$$

$$\Big\downarrow \; 1083$$

$$d\left(\frac{\int \frac{b^2+cx^2b-6ac}{cx^4+bx^2+a}dx}{2a(b^2-4ac)}+\frac{x(-2ac+b^2+bcx^2)}{2a(b^2-4ac)(a+bx^2+cx^4)}\right)+$$
$$\frac{1}{2}e\left(\frac{4c\int \frac{1}{-x^4+b^2-4ac}d(2cx^2+b)}{b^2-4ac}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)$$

$$\Big\downarrow \; 219$$

3.36. $\int \frac{d+ex}{(a+bx^2+cx^4)^2}\,dx$

$$d\left(\frac{\int \frac{b^2+cx^2b-6ac}{cx^4+bx^2+a}dx}{2a\left(b^2-4ac\right)}+\frac{x\left(-2ac+b^2+bcx^2\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)+\frac{1}{2}e\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{b+2cx^2}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)$$

$\downarrow$ 1480

$$d\left(\frac{\frac{1}{2}c\left(\frac{b^2-12ac}{\sqrt{b^2-4ac}}+b\right)\int \frac{1}{cx^2+\frac{1}{2}\left(b-\sqrt{b^2-4ac}\right)}dx+\frac{1}{2}c\left(b-\frac{b^2-12ac}{\sqrt{b^2-4ac}}\right)\int \frac{1}{cx^2+\frac{1}{2}\left(b+\sqrt{b^2-4ac}\right)}dx}{2a\left(b^2-4ac\right)}+\frac{x\left(-2ac+b^2+bcx^2\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+\right.}\right.$$

$$\frac{1}{2}e\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{b+2cx^2}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)$$

$\downarrow$ 218

$$d\left(\frac{\frac{\sqrt{c}\left(\frac{b^2-12ac}{\sqrt{b^2-4ac}}+b\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\left(b-\frac{b^2-12ac}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)}{\sqrt{2}\sqrt{\sqrt{b^2-4ac}+b}}}{2a\left(b^2-4ac\right)}+\frac{x\left(-2ac+b^2+bcx^2\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)+\frac{1}{2}e\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{b+2cx^2}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)$$

input
```
Int[(d + e*x)/(a + b*x^2 + c*x^4)^2,x]
```

output
```
d*((x*(b^2 - 2*a*c + b*c*x^2))/(2*a*(b^2 - 4*a*c)*(a + b*x^2 + c*x^4)) + (
(Sqrt[c]*(b + (b^2 - 12*a*c)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)
/Sqrt[b - Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + (Sq
rt[c]*(b - (b^2 - 12*a*c)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sq
rt[b + Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[b + Sqrt[b^2 - 4*a*c]]))/(2*a*(b
^2 - 4*a*c))) + (e*(-((b + 2*c*x^2)/((b^2 - 4*a*c)*(a + b*x^2 + c*x^4))) +
 (4*c*ArcTanh[(b + 2*c*x^2)/Sqrt[b^2 - 4*a*c]])/(b^2 - 4*a*c)^(3/2)))/2
```

3.36. $\int \frac{d+ex}{(a+bx^2+cx^4)^2}\,dx$

3.36.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1086
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] - Simp[2*c*((2*p +
 3)/((p + 1)*(b^2 - 4*a*c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; Fre
eQ[{a, b, c}, x] && ILtQ[p, -1]
```

rule 1405
```
Int[((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[(-x)*(b^2
- 2*a*c + b*c*x^2)*((a + b*x^2 + c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))
), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(b^2 - 2*a*c + 2*(p + 1)*(
b^2 - 4*a*c) + b*c*(4*p + 7)*x^2)*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; Fr
eeQ[{a, b, c}, x] && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1] && IntegerQ[2*p]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.36.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.25 (sec) , antiderivative size = 209, normalized size of antiderivative = 0.63

method	result
risch	$\frac{-\frac{bcd\,x^3}{2a\left(4ac-b^2\right)}+\frac{c\,x^2e}{4ac-b^2}+\frac{\left(2ac-b^2\right)dx}{2a\left(4ac-b^2\right)}+\frac{be}{8ac-2b^2}}{c\,x^4+b\,x^2+a}+\frac{\left(\sum_{\_R=\operatorname{RootOf}\left(c\_Z^4+\_Z^2b+a\right)}\frac{\left(-\frac{c\_R^2bd}{a\left(4ac-b^2\right)}+\frac{4ce\_R}{4ac-b^2}+\frac{d\left(6ac-b^2\right)}{a\left(4ac-b^2\right)}\right)\ln\left(x-\_R\right.}{2c\_R^3+\_Rb}\right.}{4}$
default	$16c^2\left(-\frac{\frac{\frac{\left(4ac\sqrt{-4ac+b^2}-b^2\sqrt{-4ac+b^2}+4abc-b^3\right)dx}{16a\,c^2}-\frac{e\left(4ac-b^2\right)}{8c^2}}{x^2+\frac{b}{2c}-\frac{\sqrt{-4ac+b^2}}{2c}}+\frac{2ae\sqrt{-4ac+b^2}\,\ln\left(-2c\,x^2+\sqrt{-4ac+b^2}-b\right)+\frac{\left(-12acd\sqrt{-4ac+b^2}+b^2d\right.}{}}{8ac}}{4(4ac-b^2)^2}\right.$

input
```
int((e*x+d)/(c*x^4+b*x^2+a)^2,x,method=_RETURNVERBOSE)
```

output
```
(-1/2/a*b*c*d/(4*a*c-b^2)*x^3+c/(4*a*c-b^2)*x^2*e+1/2*(2*a*c-b^2)*d/a/(4*a
*c-b^2)*x+1/2/(4*a*c-b^2)*b*e)/(c*x^4+b*x^2+a)+1/4*sum((-c/a/(4*a*c-b^2)*_
R^2*b*d+4*c/(4*a*c-b^2)*e*_R+d*(6*a*c-b^2)/a/(4*a*c-b^2))/(2*_R^3*c+_R*b)*
ln(x-_R),_R=RootOf(_Z^4*c+_Z^2*b+a))
```

3.36.5 Fricas [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 235.03 (sec) , antiderivative size = 1678440, normalized size of antiderivative = 5086.18

$$\int \frac{d+ex}{(a+bx^2+cx^4)^2}\,dx = \text{Too large to display}$$

input
```
integrate((e*x+d)/(c*x^4+b*x^2+a)^2,x, algorithm="fricas")
```

output
```
Too large to include
```

3.36.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex}{(a+bx^2+cx^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((e*x+d)/(c*x**4+b*x**2+a)**2,x)
```

output
```
Timed out
```

3.36.7 Maxima [F]

$$\int \frac{d+ex}{(a+bx^2+cx^4)^2}\,dx = \int \frac{ex+d}{(cx^4+bx^2+a)^2}\,dx$$

input
```
integrate((e*x+d)/(c*x^4+b*x^2+a)^2,x, algorithm="maxima")
```

output
```
1/2*(b*c*d*x^3 - 2*a*c*e*x^2 - a*b*e + (b^2 - 2*a*c)*d*x)/((a*b^2*c - 4*a^
2*c^2)*x^4 + a^2*b^2 - 4*a^3*c + (a*b^3 - 4*a^2*b*c)*x^2) + 1/2*integrate(
(b*c*d*x^2 - 4*a*c*e*x + (b^2 - 6*a*c)*d)/(c*x^4 + b*x^2 + a), x)/(a*b^2 -
 4*a^2*c)
```

3.36.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 3426 vs. $2(278) = 556$.

Time = 1.34 (sec) , antiderivative size = 3426, normalized size of antiderivative = 10.38

$$\int \frac{d+ex}{(a+bx^2+cx^4)^2}\,dx = \text{Too large to display}$$

input
```
integrate((e*x+d)/(c*x^4+b*x^2+a)^2,x, algorithm="giac")
```

output
```
1/2*(b*c*d*x^3 - 2*a*c*e*x^2 + b^2*d*x - 2*a*c*d*x - a*b*e)/((c*x^4 + b*x^
2 + a)*(a*b^2 - 4*a^2*c)) - 1/16*((2*b^3*c^2 - 8*a*b*c^3 - sqrt(2)*sqrt(b^
2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^3 + 4*sqrt(2)*sqrt(b^2 - 4*a*
c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqr
t(b*c + sqrt(b^2 - 4*a*c)*c)*b^2*c - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c +
sqrt(b^2 - 4*a*c)*c)*b*c^2 - 2*(b^2 - 4*a*c)*b*c^2)*(a*b^2 - 4*a^2*c)^2*d
- 2*(sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^6 - 14*sqrt(2)*sqrt(b*c +
 sqrt(b^2 - 4*a*c)*c)*a^2*b^4*c - 2*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c
)*a*b^5*c - 2*a*b^6*c + 64*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*b^2
*c^2 + 20*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^3*c^2 + sqrt(2)*sq
rt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^4*c^2 + 28*a^2*b^4*c^2 - 96*sqrt(2)*sqrt
(b*c + sqrt(b^2 - 4*a*c)*c)*a^4*c^3 - 48*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a
*c)*c)*a^3*b*c^3 - 10*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^2*c^3
- 128*a^3*b^2*c^3 + 24*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*c^4 + 1
92*a^4*c^4 + 2*(b^2 - 4*a*c)*a*b^4*c - 20*(b^2 - 4*a*c)*a^2*b^2*c^2 + 48*(
b^2 - 4*a*c)*a^3*c^3)*d*abs(a*b^2 - 4*a^2*c) + (2*a^2*b^7*c^2 - 40*a^3*b^5
*c^3 + 224*a^4*b^3*c^4 - 384*a^5*b*c^5 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*
c + sqrt(b^2 - 4*a*c)*c)*a^2*b^7 + 20*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c +
 sqrt(b^2 - 4*a*c)*c)*a^3*b^5*c + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + s
qrt(b^2 - 4*a*c)*c)*a^2*b^6*c - 112*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c ...
```

3.36.9 Mupad [B] (verification not implemented)

Time = 8.37 (sec) , antiderivative size = 2382, normalized size of antiderivative = 7.22

$$\int \frac{d+ex}{(a+bx^2+cx^4)^2}\,dx = \text{Too large to display}$$

input
```
int((d + e*x)/(a + b*x^2 + c*x^4)^2,x)
```

output
```
((b*e)/(2*(4*a*c - b^2)) + (c*e*x^2)/(4*a*c - b^2) + (d*x*(2*a*c - b^2))/(
2*a*(4*a*c - b^2)) - (b*c*d*x^3)/(2*a*(4*a*c - b^2)))/(a + b*x^2 + c*x^4)
+ symsum(log((5*b^3*c^4*d^3 - 96*a^2*c^5*d*e^2 - 36*a*b*c^5*d^3 + 16*a*b^2
*c^4*d*e^2)/(8*(a^2*b^6 - 64*a^5*c^3 - 12*a^3*b^4*c + 48*a^4*b^2*c^2)) - r
oot(1572864*a^8*b^2*c^5*z^4 - 983040*a^7*b^4*c^4*z^4 + 327680*a^6*b^6*c^3*
z^4 - 61440*a^5*b^8*c^2*z^4 + 6144*a^4*b^10*c*z^4 - 1048576*a^9*c^6*z^4 -
256*a^3*b^12*z^4 + 61440*a^5*b*c^5*d^2*z^2 + 432*a*b^9*c*d^2*z^2 + 24576*a
^5*b^2*c^4*e^2*z^2 - 6144*a^4*b^4*c^3*e^2*z^2 + 512*a^3*b^6*c^2*e^2*z^2 -
61440*a^4*b^3*c^4*d^2*z^2 + 24064*a^3*b^5*c^3*d^2*z^2 - 4608*a^2*b^7*c^2*d
^2*z^2 - 32768*a^6*c^5*e^2*z^2 - 16*b^11*d^2*z^2 - 672*a*b^6*c^2*d^2*e*z -
 15872*a^3*b^2*c^4*d^2*e*z + 4992*a^2*b^4*c^3*d^2*e*z + 18432*a^4*c^5*d^2*
e*z + 32*b^8*c*d^2*e*z - 960*a^2*b*c^4*d^2*e^2 + 240*a*b^3*c^3*d^2*e^2 - 1
6*b^5*c^2*d^2*e^2 + 360*a*b^2*c^4*d^4 - 256*a^3*c^4*e^4 - 25*b^4*c^3*d^4 -
 1296*a^2*c^5*d^4, z, k)*(root(1572864*a^8*b^2*c^5*z^4 - 983040*a^7*b^4*c^
4*z^4 + 327680*a^6*b^6*c^3*z^4 - 61440*a^5*b^8*c^2*z^4 + 6144*a^4*b^10*c*z
^4 - 1048576*a^9*c^6*z^4 - 256*a^3*b^12*z^4 + 61440*a^5*b*c^5*d^2*z^2 + 43
2*a*b^9*c*d^2*z^2 + 24576*a^5*b^2*c^4*e^2*z^2 - 6144*a^4*b^4*c^3*e^2*z^2 +
 512*a^3*b^6*c^2*e^2*z^2 - 61440*a^4*b^3*c^4*d^2*z^2 + 24064*a^3*b^5*c^3*d
^2*z^2 - 4608*a^2*b^7*c^2*d^2*z^2 - 32768*a^6*c^5*e^2*z^2 - 16*b^11*d^2*z^
2 - 672*a*b^6*c^2*d^2*e*z - 15872*a^3*b^2*c^4*d^2*e*z + 4992*a^2*b^4*c^...
```

3.36. $\int \frac{d+ex}{(a+bx^2+cx^4)^2}\,dx$

3.37 $\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^2}\,dx$

3.37.1 Optimal result

Integrand size = 25, antiderivative size = 368

$$\begin{aligned}\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^2}\,dx = &-\frac{e(b+2cx^2)}{2(b^2-4ac)(a+bx^2+cx^4)} \\ &+\frac{x(b^2d-2acd-abf+c(bd-2af)x^2)}{2a(b^2-4ac)(a+bx^2+cx^4)} \\ &+\frac{\sqrt{c}\left(bd-2af+\frac{b^2d-12acd+4abf}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{2\sqrt{2}a(b^2-4ac)\sqrt{b-\sqrt{b^2-4ac}}} \\ &+\frac{\sqrt{c}\left(bd-2af-\frac{b^2d-12acd+4abf}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{2\sqrt{2}a(b^2-4ac)\sqrt{b+\sqrt{b^2-4ac}}} \\ &+\frac{2ce\,\mathrm{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}\end{aligned}$$

output
```
-1/2*e*(2*c*x^2+b)/(-4*a*c+b^2)/(c*x^4+b*x^2+a)+1/2*x*(b^2*d-2*a*c*d-a*b*f
+c*(-2*a*f+b*d)*x^2)/a/(-4*a*c+b^2)/(c*x^4+b*x^2+a)+2*c*e*arctanh((2*c*x^2
+b)/(-4*a*c+b^2)^(1/2))/(-4*a*c+b^2)^(3/2)+1/4*arctan(x*2^(1/2)*c^(1/2)/(b
-(-4*a*c+b^2)^(1/2))^(1/2))*c^(1/2)*(b*d-2*a*f+(4*a*b*f-12*a*c*d+b^2*d)/(-
4*a*c+b^2)^(1/2))/a/(-4*a*c+b^2)*2^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2)+1/4*
arctan(x*2^(1/2)*c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2))*c^(1/2)*(b*d-2*a*f+
(-4*a*b*f+12*a*c*d-b^2*d)/(-4*a*c+b^2)^(1/2))/a/(-4*a*c+b^2)*2^(1/2)/(b+(-
4*a*c+b^2)^(1/2))^(1/2)
```

3.37.2 Mathematica [A] (verified)

Time = 0.72 (sec) , antiderivative size = 398, normalized size of antiderivative = 1.08

$$\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^2}\,dx$$
$$= \frac{1}{4}\left(\frac{2ab(e+fx)-2bdx(b+cx^2)+4acx(d+x(e+fx))}{a(-b^2+4ac)(a+bx^2+cx^4)} + \frac{\sqrt{2}\sqrt{c}\left(b^2d+b\left(\sqrt{b^2-4ac}d+4af\right)-2a\left(6cd+\sqrt{b^2-4ac}f\right)\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{a(b^2-4ac)^{3/2}\sqrt{b-\sqrt{b^2-4ac}}} + \frac{\sqrt{2}\sqrt{c}\left(-b^2d+12acd+b\sqrt{b^2-4ac}d-4abf-2a\sqrt{b^2-4ac}f\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{a(b^2-4ac)^{3/2}\sqrt{b+\sqrt{b^2-4ac}}} - \frac{4ce\log\left(-b+\sqrt{b^2-4ac}-2cx^2\right)}{(b^2-4ac)^{3/2}} + \frac{4ce\log\left(b+\sqrt{b^2-4ac}+2cx^2\right)}{(b^2-4ac)^{3/2}}\right)$$

input
```
Integrate[(d + e*x + f*x^2)/(a + b*x^2 + c*x^4)^2,x]
```

output
```
((2*a*b*(e + f*x) - 2*b*d*x*(b + c*x^2) + 4*a*c*x*(d + x*(e + f*x)))/(a*(-
b^2 + 4*a*c)*(a + b*x^2 + c*x^4)) + (Sqrt[2]*Sqrt[c]*(b^2*d + b*(Sqrt[b^2
- 4*a*c]*d + 4*a*f) - 2*a*(6*c*d + Sqrt[b^2 - 4*a*c]*f))*ArcTan[(Sqrt[2]*S
qrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]])/(a*(b^2 - 4*a*c)^(3/2)*Sqrt[b - Sq
rt[b^2 - 4*a*c]]) + (Sqrt[2]*Sqrt[c]*(-(b^2*d) + 12*a*c*d + b*Sqrt[b^2 - 4
*a*c]*d - 4*a*b*f - 2*a*Sqrt[b^2 - 4*a*c]*f)*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sq
rt[b + Sqrt[b^2 - 4*a*c]]])/(a*(b^2 - 4*a*c)^(3/2)*Sqrt[b + Sqrt[b^2 - 4*a
*c]]) - (4*c*e*Log[-b + Sqrt[b^2 - 4*a*c] - 2*c*x^2])/(b^2 - 4*a*c)^(3/2)
+ (4*c*e*Log[b + Sqrt[b^2 - 4*a*c] + 2*c*x^2])/(b^2 - 4*a*c)^(3/2))/4
```

3.37.3 Rubi [A] (verified)

Time = 0.67 (sec) , antiderivative size = 356, normalized size of antiderivative = 0.97, number of steps used = 11, number of rules used = 10, $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$, Rules used = {2202, 27, 1432, 1086, 1083, 219, 1492, 25, 1480, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^2}dx$$

$$\Big\downarrow \ \textbf{2202}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^2}dx+\int \frac{ex}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \ \textbf{27}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^2}dx+e\int \frac{x}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \ \textbf{1432}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^2}dx+\frac{1}{2}e\int \frac{1}{(cx^4+bx^2+a)^2}dx^2$$

$$\Big\downarrow \ \textbf{1086}$$

$$\frac{1}{2}e\left(-\frac{2c\int \frac{1}{cx^4+bx^2+a}dx^2}{b^2-4ac}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)+\int \frac{fx^2+d}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \ \textbf{1083}$$

$$\frac{1}{2}e\left(\frac{4c\int \frac{1}{-x^4+b^2-4ac}d(2cx^2+b)}{b^2-4ac}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)+\int \frac{fx^2+d}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \ \textbf{219}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^2}dx+\frac{1}{2}e\left(\frac{4c\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)$$

$$\Big\downarrow \ \textbf{1492}$$

$$-\frac{\int -\frac{db^2+afb+c(bd-2af)x^2-6acd}{cx^4+bx^2+a}dx}{2a(b^2-4ac)}+\frac{1}{2}e\left(\frac{4c\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{2a(b^2-4ac)(a+bx^2+cx^4)}$$

$$\Big\downarrow \ \textbf{25}$$

$$\frac{\int \frac{db^2+afb+c(bd-2af)x^2-6acd}{cx^4+bx^2+a}dx}{2a(b^2-4ac)}+\frac{1}{2}e\left(\frac{4c\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{2a(b^2-4ac)(a+bx^2+cx^4)}$$

$$\Big\downarrow \ \textbf{1480}$$

3.37. $\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^2}\,dx$

$$\frac{\frac{1}{2}c\left(\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)\int\frac{1}{cx^2+\frac{1}{2}\left(b-\sqrt{b^2-4ac}\right)}dx+\frac{1}{2}c\left(-\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)\int\frac{1}{cx^2+\frac{1}{2}\left(b+\sqrt{b^2-4ac}\right)}dx}{2a\left(b^2-4ac\right)}+$$

$$\frac{1}{2}e\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{b+2cx^2}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

$$\Big\downarrow \text{218}$$

$$\frac{\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)}{\sqrt{2}\sqrt{\sqrt{b^2-4ac}+b}}}{2a\left(b^2-4ac\right)}+$$

$$\frac{1}{2}e\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{b+2cx^2}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

input
```
Int[(d + e*x + f*x^2)/(a + b*x^2 + c*x^4)^2,x]
```

output
```
(x*(b^2*d - 2*a*c*d - a*b*f + c*(b*d - 2*a*f)*x^2))/(2*a*(b^2 - 4*a*c)*(a
+ b*x^2 + c*x^4)) + ((Sqrt[c]*(b*d - 2*a*f + (b^2*d - 12*a*c*d + 4*a*b*f)/
Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]]
)/(Sqrt[2]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + (Sqrt[c]*(b*d - 2*a*f - (b^2*d -
 12*a*c*d + 4*a*b*f)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b
+ Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[b + Sqrt[b^2 - 4*a*c]]))/(2*a*(b^2 -
4*a*c)) + (e*(-((b + 2*c*x^2)/((b^2 - 4*a*c)*(a + b*x^2 + c*x^4))) + (4*c*
ArcTan h[(b + 2*c*x^2)/Sqrt[b^2 - 4*a*c]])/(b^2 - 4*a*c)^(3/2)))/2
```

3.37.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1086
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] - Simp[2*c*((2*p +
 3)/((p + 1)*(b^2 - 4*a*c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; Fre
eQ[{a, b, c}, x] && ILtQ[p, -1]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.37. $\int \frac{d+ex+fx^2}{\left(a+bx^2+cx^4\right)^2}\,dx$

3.37.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.26 (sec) , antiderivative size = 232, normalized size of antiderivative = 0.63

method	result
risch	$\frac{\frac{c(2af-bd)x^3}{2a(4ac-b^2)}+\frac{c\,x^2e}{4ac-b^2}+\frac{(abf+2acd-b^2d)x}{2a(4ac-b^2)}+\frac{be}{8ac-2b^2}}{c\,x^4+b\,x^2+a}+\frac{\left(\sum_{\_R=\mathrm{RootOf}(c\_Z^4+\_Z^2b+a)}\frac{\left(\frac{c(2af-bd)\_R^2}{a(4ac-b^2)}+\frac{4ce\_R}{4ac-b^2}-\frac{abf-6acd+b^2d}{a(4ac-b^2)}\right.}{2c\_R^3+\_Rb}\right.}{4}$
default	$16c^2\left(-\frac{\frac{-\frac{\left(-4acd\sqrt{-4ac+b^2}+b^2d\sqrt{-4ac+b^2}+8a^2cf-2a\,b^2f-4abcd+b^3d\right)x}{16ac}-\frac{e(4ac-b^2)}{8c}}{x^2+\frac{b}{2c}-\frac{\sqrt{-4ac+b^2}}{2c}}+\frac{2ae\sqrt{-4ac+b^2}\,\ln\left(-2c\,x^2+\sqrt{-4ac+b^2}-b\right)+\frac{(4\sqrt{}}{}}{}}{4c(4ac-b^2)^2}\right.$

input

```
int((f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x,method=_RETURNVERBOSE)
```

output

```
(1/2*c*(2*a*f-b*d)/a/(4*a*c-b^2)*x^3+c/(4*a*c-b^2)*x^2*e+1/2*(a*b*f+2*a*c*
d-b^2*d)/a/(4*a*c-b^2)*x+1/2/(4*a*c-b^2)*b*e)/(c*x^4+b*x^2+a)+1/4*sum((c*(
2*a*f-b*d)/a/(4*a*c-b^2)*_R^2+4*c/(4*a*c-b^2)*e*_R-(a*b*f-6*a*c*d+b^2*d)/a
/(4*a*c-b^2))/(2*_R^3*c+_R*b)*ln(x-_R),_R=RootOf(_Z^4*c+_Z^2*b+a))
```

3.37.5 Fricas [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^2}\,dx = \text{Timed out}$$

input

```
integrate((f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x, algorithm="fricas")
```

output

```
Timed out
```

3.37.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((f*x**2+e*x+d)/(c*x**4+b*x**2+a)**2,x)
```

output
```
Timed out
```

3.37.7 Maxima [F]

$$\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^2}\,dx = \int \frac{fx^2+ex+d}{(cx^4+bx^2+a)^2}\,dx$$

input
```
integrate((f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x, algorithm="maxima")
```

output
```
-1/2*(2*a*c*e*x^2 - (b*c*d - 2*a*c*f)*x^3 + a*b*e + (a*b*f - (b^2 - 2*a*c)
*d)*x)/((a*b^2*c - 4*a^2*c^2)*x^4 + a^2*b^2 - 4*a^3*c + (a*b^3 - 4*a^2*b*c
)*x^2) - 1/2*integrate((4*a*c*e*x - a*b*f - (b*c*d - 2*a*c*f)*x^2 - (b^2 -
 6*a*c)*d)/(c*x^4 + b*x^2 + a), x)/(a*b^2 - 4*a^2*c)
```

3.37.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 5156 vs. 2(320) = 640.

Time = 1.74 (sec) , antiderivative size = 5156, normalized size of antiderivative = 14.01

$$\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^2}\,dx = \text{Too large to display}$$

input
```
integrate((f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x, algorithm="giac")
```

output
```
1/2*(b*c*d*x^3 - 2*a*c*f*x^3 - 2*a*c*e*x^2 + b^2*d*x - 2*a*c*d*x - a*b*f*x
 - a*b*e)/((c*x^4 + b*x^2 + a)*(a*b^2 - 4*a^2*c)) + 1/16*((2*b^3*c^2 - 8*a
*b*c^3 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^3 + 4
*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c + 2*sqrt(
2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^2*c - sqrt(2)*sqrt(
b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b*c^2 - 2*(b^2 - 4*a*c)*b*c^2
)*(a*b^2 - 4*a^2*c)^2*d - 2*(2*a*b^2*c^2 - 8*a^2*c^3 - sqrt(2)*sqrt(b^2 -
4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^2 + 4*sqrt(2)*sqrt(b^2 - 4*a*c)
*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*c + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(
b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sq
rt(b^2 - 4*a*c)*c)*a*c^2 - 2*(b^2 - 4*a*c)*a*c^2)*(a*b^2 - 4*a^2*c)^2*f +
2*(sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^6 - 14*sqrt(2)*sqrt(b*c + s
qrt(b^2 - 4*a*c)*c)*a^2*b^4*c - 2*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*
a*b^5*c - 2*a*b^6*c + 64*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*b^2*c
^2 + 20*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^3*c^2 + sqrt(2)*sqrt
(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^4*c^2 + 28*a^2*b^4*c^2 - 96*sqrt(2)*sqrt(b
*c + sqrt(b^2 - 4*a*c)*c)*a^4*c^3 - 48*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c
)*c)*a^3*b*c^3 - 10*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^2*c^3 -
128*a^3*b^2*c^3 + 24*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*c^4 + 192
*a^4*c^4 + 2*(b^2 - 4*a*c)*a*b^4*c - 20*(b^2 - 4*a*c)*a^2*b^2*c^2 + 48*...
```

3.37.9 Mupad [B] (verification not implemented)

Time = 8.40 (sec) , antiderivative size = 4707, normalized size of antiderivative = 12.79

$$\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^2}\,dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2)/(a + b*x^2 + c*x^4)^2,x)
```

output
```
symsum(log((5*b^3*c^4*d^3 + 8*a^3*c^4*f^3 - 96*a^2*c^5*d*e^2 + 72*a^2*c^5*
d^2*f - 3*b^4*c^3*d^2*f + 6*a^2*b^2*c^3*f^3 - 36*a*b*c^5*d^3 + 16*a*b^2*c^
4*d*e^2 + 18*a*b^2*c^4*d^2*f + 3*a*b^3*c^3*d*f^2 - 60*a^2*b*c^4*d*f^2 + 16
*a^2*b*c^4*e^2*f)/(8*(a^2*b^6 - 64*a^5*c^3 - 12*a^3*b^4*c + 48*a^4*b^2*c^2
)) - root(1572864*a^8*b^2*c^5*z^4 - 983040*a^7*b^4*c^4*z^4 + 327680*a^6*b^
6*c^3*z^4 - 61440*a^5*b^8*c^2*z^4 + 6144*a^4*b^10*c*z^4 - 1048576*a^9*c^6*
z^4 - 256*a^3*b^12*z^4 + 576*a^2*b^8*c*d*f*z^2 + 24576*a^5*b^2*c^4*d*f*z^2
 - 3072*a^3*b^6*c^2*d*f*z^2 + 2048*a^4*b^4*c^3*d*f*z^2 + 12288*a^6*b*c^4*f
^2*z^2 + 61440*a^5*b*c^5*d^2*z^2 - 49152*a^6*c^5*d*f*z^2 + 432*a*b^9*c*d^2
*z^2 - 8192*a^5*b^3*c^3*f^2*z^2 + 1536*a^4*b^5*c^2*f^2*z^2 + 24576*a^5*b^2
*c^4*e^2*z^2 - 6144*a^4*b^4*c^3*e^2*z^2 + 512*a^3*b^6*c^2*e^2*z^2 - 61440*
a^4*b^3*c^4*d^2*z^2 + 24064*a^3*b^5*c^3*d^2*z^2 - 4608*a^2*b^7*c^2*d^2*z^2
 - 32*a*b^10*d*f*z^2 - 32768*a^6*c^5*e^2*z^2 - 16*a^2*b^9*f^2*z^2 - 16*b^1
1*d^2*z^2 - 4096*a^4*b*c^4*d*e*f*z + 64*a*b^7*c*d*e*f*z + 3072*a^3*b^3*c^3
*d*e*f*z - 768*a^2*b^5*c^2*d*e*f*z + 32*a^2*b^6*c*e*f^2*z - 672*a*b^6*c^2*
d^2*e*z + 1536*a^4*b^2*c^3*e*f^2*z - 384*a^3*b^4*c^2*e*f^2*z - 15872*a^3*b
^2*c^4*d^2*e*z + 4992*a^2*b^4*c^3*d^2*e*z - 2048*a^5*c^4*e*f^2*z + 18432*a
^4*c^5*d^2*e*z + 32*b^8*c*d^2*e*z - 32*a*b^4*c^2*d*e^2*f + 192*a^2*b^2*c^3
*d*e^2*f - 192*a^3*b*c^3*e^2*f^2 + 198*a*b^4*c^2*d^2*f^2 + 144*a^2*b^3*c^2
*d*f^3 - 960*a^2*b*c^4*d^2*e^2 + 240*a*b^3*c^3*d^2*e^2 + 768*a^3*c^4*d*...
```

3.37. $\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^2}\,dx$

3.38 $\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^2}\,dx$

3.38.1 Optimal result

Integrand size = 30, antiderivative size = 386

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^2}\,dx = \frac{x(b^2d-2acd-abf+c(bd-2af)x^2)}{2a(b^2-4ac)(a+bx^2+cx^4)} - \frac{be-2ag+(2ce-bg)x^2}{2(b^2-4ac)(a+bx^2+cx^4)} + \frac{\sqrt{c}\left(bd-2af+\frac{b^2d-12acd+4abf}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{2\sqrt{2}a(b^2-4ac)\sqrt{b-\sqrt{b^2-4ac}}} + \frac{\sqrt{c}\left(bd-2af-\frac{b^2d-12acd+4abf}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{2\sqrt{2}a(b^2-4ac)\sqrt{b+\sqrt{b^2-4ac}}} + \frac{(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}$$

output
```
1/2*x*(b^2*d-2*a*c*d-a*b*f+c*(-2*a*f+b*d)*x^2)/a/(-4*a*c+b^2)/(c*x^4+b*x^2
+a)+1/2*(-b*e+2*a*g-(-b*g+2*c*e)*x^2)/(-4*a*c+b^2)/(c*x^4+b*x^2+a)+(-b*g+2
*c*e)*arctanh((2*c*x^2+b)/(-4*a*c+b^2)^(1/2))/(-4*a*c+b^2)^(3/2)+1/4*arcta
n(x*2^(1/2)*c^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2))*c^(1/2)*(b*d-2*a*f+(4*a*
b*f-12*a*c*d+b^2*d)/(-4*a*c+b^2)^(1/2))/a/(-4*a*c+b^2)*2^(1/2)/(b-(-4*a*c+
b^2)^(1/2))^(1/2)+1/4*arctan(x*2^(1/2)*c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2
))*c^(1/2)*(b*d-2*a*f+(-4*a*b*f+12*a*c*d-b^2*d)/(-4*a*c+b^2)^(1/2))/a/(-4*
a*c+b^2)*2^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2)
```

3.38.2 Mathematica [A] (verified)

Time = 0.74 (sec) , antiderivative size = 421, normalized size of antiderivative = 1.09

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^2}\,dx$$
$$= \frac{1}{4}\left(\frac{-4a^2g-2bdx(b+cx^2)+4acx(d+x(e+fx))+2ab(e+x(f-gx))}{a(-b^2+4ac)(a+bx^2+cx^4)} + \frac{\sqrt{2}\sqrt{c}\left(b^2d+b\left(\sqrt{b^2-4ac}d+4af\right)-2a\left(6cd+\sqrt{b^2-4ac}f\right)\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{a(b^2-4ac)^{3/2}\sqrt{b-\sqrt{b^2-4ac}}} + \frac{\sqrt{2}\sqrt{c}\left(-b^2d+12acd+b\sqrt{b^2-4ac}d-4abf-2a\sqrt{b^2-4ac}f\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{a(b^2-4ac)^{3/2}\sqrt{b+\sqrt{b^2-4ac}}} + \frac{2(-2ce+bg)\log\left(-b+\sqrt{b^2-4ac}-2cx^2\right)}{(b^2-4ac)^{3/2}} - \frac{2(-2ce+bg)\log\left(b+\sqrt{b^2-4ac}+2cx^2\right)}{(b^2-4ac)^{3/2}}\right)$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4)^2,x]
```

output
```
((-4*a^2*g - 2*b*d*x*(b + c*x^2) + 4*a*c*x*(d + x*(e + f*x)) + 2*a*b*(e +
x*(f - g*x)))/(a*(-b^2 + 4*a*c)*(a + b*x^2 + c*x^4)) + (Sqrt[2]*Sqrt[c]*(b
^2*d + b*(Sqrt[b^2 - 4*a*c]*d + 4*a*f) - 2*a*(6*c*d + Sqrt[b^2 - 4*a*c]*f)
)*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]])/(a*(b^2 - 4*a*c
)^(3/2)*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + (Sqrt[2]*Sqrt[c]*(-(b^2*d) + 12*a*c
*d + b*Sqrt[b^2 - 4*a*c]*d - 4*a*b*f - 2*a*Sqrt[b^2 - 4*a*c]*f)*ArcTan[(Sq
rt[2]*Sqrt[c]*x)/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(a*(b^2 - 4*a*c)^(3/2)*Sqrt
[b + Sqrt[b^2 - 4*a*c]]) + (2*(-2*c*e + b*g)*Log[-b + Sqrt[b^2 - 4*a*c] -
2*c*x^2])/(b^2 - 4*a*c)^(3/2) - (2*(-2*c*e + b*g)*Log[b + Sqrt[b^2 - 4*a*c
] + 2*c*x^2])/(b^2 - 4*a*c)^(3/2))/4
```

3.38.3 Rubi [A] (verified)

Time = 0.66 (sec) , antiderivative size = 376, normalized size of antiderivative = 0.97, number of steps used = 10, number of rules used = 9, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.300, Rules used = {2202, 1492, 25, 1480, 218, 1576, 1159, 1083, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^2}\,dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^2}dx+\int \frac{x(gx^2+e)}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \text{1492}$$

$$-\frac{\int -\frac{db^2+afb+c(bd-2af)x^2-6acd}{cx^4+bx^2+a}dx}{2a(b^2-4ac)}+\int \frac{x(gx^2+e)}{(cx^4+bx^2+a)^2}dx+\frac{x(cx^2(bd-2af)-abf-2acd+b^2d)}{2a(b^2-4ac)(a+bx^2+cx^4)}$$

$$\Big\downarrow \text{25}$$

$$\frac{\int \frac{db^2+afb+c(bd-2af)x^2-6acd}{cx^4+bx^2+a}dx}{2a(b^2-4ac)}+\int \frac{x(gx^2+e)}{(cx^4+bx^2+a)^2}dx+\frac{x(cx^2(bd-2af)-abf-2acd+b^2d)}{2a(b^2-4ac)(a+bx^2+cx^4)}$$

$$\Big\downarrow \text{1480}$$

$$\frac{\frac{1}{2}c\left(\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)\int \frac{1}{cx^2+\frac{1}{2}\left(b-\sqrt{b^2-4ac}\right)}dx+\frac{1}{2}c\left(-\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)\int \frac{1}{cx^2+\frac{1}{2}\left(b+\sqrt{b^2-4ac}\right)}dx}{2a(b^2-4ac)}+$$
$$\int \frac{x(gx^2+e)}{(cx^4+bx^2+a)^2}dx+\frac{x(cx^2(bd-2af)-abf-2acd+b^2d)}{2a(b^2-4ac)(a+bx^2+cx^4)}$$

$$\Big\downarrow \text{218}$$

$$\int \frac{x(gx^2+e)}{(cx^4+bx^2+a)^2}dx+$$
$$\frac{\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)}{\sqrt{2}\sqrt{\sqrt{b^2-4ac}+b}}}{2a(b^2-4ac)}+$$
$$\frac{x(cx^2(bd-2af)-abf-2acd+b^2d)}{2a(b^2-4ac)(a+bx^2+cx^4)}$$

$$\Big\downarrow \text{1576}$$

$$\frac{1}{2}\int \frac{gx^2+e}{\left(cx^4+bx^2+a\right)^2}dx^2+$$

$$\frac{\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)}{\sqrt{2}\sqrt{\sqrt{b^2-4ac}+b}}}{2a\left(b^2-4ac\right)}+$$

$$\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

$$\Big\downarrow \text{1159}$$

$$\frac{1}{2}\left(-\frac{(2ce-bg)\int\frac{1}{cx^4+bx^2+a}dx^2}{b^2-4ac}-\frac{-2ag+x^2(2ce-bg)+be}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)+$$

$$\frac{\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)}{\sqrt{2}\sqrt{\sqrt{b^2-4ac}+b}}}{2a\left(b^2-4ac\right)}+$$

$$\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

$$\Big\downarrow \text{1083}$$

$$\frac{1}{2}\left(\frac{2(2ce-bg)\int\frac{1}{-x^4+b^2-4ac}d\left(2cx^2+b\right)}{b^2-4ac}-\frac{-2ag+x^2(2ce-bg)+be}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)+$$

$$\frac{\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)}{\sqrt{2}\sqrt{\sqrt{b^2-4ac}+b}}}{2a\left(b^2-4ac\right)}+$$

$$\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

$$\Big\downarrow \text{219}$$

$$\frac{\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)}{\sqrt{2}\sqrt{\sqrt{b^2-4ac}+b}}}{2a\left(b^2-4ac\right)}+$$

$$\frac{1}{2}\left(\frac{2(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{-2ag+x^2(2ce-bg)+be}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)+$$

$$\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4)^2,x]
```

3.38. $\int \frac{d+ex+fx^2+gx^3}{\left(a+bx^2+cx^4\right)^2}\,dx$

output
```
(x*(b^2*d - 2*a*c*d - a*b*f + c*(b*d - 2*a*f)*x^2))/(2*a*(b^2 - 4*a*c)*(a
+ b*x^2 + c*x^4)) + ((Sqrt[c]*(b*d - 2*a*f + (b^2*d - 12*a*c*d + 4*a*b*f)/
Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]]
)/(Sqrt[2]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + (Sqrt[c]*(b*d - 2*a*f - (b^2*d -
 12*a*c*d + 4*a*b*f)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b
+ Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[b + Sqrt[b^2 - 4*a*c]]))/(2*a*(b^2 -
4*a*c)) + (-((b*e - 2*a*g + (2*c*e - b*g)*x^2)/((b^2 - 4*a*c)*(a + b*x^2 +
 c*x^4))) + (2*(2*c*e - b*g)*ArcTanh[(b + 2*c*x^2)/Sqrt[b^2 - 4*a*c]])/(b^
2 - 4*a*c)^(3/2))/2
```

3.38.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1159
```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[((b*d - 2*a*e + (2*c*d - b*e)*x)/((p + 1)*(b^2 - 4*a*c)))*(a + b*
x + c*x^2)^(p + 1), x] - Simp[(2*p + 3)*((2*c*d - b*e)/((p + 1)*(b^2 - 4*a*
c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; FreeQ[{a, b, c, d, e}, x] &
& LtQ[p, -1] && NeQ[p, -3/2]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.38.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.28 (sec) , antiderivative size = 252, normalized size of antiderivative = 0.65

method	result
risch	$\frac{\frac{c(2af-bd)x^3}{2a\left(4ac-b^2\right)}-\frac{(bg-2ec)x^2}{2\left(4ac-b^2\right)}+\frac{\left(abf+2acd-b^2d\right)x}{2a\left(4ac-b^2\right)}-\frac{2ag-be}{2\left(4ac-b^2\right)}}{cx^4+bx^2+a}+\frac{\left(\sum_{\_R=\mathrm{RootOf}\left(c\_Z^4+\_Z^2b+a\right)}\frac{\left(\frac{c(2af-bd)\_R^2}{a\left(4ac-b^2\right)}-\frac{2(bg-2ec)\_R}{4ac-b^2}-\frac{ab}{}\right.}{2c\_R^3+\_R}\right.}{4}$
default	$16c^2\left(-\frac{-\frac{\left(-4acd\sqrt{-4ac+b^2}+b^2d\sqrt{-4ac+b^2}+8a^2cf-2ab^2f-4abcd+b^3d\right)x}{16ac}-\frac{4\sqrt{-4ac+b^2}\,acg-\sqrt{-4ac+b^2}\,b^2g-4abgc+8ac^2e+b^3g-2b^2ce}{16c^2}}{x^2+\frac{b}{2c}-\frac{\sqrt{-4ac+b^2}}{2c}}\right.$

input
```
int((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x,method=_RETURNVERBOSE)
```

output
```
(1/2*c*(2*a*f-b*d)/a/(4*a*c-b^2)*x^3-1/2*(b*g-2*c*e)/(4*a*c-b^2)*x^2+1/2*(
a*b*f+2*a*c*d-b^2*d)/a/(4*a*c-b^2)*x-1/2*(2*a*g-b*e)/(4*a*c-b^2))/(c*x^4+b
*x^2+a)+1/4*sum((c*(2*a*f-b*d)/a/(4*a*c-b^2)*_R^2-2*(b*g-2*c*e)/(4*a*c-b^2
)*_R-(a*b*f-6*a*c*d+b^2*d)/a/(4*a*c-b^2))/(2*_R^3*c+_R*b)*ln(x-_R),_R=Root
Of(_Z^4*c+_Z^2*b+a))
```

3.38.5 Fricas [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x, algorithm="fricas")
```

output
```
Timed out
```

3.38.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((g*x**3+f*x**2+e*x+d)/(c*x**4+b*x**2+a)**2,x)
```

output
```
Timed out
```

3.38.7 Maxima [F]

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^2}\,dx = \int \frac{gx^3+fx^2+ex+d}{(cx^4+bx^2+a)^2}\,dx$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x, algorithm="maxima")
```

output
```
1/2*((b*c*d - 2*a*c*f)*x^3 - a*b*e + 2*a^2*g - (2*a*c*e - a*b*g)*x^2 - (a*
b*f - (b^2 - 2*a*c)*d)*x)/((a*b^2*c - 4*a^2*c^2)*x^4 + a^2*b^2 - 4*a^3*c +
 (a*b^3 - 4*a^2*b*c)*x^2) - 1/2*integrate(-(a*b*f + (b*c*d - 2*a*c*f)*x^2
+ (b^2 - 6*a*c)*d - 2*(2*a*c*e - a*b*g)*x)/(c*x^4 + b*x^2 + a), x)/(a*b^2
- 4*a^2*c)
```

3.38.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 5573 vs. $2(338) = 676$.

Time = 1.79 (sec) , antiderivative size = 5573, normalized size of antiderivative = 14.44

$$\int \frac{d + ex + fx^2 + gx^3}{(a + bx^2 + cx^4)^2}\,dx = \text{Too large to display}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x, algorithm="giac")
```

output
```
1/2*(b*c*d*x^3 - 2*a*c*f*x^3 - 2*a*c*e*x^2 + a*b*g*x^2 + b^2*d*x - 2*a*c*d
*x - a*b*f*x - a*b*e + 2*a^2*g)/((c*x^4 + b*x^2 + a)*(a*b^2 - 4*a^2*c)) +
1/16*((2*b^3*c^2 - 8*a*b*c^3 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b
^2 - 4*a*c)*c)*b^3 + 4*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a
*c)*c)*a*b*c + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)
*b^2*c - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b*c^2 -
 2*(b^2 - 4*a*c)*b*c^2)*(a*b^2 - 4*a^2*c)^2*d - 2*(2*a*b^2*c^2 - 8*a^2*c^3
 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^2 + 4*sqr
t(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*c + 2*sqrt(2)*s
qrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c - sqrt(2)*sqrt(b^2
- 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*c^2 - 2*(b^2 - 4*a*c)*a*c^2)*(a
*b^2 - 4*a^2*c)^2*f + 2*(sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^6 - 1
4*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^4*c - 2*sqrt(2)*sqrt(b*c +
 sqrt(b^2 - 4*a*c)*c)*a*b^5*c - 2*a*b^6*c + 64*sqrt(2)*sqrt(b*c + sqrt(b^2
 - 4*a*c)*c)*a^3*b^2*c^2 + 20*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*
b^3*c^2 + sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^4*c^2 + 28*a^2*b^4*c
^2 - 96*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^4*c^3 - 48*sqrt(2)*sqrt(
b*c + sqrt(b^2 - 4*a*c)*c)*a^3*b*c^3 - 10*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*
a*c)*c)*a^2*b^2*c^3 - 128*a^3*b^2*c^3 + 24*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4
*a*c)*c)*a^3*c^4 + 192*a^4*c^4 + 2*(b^2 - 4*a*c)*a*b^4*c - 20*(b^2 - 4*...
```

3.38.9 Mupad [B] (verification not implemented)

Time = 8.61 (sec) , antiderivative size = 7373, normalized size of antiderivative = 19.10

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^2}\,dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4)^2,x)
```

output
```
symsum(log((5*b^3*c^4*d^3 + 8*a^3*c^4*f^3 - 96*a^2*c^5*d*e^2 + 72*a^2*c^5*
d^2*f - 3*b^4*c^3*d^2*f + 6*a^2*b^2*c^3*f^3 - 36*a*b*c^5*d^3 + 16*a*b^2*c^
4*d*e^2 + 18*a*b^2*c^4*d^2*f + 3*a*b^3*c^3*d*f^2 - 60*a^2*b*c^4*d*f^2 + 4*
a*b^4*c^2*d*g^2 + 16*a^2*b*c^4*e^2*f - 24*a^2*b^2*c^3*d*g^2 + 4*a^2*b^3*c^
2*f*g^2 - 16*a*b^3*c^3*d*e*g + 96*a^2*b*c^4*d*e*g - 16*a^2*b^2*c^3*e*f*g)/
(8*(a^2*b^6 - 64*a^5*c^3 - 12*a^3*b^4*c + 48*a^4*b^2*c^2)) - root(1572864*
a^8*b^2*c^5*z^4 - 983040*a^7*b^4*c^4*z^4 + 327680*a^6*b^6*c^3*z^4 - 61440*
a^5*b^8*c^2*z^4 + 6144*a^4*b^10*c*z^4 - 1048576*a^9*c^6*z^4 - 256*a^3*b^12
*z^4 + 32768*a^6*b*c^4*e*g*z^2 - 512*a^3*b^7*c*e*g*z^2 + 576*a^2*b^8*c*d*f
*z^2 - 24576*a^5*b^3*c^3*e*g*z^2 + 6144*a^4*b^5*c^2*e*g*z^2 + 24576*a^5*b^
2*c^4*d*f*z^2 - 3072*a^3*b^6*c^2*d*f*z^2 + 2048*a^4*b^4*c^3*d*f*z^2 - 1536
*a^4*b^6*c*g^2*z^2 + 12288*a^6*b*c^4*f^2*z^2 + 61440*a^5*b*c^5*d^2*z^2 - 4
9152*a^6*c^5*d*f*z^2 + 432*a*b^9*c*d^2*z^2 - 8192*a^6*b^2*c^3*g^2*z^2 + 61
44*a^5*b^4*c^2*g^2*z^2 - 8192*a^5*b^3*c^3*f^2*z^2 + 1536*a^4*b^5*c^2*f^2*z
^2 + 24576*a^5*b^2*c^4*e^2*z^2 - 6144*a^4*b^4*c^3*e^2*z^2 + 512*a^3*b^6*c^
2*e^2*z^2 - 61440*a^4*b^3*c^4*d^2*z^2 + 24064*a^3*b^5*c^3*d^2*z^2 - 4608*a
^2*b^7*c^2*d^2*z^2 - 32*a*b^10*d*f*z^2 + 128*a^3*b^8*g^2*z^2 - 32768*a^6*c
^5*e^2*z^2 - 16*a^2*b^9*f^2*z^2 - 16*b^11*d^2*z^2 + 384*a^2*b^6*c*d*f*g*z
- 4096*a^4*b*c^4*d*e*f*z + 64*a*b^7*c*d*e*f*z + 2048*a^4*b^2*c^3*d*f*g*z -
 1536*a^3*b^4*c^2*d*f*g*z + 3072*a^3*b^3*c^3*d*e*f*z - 768*a^2*b^5*c^2*...
```

3.39 $\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^2} dx$

3.39.1 Optimal result

Integrand size = 35, antiderivative size = 439

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^2}\,dx &= -\frac{be-2ag+(2ce-bg)x^2}{2(b^2-4ac)(a+bx^2+cx^4)}+\frac{x(b^2d-abf-2a(cd-ah)+(bcd-2acf+abh)x^2)}{2a(b^2-4ac)(a+bx^2+cx^4)}\\ &+\frac{\left(bcd-2acf+abh+\frac{4abcf+b^2(cd-ah)-4ac(3cd+ah)}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{2\sqrt{2}a\sqrt{c}(b^2-4ac)\sqrt{b-\sqrt{b^2-4ac}}}\\ &+\frac{\left(bcd-2acf+abh-\frac{4abcf+b^2(cd-ah)-4ac(3cd+ah)}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{2\sqrt{2}a\sqrt{c}(b^2-4ac)\sqrt{b+\sqrt{b^2-4ac}}}\\ &+\frac{(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}\end{aligned}$$

output
```
1/2*(-b*e+2*a*g-(-b*g+2*c*e)*x^2)/(-4*a*c+b^2)/(c*x^4+b*x^2+a)+1/2*x*(b^2*
d-a*b*f-2*a*(-a*h+c*d)+(a*b*h-2*a*c*f+b*c*d)*x^2)/a/(-4*a*c+b^2)/(c*x^4+b*
x^2+a)+(-b*g+2*c*e)*arctanh((2*c*x^2+b)/(-4*a*c+b^2)^(1/2))/(-4*a*c+b^2)^(
3/2)+1/4*arctan(x*2^(1/2)*c^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2))*(b*c*d-2*a
*c*f+a*b*h+(4*a*b*c*f+b^2*(-a*h+c*d)-4*a*c*(a*h+3*c*d))/(-4*a*c+b^2)^(1/2)
)/a/(-4*a*c+b^2)*2^(1/2)/c^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2)+1/4*arctan(x
*2^(1/2)*c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2))*(b*c*d-2*a*c*f+a*b*h+(-4*a*
b*c*f-b^2*(-a*h+c*d)+4*a*c*(a*h+3*c*d))/(-4*a*c+b^2)^(1/2))/a/(-4*a*c+b^2)
*2^(1/2)/c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2)
```

3.39.2 Mathematica [A] (verified)

Time = 1.09 (sec) , antiderivative size = 489, normalized size of antiderivative = 1.11

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^2}\,dx$$
$$=\frac{1}{4}\left(\frac{-4a^2(g+hx)-2bdx(b+cx^2)+4acx(d+x(e+fx))+2ab(e+x(f-x(g+hx)))}{a(-b^2+4ac)(a+bx^2+cx^4)}\right.$$
$$+\frac{\sqrt{2}\left(b^2(cd-ah)-2ac\left(6cd+\sqrt{b^2-4ac}f+2ah\right)+b\left(c\sqrt{b^2-4ac}d+4acf+a\sqrt{b^2-4ac}h\right)\right)\arctan\left(\right.}{a\sqrt{c}\,(b^2-4ac)^{3/2}\sqrt{b-\sqrt{b^2-4ac}}}$$
$$+\frac{\sqrt{2}\left(b^2(-cd+ah)+2ac\left(6cd-\sqrt{b^2-4ac}f+2ah\right)+b\left(c\sqrt{b^2-4ac}d-4acf+a\sqrt{b^2-4ac}h\right)\right)\arctan}{a\sqrt{c}\,(b^2-4ac)^{3/2}\sqrt{b+\sqrt{b^2-4ac}}}$$
$$+\frac{2(-2ce+bg)\log\left(-b+\sqrt{b^2-4ac}-2cx^2\right)}{(b^2-4ac)^{3/2}}$$
$$\left.-\frac{2(-2ce+bg)\log\left(b+\sqrt{b^2-4ac}+2cx^2\right)}{(b^2-4ac)^{3/2}}\right)$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4)/(a + b*x^2 + c*x^4)^2,x]
```

output
```
((-4*a^2*(g + h*x) - 2*b*d*x*(b + c*x^2) + 4*a*c*x*(d + x*(e + f*x)) + 2*a
*b*(e + x*(f - x*(g + h*x))))/(a*(-b^2 + 4*a*c)*(a + b*x^2 + c*x^4)) + (Sq
rt[2]*(b^2*(c*d - a*h) - 2*a*c*(6*c*d + Sqrt[b^2 - 4*a*c]*f + 2*a*h) + b*(
c*Sqrt[b^2 - 4*a*c]*d + 4*a*c*f + a*Sqrt[b^2 - 4*a*c]*h))*ArcTan[(Sqrt[2]*
Sqrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]])/(a*Sqrt[c]*(b^2 - 4*a*c)^(3/2)*Sq
rt[b - Sqrt[b^2 - 4*a*c]]) + (Sqrt[2]*(b^2*(-(c*d) + a*h) + 2*a*c*(6*c*d -
 Sqrt[b^2 - 4*a*c]*f + 2*a*h) + b*(c*Sqrt[b^2 - 4*a*c]*d - 4*a*c*f + a*Sqr
t[b^2 - 4*a*c]*h))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqrt[b^2 - 4*a*c]]]
)/(a*Sqrt[c]*(b^2 - 4*a*c)^(3/2)*Sqrt[b + Sqrt[b^2 - 4*a*c]]) + (2*(-2*c*e
 + b*g)*Log[-b + Sqrt[b^2 - 4*a*c] - 2*c*x^2])/(b^2 - 4*a*c)^(3/2) - (2*(-
2*c*e + b*g)*Log[b + Sqrt[b^2 - 4*a*c] + 2*c*x^2])/(b^2 - 4*a*c)^(3/2))/4
```

3.39.3 Rubi [A] (verified)

Time = 0.82 (sec) , antiderivative size = 429, normalized size of antiderivative = 0.98, number of steps used = 10, number of rules used = 9, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.257, Rules used = {2202, 1576, 1159, 1083, 219, 2206, 25, 1480, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^2}dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{hx^4+fx^2+d}{(cx^4+bx^2+a)^2}dx+\int \frac{x(gx^2+e)}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \text{1576}$$

$$\int \frac{hx^4+fx^2+d}{(cx^4+bx^2+a)^2}dx+\frac{1}{2}\int \frac{gx^2+e}{(cx^4+bx^2+a)^2}dx^2$$

$$\Big\downarrow \text{1159}$$

$$\frac{1}{2}\left(-\frac{(2ce-bg)\int \frac{1}{cx^4+bx^2+a}dx^2}{b^2-4ac}-\frac{-2ag+x^2(2ce-bg)+be}{(b^2-4ac)(a+bx^2+cx^4)}\right)+\int \frac{hx^4+fx^2+d}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \text{1083}$$

$$\frac{1}{2}\left(\frac{2(2ce-bg)\int \frac{1}{-x^4+b^2-4ac}d(2cx^2+b)}{b^2-4ac}-\frac{-2ag+x^2(2ce-bg)+be}{(b^2-4ac)(a+bx^2+cx^4)}\right)+\int \frac{hx^4+fx^2+d}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \text{219}$$

$$\int \frac{hx^4+fx^2+d}{(cx^4+bx^2+a)^2}dx+\frac{1}{2}\left(\frac{2(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{-2ag+x^2(2ce-bg)+be}{(b^2-4ac)(a+bx^2+cx^4)}\right)$$

$$\Big\downarrow \text{2206}$$

$$-\frac{\int -\frac{db^2+afb+(bcd-2acf+abh)x^2-2a(3cd+ah)}{cx^4+bx^2+a}dx}{2a(b^2-4ac)}+\frac{1}{2}\left(\frac{2(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{-2ag+x^2(2ce-bg)+be}{(b^2-4ac)(a+bx^2+cx^4)}\right)+\frac{x(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d)}{2a(b^2-4ac)(a+bx^2+cx^4)}$$

$$\Big\downarrow \text{25}$$

$$\frac{\int \frac{db^2+afb+(bcd-2acf+abh)x^2-2a(3cd+ah)}{cx^4+bx^2+a}dx}{2a\left(b^2-4ac\right)}+\frac{1}{2}\left(\frac{2(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{-2ag+x^2(2ce-bg)+be}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)+\frac{x\left(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

$$\Big\downarrow \text{1480}$$

$$\frac{\frac{1}{2}\left(\frac{b^2(cd-ah)+4abcf-4ac(ah+3cd)}{\sqrt{b^2-4ac}}+abh-2acf+bcd\right)\int \frac{1}{cx^2+\frac{1}{2}\left(b-\sqrt{b^2-4ac}\right)}dx+\frac{1}{2}\left(-\frac{b^2(cd-ah)+4abcf-4ac(ah+3cd)}{\sqrt{b^2-4ac}}+abh\right.}{2a\left(b^2-4ac\right)}$$

$$\frac{1}{2}\left(\frac{2(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{-2ag+x^2(2ce-bg)+be}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)+\frac{x\left(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

$$\Big\downarrow \text{218}$$

$$\frac{\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{b^2(cd-ah)+4abcf-4ac(ah+3cd)}{\sqrt{b^2-4ac}}+abh-2acf+bcd\right)}{\sqrt{2}\sqrt{c}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{b^2(cd-ah)+4abcf-4ac(ah+3cd)}{\sqrt{b^2-4ac}}+abh-2a\right.}{\sqrt{2}\sqrt{c}\sqrt{\sqrt{b^2-4ac}+b}}}{2a\left(b^2-4ac\right)}$$

$$\frac{1}{2}\left(\frac{2(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{-2ag+x^2(2ce-bg)+be}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)+\frac{x\left(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4)/(a + b*x^2 + c*x^4)^2,x]
```

3.39. $\int \frac{d+ex+fx^2+gx^3+hx^4}{\left(a+bx^2+cx^4\right)^2}\,dx$

output
```
(x*(b^2*d - a*b*f - 2*a*(c*d - a*h) + (b*c*d - 2*a*c*f + a*b*h)*x^2))/(2*a
*(b^2 - 4*a*c)*(a + b*x^2 + c*x^4)) + (((b*c*d - 2*a*c*f + a*b*h + (4*a*b*
c*f + b^2*(c*d - a*h) - 4*a*c*(3*c*d + a*h))/Sqrt[b^2 - 4*a*c])*ArcTan[(Sq
rt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[c]*Sqrt[b - S
qrt[b^2 - 4*a*c]]) + ((b*c*d - 2*a*c*f + a*b*h - (4*a*b*c*f + b^2*(c*d - a
*h) - 4*a*c*(3*c*d + a*h))/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/S
qrt[b + Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[c]*Sqrt[b + Sqrt[b^2 - 4*a*c]])
)/(2*a*(b^2 - 4*a*c)) + (-((b*e - 2*a*g + (2*c*e - b*g)*x^2)/((b^2 - 4*a*c
)*(a + b*x^2 + c*x^4))) + (2*(2*c*e - b*g)*ArcTanh[(b + 2*c*x^2)/Sqrt[b^2
- 4*a*c]])/(b^2 - 4*a*c)^(3/2))/2
```

3.39.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1159
```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[((b*d - 2*a*e + (2*c*d - b*e)*x)/((p + 1)*(b^2 - 4*a*c)))*(a + b*
x + c*x^2)^(p + 1), x] - Simp[(2*p + 3)*((2*c*d - b*e)/((p + 1)*(b^2 - 4*a*
c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; FreeQ[{a, b, c, d, e}, x] &
& LtQ[p, -1] && NeQ[p, -3/2]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2206
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> With[{d =
 Coeff[PolynomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 0], e = Coeff[Poly
nomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 2]}, Simp[x*(a + b*x^2 + c*x^
4)^(p + 1)*((a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)/(2*a*(p + 1)*(b
^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(a + b*x^2 + c
*x^4)^(p + 1)*ExpandToSum[2*a*(p + 1)*(b^2 - 4*a*c)*PolynomialQuotient[Px,
a + b*x^2 + c*x^4, x] + b^2*d*(2*p + 3) - 2*a*c*d*(4*p + 5) - a*b*e + c*(4*
p + 7)*(b*d - 2*a*e)*x^2, x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x
^2] && Expon[Px, x^2] > 1 && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1]
```

3.39.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.15 (sec) , antiderivative size = 274, normalized size of antiderivative = 0.62

method	result
risch	$\frac{-\frac{(abh-2acf+bcd)x^3}{2a\left(4ac-b^2\right)}-\frac{(bg-2ec)x^2}{2\left(4ac-b^2\right)}-\frac{\left(2a^2h-abf-2acd+b^2d\right)x}{2a\left(4ac-b^2\right)}-\frac{2ag-be}{2\left(4ac-b^2\right)}}{c\,x^4+b\,x^2+a}+\frac{\left(\sum_{\_R=\mathrm{RootOf}\left(c\_Z^4+\_Z^2b+a\right)}\frac{\left(-\frac{(abh-2acf+bcd)\_R^2}{a\left(4ac-b^2\right)}\right.}{}\right.}{4}$
default	$\frac{-\frac{(abh-2acf+bcd)x^3}{2a\left(4ac-b^2\right)}-\frac{(bg-2ec)x^2}{2\left(4ac-b^2\right)}-\frac{\left(2a^2h-abf-2acd+b^2d\right)x}{2a\left(4ac-b^2\right)}-\frac{2ag-be}{2\left(4ac-b^2\right)}}{c\,x^4+b\,x^2+a}+\frac{2c\left(\frac{\left(-4\sqrt{-4ac+b^2}\,abcg+8\sqrt{-4ac+b^2}\,a\,c^2e\right)\ln\left(2c\,x^2+\sqrt{-4\right.\right.}{4c}\right.}{}$

input
```
int((h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x,method=_RETURNVERBOSE)
```

output
```
(-1/2/a*(a*b*h-2*a*c*f+b*c*d)/(4*a*c-b^2)*x^3-1/2*(b*g-2*c*e)/(4*a*c-b^2)*
x^2-1/2*(2*a^2*h-a*b*f-2*a*c*d+b^2*d)/a/(4*a*c-b^2)*x-1/2*(2*a*g-b*e)/(4*a
*c-b^2))/(c*x^4+b*x^2+a)+1/4*sum((-1/a*(a*b*h-2*a*c*f+b*c*d)/(4*a*c-b^2)*_
R^2-2*(b*g-2*c*e)/(4*a*c-b^2)*_R+(2*a^2*h-a*b*f+6*a*c*d-b^2*d)/a/(4*a*c-b^
2))/(2*_R^3*c+_R*b)*ln(x-_R),_R=RootOf(_Z^4*c+_Z^2*b+a))
```

3.39.5 Fricas [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{\left(a+bx^2+cx^4\right)^2}\,dx = \text{Timed out}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x, algorithm="fricas
")
```

output
```
Timed out
```

3.39.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^2}\,dx=\text{Timed out}$$

input
```
integrate((h*x**4+g*x**3+f*x**2+e*x+d)/(c*x**4+b*x**2+a)**2,x)
```

output
```
Timed out
```

3.39.7 Maxima [F]

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^2}\,dx=\int \frac{hx^4+gx^3+fx^2+ex+d}{(cx^4+bx^2+a)^2}\,dx$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x, algorithm="maxima
")
```

output
```
1/2*((b*c*d - 2*a*c*f + a*b*h)*x^3 - a*b*e + 2*a^2*g - (2*a*c*e - a*b*g)*x
^2 - (a*b*f - 2*a^2*h - (b^2 - 2*a*c)*d)*x)/((a*b^2*c - 4*a^2*c^2)*x^4 + a
^2*b^2 - 4*a^3*c + (a*b^3 - 4*a^2*b*c)*x^2) + 1/2*integrate((a*b*f - 2*a^2
*h + (b*c*d - 2*a*c*f + a*b*h)*x^2 + (b^2 - 6*a*c)*d - 2*(2*a*c*e - a*b*g)
*x)/(c*x^4 + b*x^2 + a), x)/(a*b^2 - 4*a^2*c)
```

3.39.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 7495 vs. $2(391) = 782$.

Time = 1.73 (sec) , antiderivative size = 7495, normalized size of antiderivative = 17.07

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^2}\,dx=\text{Too large to display}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x, algorithm="giac")
```

output
```
1/2*(b*c*d*x^3 - 2*a*c*f*x^3 + a*b*h*x^3 - 2*a*c*e*x^2 + a*b*g*x^2 + b^2*d
*x - 2*a*c*d*x - a*b*f*x + 2*a^2*h*x - a*b*e + 2*a^2*g)/((c*x^4 + b*x^2 +
a)*(a*b^2 - 4*a^2*c)) + 1/16*((2*b^3*c^3 - 8*a*b*c^4 - sqrt(2)*sqrt(b^2 -
4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^3*c + 4*sqrt(2)*sqrt(b^2 - 4*a*c)
*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c^2 + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqr
t(b*c + sqrt(b^2 - 4*a*c)*c)*b^2*c^2 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c
+ sqrt(b^2 - 4*a*c)*c)*b*c^3 - 2*(b^2 - 4*a*c)*b*c^3)*(a*b^2 - 4*a^2*c)^2*
d - 2*(2*a*b^2*c^3 - 8*a^2*c^4 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt
(b^2 - 4*a*c)*c)*a*b^2*c + 4*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2
 - 4*a*c)*c)*a^2*c^2 + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4
*a*c)*c)*a*b*c^2 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*
c)*a*c^3 - 2*(b^2 - 4*a*c)*a*c^3)*(a*b^2 - 4*a^2*c)^2*f + (2*a*b^3*c^2 - 8
*a^2*b*c^3 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b
^3 + 4*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b*c +
 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^2*c - sqr
t(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c^2 - 2*(b^2 -
4*a*c)*a*b*c^2)*(a*b^2 - 4*a^2*c)^2*h + 2*(sqrt(2)*sqrt(b*c + sqrt(b^2 - 4
*a*c)*c)*a*b^6*c - 14*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^4*c^2
- 2*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^5*c^2 - 2*a*b^6*c^2 + 64*s
qrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*b^2*c^3 + 20*sqrt(2)*sqrt(b*...
```

3.39.9 Mupad [B] (verification not implemented)

Time = 8.80 (sec) , antiderivative size = 13024, normalized size of antiderivative = 29.67

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^2}\,dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4)/(a + b*x^2 + c*x^4)^2,x)
```

output
```
((b*e - 2*a*g)/(2*(4*a*c - b^2)) + (x^2*(2*c*e - b*g))/(2*(4*a*c - b^2)) -
 (x*(b^2*d + 2*a^2*h - 2*a*c*d - a*b*f))/(2*a*(4*a*c - b^2)) - (x^3*(b*c*d
 - 2*a*c*f + a*b*h))/(2*a*(4*a*c - b^2)))/(a + b*x^2 + c*x^4) + symsum(log
((5*b^3*c^4*d^3 + 8*a^3*c^4*f^3 - 96*a^2*c^5*d*e^2 + 72*a^2*c^5*d^2*f - 3*
a^3*b^3*c*h^3 - 4*a^4*b*c^2*h^3 - 3*b^4*c^3*d^2*f - 32*a^3*c^4*e^2*h + b^5
*c^2*d^2*h + 8*a^4*c^3*f*h^2 + 6*a^2*b^2*c^3*f^3 - 36*a*b*c^5*d^3 + a*b^5*
c*d*h^2 + 48*a^3*c^4*d*f*h + 16*a*b^2*c^4*d*e^2 + 18*a*b^2*c^4*d^2*f + 3*a
*b^3*c^3*d*f^2 - 60*a^2*b*c^4*d*f^2 + 4*a*b^4*c^2*d*g^2 + 16*a^2*b*c^4*e^2
*f - a*b^3*c^3*d^2*h - 60*a^2*b*c^4*d^2*h - 28*a^3*b*c^3*d*h^2 + a^2*b^4*c
*f*h^2 - 28*a^3*b*c^3*f^2*h - 24*a^2*b^2*c^3*d*g^2 - 9*a^2*b^3*c^2*d*h^2 +
 4*a^2*b^3*c^2*f*g^2 - 5*a^2*b^3*c^2*f^2*h + 18*a^3*b^2*c^2*f*h^2 - 8*a^3*
b^2*c^2*g^2*h - 16*a*b^3*c^3*d*e*g + 96*a^2*b*c^4*d*e*g - 4*a*b^4*c^2*d*f*
h + 32*a^3*b*c^3*e*g*h + 52*a^2*b^2*c^3*d*f*h - 16*a^2*b^2*c^3*e*f*g)/(8*(
a^2*b^6 - 64*a^5*c^3 - 12*a^3*b^4*c + 48*a^4*b^2*c^2)) - root(1572864*a^8*
b^2*c^6*z^4 - 983040*a^7*b^4*c^5*z^4 + 327680*a^6*b^6*c^4*z^4 - 61440*a^5*
b^8*c^3*z^4 + 6144*a^4*b^10*c^2*z^4 - 256*a^3*b^12*c*z^4 - 1048576*a^9*c^7
*z^4 + 192*a^3*b^8*c*f*h*z^2 + 57344*a^6*b*c^5*d*h*z^2 + 32768*a^6*b*c^5*e
*g*z^2 + 96*a^2*b^9*c*d*h*z^2 - 32*a*b^10*c*d*f*z^2 + 6144*a^5*b^4*c^3*f*h
*z^2 - 2048*a^4*b^6*c^2*f*h*z^2 - 49152*a^5*b^3*c^4*d*h*z^2 - 24576*a^5*b^
3*c^4*e*g*z^2 + 15360*a^4*b^5*c^3*d*h*z^2 + 6144*a^4*b^5*c^3*e*g*z^2 - ...
```

3.39. $\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^2}\,dx$

3.40 $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^2}\,dx$

3.40.1 Optimal result

Integrand size = 40, antiderivative size = 468

$$\begin{aligned}
&\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^2}\,dx \\
&= \frac{x(b^2d-abf-2a(cd-ah)+(bcd-2acf+abh)x^2)}{2a(b^2-4ac)(a+bx^2+cx^4)} \\
&+ \frac{2acg-b(ce+ai)-(2c^2e-bcg+b^2i-2aci)x^2}{2c(b^2-4ac)(a+bx^2+cx^4)} \\
&+ \frac{\left(bcd-2acf+abh+\frac{4abcf+b^2(cd-ah)-4ac(3cd+ah)}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{2\sqrt{2}a\sqrt{c}(b^2-4ac)\sqrt{b-\sqrt{b^2-4ac}}} \\
&+ \frac{\left(bcd-2acf+abh-\frac{4abcf+b^2(cd-ah)-4ac(3cd+ah)}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{2\sqrt{2}a\sqrt{c}(b^2-4ac)\sqrt{b+\sqrt{b^2-4ac}}} \\
&+ \frac{(2ce-bg+2ai)\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}
\end{aligned}$$

output
```
1/2*x*(b^2*d-a*b*f-2*a*(-a*h+c*d)+(a*b*h-2*a*c*f+b*c*d)*x^2)/a/(-4*a*c+b^2
)/(c*x^4+b*x^2+a)+1/2*(2*a*c*g-b*(a*i+c*e)-(-2*a*c*i+b^2*i-b*c*g+2*c^2*e)*
x^2)/c/(-4*a*c+b^2)/(c*x^4+b*x^2+a)+(2*a*i-b*g+2*c*e)*arctanh((2*c*x^2+b)/
(-4*a*c+b^2)^(1/2))/(-4*a*c+b^2)^(3/2)+1/4*arctan(x*2^(1/2)*c^(1/2)/(b-(-4
*a*c+b^2)^(1/2))^(1/2))*(b*c*d-2*a*c*f+a*b*h+(4*a*b*c*f+b^2*(-a*h+c*d)-4*a
*c*(a*h+3*c*d))/(-4*a*c+b^2)^(1/2))/a/(-4*a*c+b^2)*2^(1/2)/c^(1/2)/(b-(-4*
a*c+b^2)^(1/2))^(1/2)+1/4*arctan(x*2^(1/2)*c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^
(1/2))*(b*c*d-2*a*c*f+a*b*h+(-4*a*b*c*f-b^2*(-a*h+c*d)+4*a*c*(a*h+3*c*d))/
(-4*a*c+b^2)^(1/2))/a/(-4*a*c+b^2)*2^(1/2)/c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^
(1/2)
```

3.40.2 Mathematica [A] (verified)

Time = 1.21 (sec) , antiderivative size = 524, normalized size of antiderivative = 1.12

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^2}\,dx$$

$$= \frac{1}{4}\left(\frac{2(-bcdx(b+cx^2)+a^2(bi-2c(g+x(h+ix)))+a(b^2ix^2+2c^2x(d+x(e+fx))+bc(e+x(f-x(g}{ac(-b^2+4ac)(a+bx^2+cx^4)}\right.$$

$$+\frac{\sqrt{2}\left(b^2(cd-ah)-2ac\left(6cd+\sqrt{b^2-4ac}f+2ah\right)+b\left(c\sqrt{b^2-4ac}d+4acf+a\sqrt{b^2-4ac}h\right)\right)\arctan\left(\right.}{a\sqrt{c}\,(b^2-4ac)^{3/2}\sqrt{b-\sqrt{b^2-4ac}}}$$

$$+\frac{\sqrt{2}\left(b^2(-cd+ah)+2ac\left(6cd-\sqrt{b^2-4ac}f+2ah\right)+b\left(c\sqrt{b^2-4ac}d-4acf+a\sqrt{b^2-4ac}h\right)\right)\arctan\left(\right.}{a\sqrt{c}\,(b^2-4ac)^{3/2}\sqrt{b+\sqrt{b^2-4ac}}}$$

$$+\frac{2(-2ce+bg-2ai)\log\left(-b+\sqrt{b^2-4ac}-2cx^2\right)}{(b^2-4ac)^{3/2}}$$

$$\left.+\frac{2(2ce-bg+2ai)\log\left(b+\sqrt{b^2-4ac}+2cx^2\right)}{(b^2-4ac)^{3/2}}\right)$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(a + b*x^2 + c*x^4)^2,
x]
```

output
```
((2*(-(b*c*d*x*(b + c*x^2)) + a^2*(b*i - 2*c*(g + x*(h + i*x))) + a*(b^2*i
*x^2 + 2*c^2*x*(d + x*(e + f*x)) + b*c*(e + x*(f - x*(g + h*x))))))/(a*c*(
-b^2 + 4*a*c)*(a + b*x^2 + c*x^4)) + (Sqrt[2]*(b^2*(c*d - a*h) - 2*a*c*(6*
c*d + Sqrt[b^2 - 4*a*c]*f + 2*a*h) + b*(c*Sqrt[b^2 - 4*a*c]*d + 4*a*c*f +
a*Sqrt[b^2 - 4*a*c]*h))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a
*c]]])/(a*Sqrt[c]*(b^2 - 4*a*c)^(3/2)*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + (Sqrt
[2]*(b^2*(-(c*d) + a*h) + 2*a*c*(6*c*d - Sqrt[b^2 - 4*a*c]*f + 2*a*h) + b*
(c*Sqrt[b^2 - 4*a*c]*d - 4*a*c*f + a*Sqrt[b^2 - 4*a*c]*h))*ArcTan[(Sqrt[2]
*Sqrt[c]*x)/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(a*Sqrt[c]*(b^2 - 4*a*c)^(3/2)*S
qrt[b + Sqrt[b^2 - 4*a*c]]) + (2*(-2*c*e + b*g - 2*a*i)*Log[-b + Sqrt[b^2
- 4*a*c] - 2*c*x^2]])/(b^2 - 4*a*c)^(3/2) + (2*(2*c*e - b*g + 2*a*i)*Log[b
+ Sqrt[b^2 - 4*a*c] + 2*c*x^2]])/(b^2 - 4*a*c)^(3/2))/4
```

3.40.3 Rubi [A] (verified)

Time = 0.82 (sec) , antiderivative size = 460, normalized size of antiderivative = 0.98, number of steps used = 11, number of rules used = 10, $\frac{\text{number of rules}}{\text{integrand size}} = 0.250$, Rules used = {2202, 2194, 2191, 27, 1083, 219, 2206, 25, 1480, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^2}\,dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{hx^4+fx^2+d}{(cx^4+bx^2+a)^2}dx+\int \frac{x(ix^4+gx^2+e)}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \text{2194}$$

$$\int \frac{hx^4+fx^2+d}{(cx^4+bx^2+a)^2}dx+\frac{1}{2}\int \frac{ix^4+gx^2+e}{(cx^4+bx^2+a)^2}dx^2$$

$$\Big\downarrow \text{2191}$$

$$\frac{1}{2}\left(\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{c\,(b^2-4ac)\,(a+bx^2+cx^4)}-\frac{\int \frac{2ce-bg+2ai}{cx^4+bx^2+a}dx^2}{b^2-4ac}\right)+\int \frac{hx^4+fx^2+d}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \text{27}$$

$$\frac{1}{2}\left(\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{c\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}-\frac{(2ai-bg+2ce)\int\frac{1}{cx^4+bx^2+a}dx^2}{b^2-4ac}\right)+\int\frac{hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^2}dx$$

$$\Big\downarrow \text{ 1083}$$

$$\frac{1}{2}\left(\frac{2(2ai-bg+2ce)\int\frac{1}{-x^4+b^2-4ac}d\left(2cx^2+b\right)}{b^2-4ac}+\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{c\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)+\int\frac{hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^2}dx$$

$$\Big\downarrow \text{ 219}$$

$$\int\frac{hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^2}dx+\frac{1}{2}\left(\frac{2\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)(2ai-bg+2ce)}{\left(b^2-4ac\right)^{3/2}}+\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{c\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)$$

$$\Big\downarrow \text{ 2206}$$

$$-\frac{\int-\frac{db^2+afb+(bcd-2acf+abh)x^2-2a(3cd+ah)}{cx^4+bx^2+a}dx}{2a\left(b^2-4ac\right)}+\frac{1}{2}\left(\frac{2\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)(2ai-bg+2ce)}{\left(b^2-4ac\right)^{3/2}}+\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{c\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)+\frac{x\left(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

$$\Big\downarrow \text{ 25}$$

$$\frac{\int\frac{db^2+afb+(bcd-2acf+abh)x^2-2a(3cd+ah)}{cx^4+bx^2+a}dx}{2a\left(b^2-4ac\right)}+\frac{1}{2}\left(\frac{2\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)(2ai-bg+2ce)}{\left(b^2-4ac\right)^{3/2}}+\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{c\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)+\frac{x\left(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

$$\Big\downarrow \text{ 1480}$$

3.40. $\int\frac{d+ex+fx^2+gx^3+hx^4+ix^5}{\left(a+bx^2+cx^4\right)^2}dx$

$$\frac{\frac{1}{2}\left(\frac{b^2(cd-ah)+4abcf-4ac(ah+3cd)}{\sqrt{b^2-4ac}}+abh-2acf+bcd\right)\int \frac{1}{cx^2+\frac{1}{2}\left(b-\sqrt{b^2-4ac}\right)}dx+\frac{1}{2}\left(-\frac{b^2(cd-ah)+4abcf-4ac(ah+3cd)}{\sqrt{b^2-4ac}}+abh\right.}{2a\left(b^2-4ac\right)}$$

$$\frac{1}{2}\left(\frac{2\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)(2ai-bg+2ce)}{\left(b^2-4ac\right)^{3/2}}+\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{c\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)+$$

$$\frac{x\left(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

$$\Big\downarrow \text{218}$$

$$\frac{\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{b^2(cd-ah)+4abcf-4ac(ah+3cd)}{\sqrt{b^2-4ac}}+abh-2acf+bcd\right)}{\sqrt{2}\sqrt{c}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{b^2(cd-ah)+4abcf-4ac(ah+3cd)}{\sqrt{b^2-4ac}}+abh-2a\right.}{\sqrt{2}\sqrt{c}\sqrt{\sqrt{b^2-4ac}+b}}}{2a\left(b^2-4ac\right)}$$

$$\frac{1}{2}\left(\frac{2\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)(2ai-bg+2ce)}{\left(b^2-4ac\right)^{3/2}}+\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{c\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)+$$

$$\frac{x\left(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(a + b*x^2 + c*x^4)^2,x]
```

output
```
(x*(b^2*d - a*b*f - 2*a*(c*d - a*h) + (b*c*d - 2*a*c*f + a*b*h)*x^2))/(2*a
*(b^2 - 4*a*c)*(a + b*x^2 + c*x^4)) + (((b*c*d - 2*a*c*f + a*b*h + (4*a*b*
c*f + b^2*(c*d - a*h) - 4*a*c*(3*c*d + a*h))/Sqrt[b^2 - 4*a*c])*ArcTan[(Sq
rt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[c]*Sqrt[b - S
qrt[b^2 - 4*a*c]]) + ((b*c*d - 2*a*c*f + a*b*h - (4*a*b*c*f + b^2*(c*d - a
*h) - 4*a*c*(3*c*d + a*h))/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/S
qrt[b + Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[c]*Sqrt[b + Sqrt[b^2 - 4*a*c]])
)/(2*a*(b^2 - 4*a*c)) + ((c*(2*a*g - b*(e + (a*i)/c)) - (2*c^2*e - b*c*g +
 b^2*i - 2*a*c*i)*x^2)/(c*(b^2 - 4*a*c)*(a + b*x^2 + c*x^4)) + (2*(2*c*e -
 b*g + 2*a*i)*ArcTanh[(b + 2*c*x^2)/Sqrt[b^2 - 4*a*c]])/(b^2 - 4*a*c)^(3/2
))/2
```

3.40. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{\left(a+bx^2+cx^4\right)^2}dx$

3.40.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 2191
```
Int[(Pq_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> With[{Q =
PolynomialQuotient[Pq, a + b*x + c*x^2, x], f = Coeff[PolynomialRemainder[P
q, a + b*x + c*x^2, x], x, 0], g = Coeff[PolynomialRemainder[Pq, a + b*x +
c*x^2, x], x, 1]}, Simp[(b*f - 2*a*g + (2*c*f - b*g)*x)*((a + b*x + c*x^2)^
(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] + Simp[1/((p + 1)*(b^2 - 4*a*c))   Int
[(a + b*x + c*x^2)^(p + 1)*ExpandToSum[(p + 1)*(b^2 - 4*a*c)*Q - (2*p + 3)*
(2*c*f - b*g), x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Pq, x] && NeQ[b^
2 - 4*a*c, 0] && LtQ[p, -1]
```

3.40. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^2}\,dx$

rule 2194
```
Int[(Pq_)*(x_)^(m_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :
> Simp[1/2   Subst[Int[x^((m - 1)/2)*SubstFor[x^2, Pq, x]*(a + b*x + c*x^2)
^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pq, x^2] && IntegerQ
[(m - 1)/2]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2206
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> With[{d =
 Coeff[PolynomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 0], e = Coeff[Poly
nomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 2]}, Simp[x*(a + b*x^2 + c*x^
4)^(p + 1)*((a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)/(2*a*(p + 1)*(b
^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(a + b*x^2 + c
*x^4)^(p + 1)*ExpandToSum[2*a*(p + 1)*(b^2 - 4*a*c)*PolynomialQuotient[Px,
a + b*x^2 + c*x^4, x] + b^2*d*(2*p + 3) - 2*a*c*d*(4*p + 5) - a*b*e + c*(4*
p + 7)*(b*d - 2*a*e)*x^2, x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x
^2] && Expon[Px, x^2] > 1 && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1]
```

3.40.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.15 (sec) , antiderivative size = 304, normalized size of antiderivative = 0.65

method	result
risch	$\frac{-\frac{(abh-2acf+bcd)x^3}{2a\left(4ac-b^2\right)}-\frac{\left(2aci-b^2i+gbc-2e\,c^2\right)x^2}{2c\left(4ac-b^2\right)}-\frac{\left(2a^2h-abf-2acd+b^2d\right)x}{2a\left(4ac-b^2\right)}+\frac{abi-2acg+ebc}{2c\left(4ac-b^2\right)}}{c\,x^4+b\,x^2+a}+\frac{\left(\sum_{\_R=\mathrm{RootOf}\left(c\_Z^4+\_Z^2b+a\right)}\frac{\left(-\frac{(abh-\cdots}{\cdots}\right.}{\cdots}\right.}{\cdots}$
default	$\frac{-\frac{(abh-2acf+bcd)x^3}{2a\left(4ac-b^2\right)}-\frac{\left(2aci-b^2i+gbc-2e\,c^2\right)x^2}{2c\left(4ac-b^2\right)}-\frac{\left(2a^2h-abf-2acd+b^2d\right)x}{2a\left(4ac-b^2\right)}+\frac{abi-2acg+ebc}{2c\left(4ac-b^2\right)}}{c\,x^4+b\,x^2+a}+\frac{2c\left(\frac{\left(8\sqrt{-4ac+b^2}\,a^2ci-4\sqrt{-4ac+b^2}\,abcg+\cdots\right.}{\cdots}\right.}{\cdots}$

3.40. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{\left(a+bx^2+cx^4\right)^2}\,dx$

input
```
int((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x,method=_RETURNVERB
OSE)
```

output
```
(-1/2/a*(a*b*h-2*a*c*f+b*c*d)/(4*a*c-b^2)*x^3-1/2*(2*a*c*i-b^2*i+b*c*g-2*c
^2*e)/c/(4*a*c-b^2)*x^2-1/2*(2*a^2*h-a*b*f-2*a*c*d+b^2*d)/a/(4*a*c-b^2)*x+
1/2/c*(a*b*i-2*a*c*g+b*c*e)/(4*a*c-b^2))/(c*x^4+b*x^2+a)+1/4*sum((-1/a*(a*
b*h-2*a*c*f+b*c*d)/(4*a*c-b^2)*_R^2+2*(2*a*i-b*g+2*c*e)/(4*a*c-b^2)*_R+(2*
a^2*h-a*b*f+6*a*c*d-b^2*d)/a/(4*a*c-b^2))/(2*_R^3*c+_R*b)*ln(x-_R),_R=Root
Of(_Z^4*c+_Z^2*b+a))
```

3.40.5 Fricas [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^2}\,dx=\text{Timed out}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x, algorithm="
fricas")
```

output
```
Timed out
```

3.40.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^2}\,dx=\text{Timed out}$$

input
```
integrate((i*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(c*x**4+b*x**2+a)**2,x)
```

output
```
Timed out
```

3.40.7 Maxima [F]

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^2}\,dx = \int \frac{ix^5+hx^4+gx^3+fx^2+ex+d}{(cx^4+bx^2+a)^2}\,dx$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x, algorithm="
maxima")
```

output
```
-1/2*(a*b*c*e - 2*a^2*c*g + a^2*b*i - (b*c^2*d - 2*a*c^2*f + a*b*c*h)*x^3
+ (2*a*c^2*e - a*b*c*g + (a*b^2 - 2*a^2*c)*i)*x^2 + (a*b*c*f - 2*a^2*c*h -
 (b^2*c - 2*a*c^2)*d)*x)/(a^2*b^2*c - 4*a^3*c^2 + (a*b^2*c^2 - 4*a^2*c^3)*
x^4 + (a*b^3*c - 4*a^2*b*c^2)*x^2) + 1/2*integrate((a*b*f - 2*a^2*h + (b*c
*d - 2*a*c*f + a*b*h)*x^2 + (b^2 - 6*a*c)*d - 2*(2*a*c*e - a*b*g + 2*a^2*i
)*x)/(c*x^4 + b*x^2 + a), x)/(a*b^2 - 4*a^2*c)
```

3.40.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 7962 vs. 2(420) = 840.

Time = 1.91 (sec) , antiderivative size = 7962, normalized size of antiderivative = 17.01

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^2}\,dx = \text{Too large to display}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x, algorithm="
giac")
```

output
```
1/2*(b*c^2*d*x^3 - 2*a*c^2*f*x^3 + a*b*c*h*x^3 - 2*a*c^2*e*x^2 + a*b*c*g*x
^2 - a*b^2*i*x^2 + 2*a^2*c*i*x^2 + b^2*c*d*x - 2*a*c^2*d*x - a*b*c*f*x + 2
*a^2*c*h*x - a*b*c*e + 2*a^2*c*g - a^2*b*i)/((c*x^4 + b*x^2 + a)*(a*b^2*c
- 4*a^2*c^2)) + 1/16*((2*b^3*c^3 - 8*a*b*c^4 - sqrt(2)*sqrt(b^2 - 4*a*c)*s
qrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^3*c + 4*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*
c + sqrt(b^2 - 4*a*c)*c)*a*b*c^2 + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c +
sqrt(b^2 - 4*a*c)*c)*b^2*c^2 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b
^2 - 4*a*c)*c)*b*c^3 - 2*(b^2 - 4*a*c)*b*c^3)*(a*b^2 - 4*a^2*c)^2*d - 2*(2
*a*b^2*c^3 - 8*a^2*c^4 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4
*a*c)*c)*a*b^2*c + 4*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c
)*c)*a^2*c^2 + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)
*a*b*c^2 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*c^3
 - 2*(b^2 - 4*a*c)*a*c^3)*(a*b^2 - 4*a^2*c)^2*f + (2*a*b^3*c^2 - 8*a^2*b*c
^3 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^3 + 4*s
qrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b*c + 2*sqrt(
2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^2*c - sqrt(2)*sqr
t(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c^2 - 2*(b^2 - 4*a*c)*a
*b*c^2)*(a*b^2 - 4*a^2*c)^2*h + 2*(sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)
*a*b^6*c - 14*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^4*c^2 - 2*sqrt
(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^5*c^2 - 2*a*b^6*c^2 + 64*sqrt(2...
```

3.40.9 Mupad [B] (verification not implemented)

Time = 9.41 (sec) , antiderivative size = 18449, normalized size of antiderivative = 39.42

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^2}\,dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(a + b*x^2 + c*x^4)^2,x)
```

output
```
((b*c*e - 2*a*c*g + a*b*i)/(2*c*(4*a*c - b^2)) - (x*(b^2*d + 2*a^2*h - 2*a
*c*d - a*b*f))/(2*a*(4*a*c - b^2)) + (x^2*(2*c^2*e + b^2*i - b*c*g - 2*a*c
*i))/(2*c*(4*a*c - b^2)) - (x^3*(b*c*d - 2*a*c*f + a*b*h))/(2*a*(4*a*c - b
^2)))/(a + b*x^2 + c*x^4) + symsum(log((5*b^3*c^4*d^3 + 8*a^3*c^4*f^3 - 96
*a^2*c^5*d*e^2 + 72*a^2*c^5*d^2*f - 3*a^3*b^3*c*h^3 - 4*a^4*b*c^2*h^3 - 3*
b^4*c^3*d^2*f - 32*a^3*c^4*e^2*h - 96*a^4*c^3*d*i^2 + b^5*c^2*d^2*h + 8*a^
4*c^3*f*h^2 - 32*a^5*c^2*h*i^2 + 6*a^2*b^2*c^3*f^3 - 36*a*b*c^5*d^3 + a*b^
5*c*d*h^2 - 192*a^3*c^4*d*e*i + 48*a^3*c^4*d*f*h - 64*a^4*c^3*e*h*i + 16*a
*b^2*c^4*d*e^2 + 18*a*b^2*c^4*d^2*f + 3*a*b^3*c^3*d*f^2 - 60*a^2*b*c^4*d*f
^2 + 4*a*b^4*c^2*d*g^2 + 16*a^2*b*c^4*e^2*f - a*b^3*c^3*d^2*h - 60*a^2*b*c
^4*d^2*h - 28*a^3*b*c^3*d*h^2 + a^2*b^4*c*f*h^2 - 28*a^3*b*c^3*f^2*h + 16*
a^4*b*c^2*f*i^2 - 24*a^2*b^2*c^3*d*g^2 - 9*a^2*b^3*c^2*d*h^2 + 4*a^2*b^3*c
^2*f*g^2 + 16*a^3*b^2*c^2*d*i^2 - 5*a^2*b^3*c^2*f^2*h + 18*a^3*b^2*c^2*f*h
^2 - 8*a^3*b^2*c^2*g^2*h - 16*a*b^3*c^3*d*e*g + 96*a^2*b*c^4*d*e*g - 4*a*b
^4*c^2*d*f*h + 96*a^3*b*c^3*d*g*i + 32*a^3*b*c^3*e*f*i + 32*a^3*b*c^3*e*g*
h + 32*a^4*b*c^2*g*h*i + 32*a^2*b^2*c^3*d*e*i + 52*a^2*b^2*c^3*d*f*h - 16*
a^2*b^2*c^3*e*f*g - 16*a^2*b^3*c^2*d*g*i - 16*a^3*b^2*c^2*f*g*i)/(8*(a^2*b
^6 - 64*a^5*c^3 - 12*a^3*b^4*c + 48*a^4*b^2*c^2)) - root(1572864*a^8*b^2*c
^6*z^4 - 983040*a^7*b^4*c^5*z^4 + 327680*a^6*b^6*c^4*z^4 - 61440*a^5*b^8*c
^3*z^4 + 6144*a^4*b^10*c^2*z^4 - 256*a^3*b^12*c*z^4 - 1048576*a^9*c^7*z...
```

3.40. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^2}\,dx$

3.41 $\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^2}\,dx$

3.41.1 Optimal result

Integrand size = 55, antiderivative size = 770

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^2}\,dx$$
$$= \frac{mx}{c^2} - \frac{bc(ce+aj)-ab^2l-2ac(cg-al)+(2c^3e-c^2(bg+2aj)-b^3l+bc(bj+3al))\,x^2}{2c^2\,(b^2-4ac)\,(a+bx^2+cx^4)}$$
$$- \frac{x(abc(cf+ak)-b^2(c^2d+a^2m)+2ac(c^2d-ach+a^2m)+(ab^2ck+2ac^2(cf-ak)-ab^3m-bc(c^2d+\cdots}{2ac^2\,(b^2-4ac)\,(a+bx^2+cx^4)}$$
$$+ \frac{\left(ab^2ck-2ac^2(cf+3ak)-3ab^3m+bc(c^2d+ach+13a^2m)-\frac{ab^3ck-4abc^2(cf+2ak)-3ab^4m-b^2c(c^2d-ach-19a^2\cdots}{\sqrt{b^2-4ac}}\right.\cdots}{2\sqrt{2}ac^{5/2}\,(b^2-4ac)\,\sqrt{b-\sqrt{b^2-4ac}}\cdots}$$
$$+ \frac{\left(ab^2ck-2ac^2(cf+3ak)-3ab^3m+bc(c^2d+ach+13a^2m)+\frac{ab^3ck-4abc^2(cf+2ak)-3ab^4m-b^2c(c^2d-ach-19a^2\cdots}{\sqrt{b^2-4ac}}\right.\cdots}{2\sqrt{2}ac^{5/2}\,(b^2-4ac)\,\sqrt{b+\sqrt{b^2-4ac}}\cdots}$$
$$+ \frac{(4c^3e-c^2(2bg-4aj)+b^3l-6abcl)\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{2c^2\,(b^2-4ac)^{3/2}} + \frac{l\log(a+bx^2+cx^4)}{4c^2}$$

output
```
m*x/c^2+1/2*(-b*c*(a*j+c*e)+a*b^2*l+2*a*c*(-a*l+c*g)-(2*c^3*e-c^2*(2*a*j+b
*g)-b^3*l+b*c*(3*a*l+b*j))*x^2)/c^2/(-4*a*c+b^2)/(c*x^4+b*x^2+a)-1/2*x*(a*
b*c*(a*k+c*f)-b^2*(a^2*m+c^2*d)+2*a*c*(a^2*m-a*c*h+c^2*d)+(a*b^2*c*k+2*a*c
^2*(-a*k+c*f)-a*b^3*m-b*c*(-3*a^2*m+a*c*h+c^2*d))*x^2)/a/c^2/(-4*a*c+b^2)/
(c*x^4+b*x^2+a)+1/2*(4*c^3*e-c^2*(-4*a*j+2*b*g)+b^3*l-6*a*b*c*l)*arctanh((
2*c*x^2+b)/(-4*a*c+b^2)^(1/2))/c^2/(-4*a*c+b^2)^(3/2)+1/4*l*ln(c*x^4+b*x^2
+a)/c^2+1/4*arctan(x*2^(1/2)*c^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2))*(a*b^2*
c*k-2*a*c^2*(3*a*k+c*f)-3*a*b^3*m+b*c*(13*a^2*m+a*c*h+c^2*d)+(-a*b^3*c*k+4
*a*b*c^2*(2*a*k+c*f)+3*a*b^4*m+b^2*c*(-19*a^2*m-a*c*h+c^2*d)-4*a*c^2*(-5*a
^2*m+a*c*h+3*c^2*d))/(-4*a*c+b^2)^(1/2))/a/c^(5/2)/(-4*a*c+b^2)*2^(1/2)/(b
-(-4*a*c+b^2)^(1/2))^(1/2)+1/4*arctan(x*2^(1/2)*c^(1/2)/(b+(-4*a*c+b^2)^(1
/2))^(1/2))*(a*b^2*c*k-2*a*c^2*(3*a*k+c*f)-3*a*b^3*m+b*c*(13*a^2*m+a*c*h+c
^2*d)+(a*b^3*c*k-4*a*b*c^2*(2*a*k+c*f)-3*a*b^4*m-b^2*c*(-19*a^2*m-a*c*h+c^
2*d)+4*a*c^2*(-5*a^2*m+a*c*h+3*c^2*d))/(-4*a*c+b^2)^(1/2))/a/c^(5/2)/(-4*a
*c+b^2)*2^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2)
```

3.41.2 Mathematica [A] (verified)

Time = 3.22 (sec) , antiderivative size = 935, normalized size of antiderivative = 1.21

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^2}\,dx$$

$$= \frac{4\sqrt{c}mx + \frac{2\sqrt{c}\left(2a^3c(l+mx)-bc^2dx\left(b+cx^2\right)+a\left(b^2cx^2(j+kx)-b^3x^2(l+mx)+2c^3x(d+x(e+fx))+bc^2(e+x(f-x(g+hx)))\right)-a^2\left(b^2(l+mx)+2c\right.\right.}{a(-b^2+4ac)(a+bx^2+cx^4)}}{}$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4 + j*x^5 + k*x^6 + l*x^7 + m*x^8
)/(a + b*x^2 + c*x^4)^2,x]
```

output
```
(4*Sqrt[c]*m*x + (2*Sqrt[c]*(2*a^3*c*(l + m*x) - b*c^2*d*x*(b + c*x^2) + a
*(b^2*c*x^2*(j + k*x) - b^3*x^2*(l + m*x) + 2*c^3*x*(d + x*(e + f*x)) + b*
c^2*(e + x*(f - x*(g + h*x)))) - a^2*(b^2*(l + m*x) + 2*c^2*(g + x*(h + x*
(j + k*x))) - b*c*(j + x*(k + 3*x*(l + m*x))))))/(a*(-b^2 + 4*a*c)*(a + b*
x^2 + c*x^4)) - (Sqrt[2]*(-3*a*b^4*m + 2*a*c^2*(6*c^2*d + c*Sqrt[b^2 - 4*a
*c]*f + 2*a*c*h + 3*a*Sqrt[b^2 - 4*a*c]*k - 10*a^2*m) + a*b^3*(c*k + 3*Sqr
t[b^2 - 4*a*c]*m) - b*c*(c^2*(Sqrt[b^2 - 4*a*c]*d + 4*a*f) + a*c*(Sqrt[b^2
 - 4*a*c]*h + 8*a*k) + 13*a^2*Sqrt[b^2 - 4*a*c]*m) + b^2*c*(-(c^2*d) + a*c
*h + a*(-(Sqrt[b^2 - 4*a*c]*k) + 19*a*m)))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt
[b - Sqrt[b^2 - 4*a*c]]])/(a*(b^2 - 4*a*c)^(3/2)*Sqrt[b - Sqrt[b^2 - 4*a*c
]]) - (Sqrt[2]*(3*a*b^4*m + 2*a*c^2*(-6*c^2*d + c*Sqrt[b^2 - 4*a*c]*f - 2*
a*c*h + 3*a*Sqrt[b^2 - 4*a*c]*k + 10*a^2*m) + a*b^3*(-(c*k) + 3*Sqrt[b^2 -
 4*a*c]*m) - b*c*(c^2*(Sqrt[b^2 - 4*a*c]*d - 4*a*f) + a*c*(Sqrt[b^2 - 4*a*
c]*h - 8*a*k) + 13*a^2*Sqrt[b^2 - 4*a*c]*m) + b^2*c*(c^2*d - a*c*h - a*(Sq
rt[b^2 - 4*a*c]*k + 19*a*m)))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqrt[b^2
 - 4*a*c]]])/(a*(b^2 - 4*a*c)^(3/2)*Sqrt[b + Sqrt[b^2 - 4*a*c]]) + (Sqrt[c
]*(-4*c^3*e + 2*c^2*(b*g - 2*a*j) + b^2*(-b + Sqrt[b^2 - 4*a*c])*l + a*c*(
6*b*l - 4*Sqrt[b^2 - 4*a*c]*l))*Log[-b + Sqrt[b^2 - 4*a*c] - 2*c*x^2])/(b^
2 - 4*a*c)^(3/2) + (Sqrt[c]*(4*c^3*e + c^2*(-2*b*g + 4*a*j) + b^2*(b + Sqr
t[b^2 - 4*a*c])*l - 2*a*c*(3*b + 2*Sqrt[b^2 - 4*a*c])*l)*Log[b + Sqrt[b...
```

3.41.3 Rubi [A] (verified)

Time = 4.26 (sec) , antiderivative size = 788, normalized size of antiderivative = 1.02, number of steps used = 12, number of rules used = 11, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.200, Rules used = {2202, 2194, 2191, 1142, 1083, 219, 1103, 2206, 25, 2205, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^2}dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{mx^8+kx^6+hx^4+fx^2+d}{(cx^4+bx^2+a)^2}dx+\int \frac{x(lx^6+jx^4+gx^2+e)}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \text{2194}$$

$$\int \frac{mx^8+kx^6+hx^4+fx^2+d}{(cx^4+bx^2+a)^2}dx+\frac{1}{2}\int \frac{lx^6+jx^4+gx^2+e}{(cx^4+bx^2+a)^2}dx^2$$

$$\Big\downarrow \text{2191}$$

$$\frac{1}{2}\left(-\frac{\int \frac{\left(4a-\frac{b^2}{c}\right)lx^2+2ce-bg+2aj-\frac{abl}{c}}{cx^4+bx^2+a}dx^2}{b^2-4ac}-\frac{x^2\left(-c^2(2aj+bg)+bc(3al+bj)+b^3(-l)+2c^3e\right)-ab^2l+bc(aj+ce)-}{c^2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right.$$

$$\int \frac{mx^8+kx^6+hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^2}dx$$

$$\Big\downarrow \text{1142}$$

$$\frac{1}{2}\left(-\frac{\frac{\left(-c^2(2bg-4aj)-6abcl+b^3l+4c^3e\right)\int \frac{1}{cx^4+bx^2+a}dx^2}{2c^2}-\frac{l\left(b^2-4ac\right)\int \frac{2cx^2+b}{cx^4+bx^2+a}dx^2}{2c^2}}{b^2-4ac}-\frac{x^2\left(-c^2(2aj+bg)+bc(3al+bj)+b^3(\right.}{c^2\left(b^2-4\right.}\right.$$

$$\int \frac{mx^8+kx^6+hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^2}dx$$

$$\Big\downarrow \text{1083}$$

$$\frac{1}{2}\left(-\frac{-\frac{l\left(b^2-4ac\right)\int \frac{2cx^2+b}{cx^4+bx^2+a}dx^2}{2c^2}-\frac{\left(-c^2(2bg-4aj)-6abcl+b^3l+4c^3e\right)\int \frac{1}{-x^4+b^2-4ac}d\left(2cx^2+b\right)}{c^2}}{b^2-4ac}-\frac{x^2\left(-c^2(2aj+bg)+bc(3al+\right.}{c}\right.$$

$$\int \frac{mx^8+kx^6+hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^2}dx$$

$$\Big\downarrow \text{219}$$

$$\frac{1}{2}\left(-\frac{-\frac{l\left(b^2-4ac\right)\int \frac{2cx^2+b}{cx^4+bx^2+a}dx^2}{2c^2}-\frac{\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-c^2(2bg-4aj)-6abcl+b^3l+4c^3e\right)}{c^2\sqrt{b^2-4ac}}}{b^2-4ac}-\frac{x^2\left(-c^2(2aj+bg)+bc(3al+bj)\right.}{c^2\left(\right.}\right.$$

$$\int \frac{mx^8+kx^6+hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^2}dx$$

$$\Big\downarrow \text{1103}$$

$$\int \frac{mx^8+kx^6+hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^2}dx+$$

$$\frac{1}{2}\left(-\frac{-\frac{\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-c^2(2bg-4aj)-6abcl+b^3l+4c^3e\right)}{c^2\sqrt{b^2-4ac}}-\frac{l\left(b^2-4ac\right)\log\left(a+bx^2+cx^4\right)}{2c^2}}{b^2-4ac}-\frac{x^2\left(-c^2(2aj+bg)+bc(3al+bj)\right.}{c^2\left(\right.}\right.$$

$$\Big\downarrow \text{2206}$$

3.41. $\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{\left(a+bx^2+cx^4\right)^2}\,dx$

$$-\frac{\int -\frac{-2a\left(4a-\frac{b^2}{c}\right)mx^4+\frac{\left(-amb^3+ackb^2+c\left(5ma^2+cha+c^2d\right)b-2ac^2(cf+3ak)\right)x^2}{c^2}+\frac{\left(c^2d-a^2m\right)b^2+ac(cf+ak)b-2ac\left(-ma^2+cha+3c^2d\right)}{c^2}}{cx^4+bx^2+a}dx}{2a\left(b^2-4ac\right)}-$$
$$\frac{x\left(-\left(b^2\left(a^2m+c^2d\right)\right)+x^2\left(-bc\left(-3a^2m+ach+c^2d\right)-ab^3m+ab^2ck+2ac^2(cf-ak)\right)+2ac\left(a^2m-ach+c^2d\right)}{2ac^2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$
$$\frac{1}{2}\left(-\frac{-\frac{\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-c^2(2bg-4aj)-6abcl+b^3l+4c^3e\right)}{c^2\sqrt{b^2-4ac}}-\frac{l\left(b^2-4ac\right)\log\left(a+bx^2+cx^4\right)}{2c^2}}{b^2-4ac}-\frac{x^2\left(-c^2(2aj+bg)+bc(3al+bj)\right.}{c^2\left(\right.}\right.$$

$\downarrow$ 25

$$\frac{\int \frac{-2a\left(4a-\frac{b^2}{c}\right)mx^4+\frac{\left(-amb^3+ackb^2+c\left(5ma^2+cha+c^2d\right)b-2ac^2(cf+3ak)\right)x^2}{c^2}+\frac{\left(c^2d-a^2m\right)b^2+ac(cf+ak)b-2ac\left(-ma^2+cha+3c^2d\right)}{c^2}}{cx^4+bx^2+a}dx}{2a\left(b^2-4ac\right)}-$$
$$\frac{x\left(-\left(b^2\left(a^2m+c^2d\right)\right)+x^2\left(-bc\left(-3a^2m+ach+c^2d\right)-ab^3m+ab^2ck+2ac^2(cf-ak)\right)+2ac\left(a^2m-ach+c^2d\right)}{2ac^2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$
$$\frac{1}{2}\left(-\frac{-\frac{\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-c^2(2bg-4aj)-6abcl+b^3l+4c^3e\right)}{c^2\sqrt{b^2-4ac}}-\frac{l\left(b^2-4ac\right)\log\left(a+bx^2+cx^4\right)}{2c^2}}{b^2-4ac}-\frac{x^2\left(-c^2(2aj+bg)+bc(3al+bj)\right.}{c^2\left(\right.}\right.$$

$\downarrow$ 2205

$$\frac{\int \left(\frac{2a\left(b^2-4ac\right)m}{c^2}+\frac{\left(c^2d-3a^2m\right)b^2+ac(cf+ak)b+\left(-3amb^3+ackb^2+c\left(13ma^2+cha+c^2d\right)b-2ac^2(cf+3ak)\right)x^2-2ac\left(-5ma^2+cha+3c^2d\right)}{c^2\left(cx^4+bx^2+a\right)}\right)dx}{2a\left(b^2-4ac\right)}$$
$$\frac{x\left(-\left(b^2\left(a^2m+c^2d\right)\right)+x^2\left(-bc\left(-3a^2m+ach+c^2d\right)-ab^3m+ab^2ck+2ac^2(cf-ak)\right)+2ac\left(a^2m-ach+c^2d\right)}{2ac^2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$
$$\frac{1}{2}\left(-\frac{-\frac{\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-c^2(2bg-4aj)-6abcl+b^3l+4c^3e\right)}{c^2\sqrt{b^2-4ac}}-\frac{l\left(b^2-4ac\right)\log\left(a+bx^2+cx^4\right)}{2c^2}}{b^2-4ac}-\frac{x^2\left(-c^2(2aj+bg)+bc(3al+bj)\right.}{c^2\left(\right.}\right.$$

$\downarrow$ 2009

3.41. $\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{\left(a+bx^2+cx^4\right)^2}\,dx$

$$\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(-\frac{-b^2c\left(-19a^2m-ach+c^2d\right)+4ac^2\left(-5a^2m+ach+3c^2d\right)-3ab^4m+ab^3ck-4abc^2(2ak+cf)}{\sqrt{b^2-4ac}}+bc(13a^2m+ach+c^2d)-3ab^3m+ab^2c\right.}{\sqrt{2}c^{5/2}\sqrt{b-\sqrt{b^2-4ac}}}$$

$$\frac{x\left(-\left(b^2\left(a^2m+c^2d\right)\right)+x^2\left(-bc\left(-3a^2m+ach+c^2d\right)-ab^3m+ab^2ck+2ac^2(cf-ak)\right)+2ac\left(a^2m-ach+c^2d\right)\right.}{2ac^2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

$$\frac{1}{2}\left(-\frac{-\frac{\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-c^2(2bg-4aj)-6abcl+b^3l+4c^3e\right)}{c^2\sqrt{b^2-4ac}}-\frac{l\left(b^2-4ac\right)\log\left(a+bx^2+cx^4\right)}{2c^2}}{b^2-4ac}-\frac{x^2\left(-c^2(2aj+bg)+bc(3al+bj)\right.}{c^2\left(\right.}\right.$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4 + j*x^5 + k*x^6 + l*x^7 + m*x^8)/(a +
 b*x^2 + c*x^4)^2,x]
```

output
```
-1/2*(x*(a*b*c*(c*f + a*k) - b^2*(c^2*d + a^2*m) + 2*a*c*(c^2*d - a*c*h +
a^2*m) + (a*b^2*c*k + 2*a*c^2*(c*f - a*k) - a*b^3*m - b*c*(c^2*d + a*c*h -
 3*a^2*m))*x^2))/(a*c^2*(b^2 - 4*a*c)*(a + b*x^2 + c*x^4)) + ((2*a*(b^2 -
4*a*c)*m*x)/c^2 + ((a*b^2*c*k - 2*a*c^2*(c*f + 3*a*k) - 3*a*b^3*m + b*c*(c
^2*d + a*c*h + 13*a^2*m) - (a*b^3*c*k - 4*a*b*c^2*(c*f + 2*a*k) - 3*a*b^4*
m - b^2*c*(c^2*d - a*c*h - 19*a^2*m) + 4*a*c^2*(3*c^2*d + a*c*h - 5*a^2*m)
)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]
]])/(Sqrt[2]*c^(5/2)*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + ((a*b^2*c*k - 2*a*c^2*
(c*f + 3*a*k) - 3*a*b^3*m + b*c*(c^2*d + a*c*h + 13*a^2*m) + (a*b^3*c*k -
4*a*b*c^2*(c*f + 2*a*k) - 3*a*b^4*m - b^2*c*(c^2*d - a*c*h - 19*a^2*m) + 4
*a*c^2*(3*c^2*d + a*c*h - 5*a^2*m))/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqr
t[c]*x)/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*c^(5/2)*Sqrt[b + Sqrt[b^2 -
 4*a*c]]))/(2*a*(b^2 - 4*a*c)) + (-((b*c*(c*e + a*j) - a*b^2*l - 2*a*c*(c*
g - a*l) + (2*c^3*e - c^2*(b*g + 2*a*j) - b^3*l + b*c*(b*j + 3*a*l))*x^2)/
(c^2*(b^2 - 4*a*c)*(a + b*x^2 + c*x^4))) - (-(((4*c^3*e - c^2*(2*b*g - 4*a
*j) + b^3*l - 6*a*b*c*l)*ArcTanh[(b + 2*c*x^2)/Sqrt[b^2 - 4*a*c]])/(c^2*Sq
rt[b^2 - 4*a*c])) - ((b^2 - 4*a*c)*l*Log[a + b*x^2 + c*x^4])/(2*c^2))/(b^2
 - 4*a*c))/2
```

3.41. $\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^2}\,dx$

3.41.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2191
```
Int[(Pq_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> With[{Q =
PolynomialQuotient[Pq, a + b*x + c*x^2, x], f = Coeff[PolynomialRemainder[P
q, a + b*x + c*x^2, x], x, 0], g = Coeff[PolynomialRemainder[Pq, a + b*x +
c*x^2, x], x, 1]}, Simp[(b*f - 2*a*g + (2*c*f - b*g)*x)*((a + b*x + c*x^2)^
(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] + Simp[1/((p + 1)*(b^2 - 4*a*c))   Int
[(a + b*x + c*x^2)^(p + 1)*ExpandToSum[(p + 1)*(b^2 - 4*a*c)*Q - (2*p + 3)*
(2*c*f - b*g), x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Pq, x] && NeQ[b^
2 - 4*a*c, 0] && LtQ[p, -1]
```

rule 2194
```
Int[(Pq_)*(x_)^(m_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :
> Simp[1/2   Subst[Int[x^((m - 1)/2)*SubstFor[x^2, Pq, x]*(a + b*x + c*x^2)
^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pq, x^2] && IntegerQ
[(m - 1)/2]
```

3.41. $\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^2}\,dx$

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2205
```
Int[(Px_)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :> Int[ExpandInte
grand[Px/(a + b*x^2 + c*x^4), x], x] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x^
2] && Expon[Px, x^2] > 1
```

rule 2206
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> With[{d =
 Coeff[PolynomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 0], e = Coeff[Poly
nomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 2]}, Simp[x*(a + b*x^2 + c*x^
4)^(p + 1)*((a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)/(2*a*(p + 1)*(b
^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(a + b*x^2 + c
*x^4)^(p + 1)*ExpandToSum[2*a*(p + 1)*(b^2 - 4*a*c)*PolynomialQuotient[Px,
a + b*x^2 + c*x^4, x] + b^2*d*(2*p + 3) - 2*a*c*d*(4*p + 5) - a*b*e + c*(4*
p + 7)*(b*d - 2*a*e)*x^2, x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x
^2] && Expon[Px, x^2] > 1 && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1]
```

3.41.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 1.01 (sec) , antiderivative size = 506, normalized size of antiderivative = 0.66

method	result
risch	$\frac{mx}{c^2} + \frac{\frac{\left(3a^2bcm-2a^2c^2k-a\,b^3m+a\,b^2ck-ab\,c^2h+2a\,c^3f-b\,c^3d\right)x^3}{2a\left(4ac-b^2\right)}+\frac{\left(3abcl-2a\,c^2j-b^3l+b^2cj-b\,c^2g+2c^3e\right)x^2}{8ac-2b^2}+\frac{\left(2a^3cm-a^2b^2m+a^2bck-2a^2\right.}{2a\left(4a\right.}}{c^2(c\,x^4+b\,x^2+a)}$
default	Expression too large to display

input
```
int((m*x^8+l*x^7+k*x^6+j*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x,
method=_RETURNVERBOSE)
```

output
```
m*x/c^2+(1/2/a*(3*a^2*b*c*m-2*a^2*c^2*k-a*b^3*m+a*b^2*c*k-a*b*c^2*h+2*a*c^
3*f-b*c^3*d)/(4*a*c-b^2)*x^3+1/2*(3*a*b*c*l-2*a*c^2*j-b^3*l+b^2*c*j-b*c^2*
g+2*c^3*e)/(4*a*c-b^2)*x^2+1/2*(2*a^3*c*m-a^2*b^2*m+a^2*b*c*k-2*a^2*c^2*h+
a*b*c^2*f+2*a*c^3*d-b^2*c^2*d)/a/(4*a*c-b^2)*x+1/2*(2*a^2*c*l-a*b^2*l+a*b*
c*j-2*a*c^2*g+b*c^2*e)/(4*a*c-b^2))/c^2/(c*x^4+b*x^2+a)+1/4/c^2*sum((2*l*c
*_R^3-(13*a^2*b*c*m-6*a^2*c^2*k-3*a*b^3*m+a*b^2*c*k+a*b*c^2*h-2*a*c^3*f+b*
c^3*d)/a/(4*a*c-b^2)*_R^2-2*c*(a*b*l-2*a*c*j+b*c*g-2*c^2*e)/(4*a*c-b^2)*_R
-(10*a^3*c*m-3*a^2*b^2*m+a^2*b*c*k-2*a^2*c^2*h+a*b*c^2*f-6*a*c^3*d+b^2*c^2
*d)/a/(4*a*c-b^2))/(2*_R^3*c+_R*b)*ln(x-_R),_R=RootOf(_Z^4*c+_Z^2*b+a))
```

3.41.5 Fricas [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((m*x^8+l*x^7+k*x^6+j*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a
)^2,x, algorithm="fricas")
```

output
```
Timed out
```

3.41.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((m*x**8+l*x**7+k*x**6+j*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(c*x**4
+b*x**2+a)**2,x)
```

output
```
Timed out
```

3.41.7 Maxima [F]

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4 + jx^5 + kx^6 + lx^7 + mx^8}{(a + bx^2 + cx^4)^2} \, dx$$
$$= \int \frac{mx^8 + lx^7 + kx^6 + jx^5 + hx^4 + gx^3 + fx^2 + ex + d}{(cx^4 + bx^2 + a)^2} \, dx$$

input
```
integrate((m*x^8+l*x^7+k*x^6+j*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a
)^2,x, algorithm="maxima")
```

output
```
-1/2*(a*b*c^2*e - 2*a^2*c^2*g + a^2*b*c*j - (b*c^3*d - 2*a*c^3*f + a*b*c^2
*h - (a*b^2*c - 2*a^2*c^2)*k + (a*b^3 - 3*a^2*b*c)*m)*x^3 + (2*a*c^3*e - a
*b*c^2*g + (a*b^2*c - 2*a^2*c^2)*j - (a*b^3 - 3*a^2*b*c)*l)*x^2 - (a^2*b^2
 - 2*a^3*c)*l + (a*b*c^2*f - 2*a^2*c^2*h + a^2*b*c*k - (b^2*c^2 - 2*a*c^3)
*d - (a^2*b^2 - 2*a^3*c)*m)*x)/(a^2*b^2*c^2 - 4*a^3*c^3 + (a*b^2*c^3 - 4*a
^2*c^4)*x^4 + (a*b^3*c^2 - 4*a^2*b*c^3)*x^2) + m*x/c^2 - 1/2*integrate(-(a
*b*c^2*f - 2*a^2*c^2*h + a^2*b*c*k + 2*(a*b^2*c - 4*a^2*c^2)*l*x^3 + (b*c^
3*d - 2*a*c^3*f + a*b*c^2*h + (a*b^2*c - 6*a^2*c^2)*k - (3*a*b^3 - 13*a^2*
b*c)*m)*x^2 + (b^2*c^2 - 6*a*c^3)*d - (3*a^2*b^2 - 10*a^3*c)*m - 2*(2*a*c^
3*e - a*b*c^2*g + 2*a^2*c^2*j - a^2*b*c*l)*x)/(c*x^4 + b*x^2 + a), x)/(a*b
^2*c^2 - 4*a^2*c^3)
```

3.41.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 20159 vs. 2(718) = 1436.

Time = 3.93 (sec) , antiderivative size = 20159, normalized size of antiderivative = 26.18

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4 + jx^5 + kx^6 + lx^7 + mx^8}{(a + bx^2 + cx^4)^2} \, dx = \text{Too large to display}$$

input
```
integrate((m*x^8+l*x^7+k*x^6+j*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a
)^2,x, algorithm="giac")
```

output
```
m*x/c^2 + 1/4*l*log(abs(c*x^4 + b*x^2 + a))/c^2 + 1/16*((a^2*b^4*c^5 - 8*a
^3*b^2*c^6 + 16*a^4*c^7)^2*(2*b^3*c^5 - 8*a*b*c^6 - sqrt(2)*sqrt(b^2 - 4*a
*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^3*c^3 + 4*sqrt(2)*sqrt(b^2 - 4*a*c)*
sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c^4 + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt
(b*c + sqrt(b^2 - 4*a*c)*c)*b^2*c^4 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c +
 sqrt(b^2 - 4*a*c)*c)*b*c^5 - 2*(b^2 - 4*a*c)*b*c^5)*d - 2*(a^2*b^4*c^5 -
8*a^3*b^2*c^6 + 16*a^4*c^7)^2*(2*a*b^2*c^5 - 8*a^2*c^6 - sqrt(2)*sqrt(b^2
- 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^2*c^3 + 4*sqrt(2)*sqrt(b^2 -
4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*c^4 + 2*sqrt(2)*sqrt(b^2 - 4*a*
c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c^4 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqr
t(b*c + sqrt(b^2 - 4*a*c)*c)*a*c^5 - 2*(b^2 - 4*a*c)*a*c^5)*f + (a^2*b^4*c
^5 - 8*a^3*b^2*c^6 + 16*a^4*c^7)^2*(2*a*b^3*c^4 - 8*a^2*b*c^5 - sqrt(2)*sq
rt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^3*c^2 + 4*sqrt(2)*sqrt
(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b*c^3 + 2*sqrt(2)*sqrt(b
^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^2*c^3 - sqrt(2)*sqrt(b^2 -
 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c^4 - 2*(b^2 - 4*a*c)*a*b*c^4)
*h + (a^2*b^4*c^5 - 8*a^3*b^2*c^6 + 16*a^4*c^7)^2*(2*a*b^4*c^3 - 20*a^2*b^
2*c^4 + 48*a^3*c^5 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c
)*c)*a*b^4*c + 10*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c
)*a^2*b^2*c^2 + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c...
```

3.41.9 Mupad [B] (verification not implemented)

Time = 22.90 (sec) , antiderivative size = 82785, normalized size of antiderivative = 107.51

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4 + jx^5 + kx^6 + lx^7 + mx^8}{(a + bx^2 + cx^4)^2} \, dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4 + j*x^5 + k*x^6 + l*x^7 + m*x^8)/(a +
 b*x^2 + c*x^4)^2,x)
```

output
```
symsum(log(root(1572864*a^8*b^2*c^10*z^4 - 983040*a^7*b^4*c^9*z^4 + 327680
*a^6*b^6*c^8*z^4 - 61440*a^5*b^8*c^7*z^4 + 6144*a^4*b^10*c^6*z^4 - 256*a^3
*b^12*c^5*z^4 - 1048576*a^9*c^11*z^4 - 1572864*a^8*b^2*c^8*l*z^3 + 983040*
a^7*b^4*c^7*l*z^3 - 327680*a^6*b^6*c^6*l*z^3 + 61440*a^5*b^8*c^5*l*z^3 - 6
144*a^4*b^10*c^4*l*z^3 + 256*a^3*b^12*c^3*l*z^3 + 1048576*a^9*c^9*l*z^3 +
96*a^3*b^12*c*k*m*z^2 + 98304*a^8*b*c^7*j*l*z^2 + 24576*a^8*b*c^7*h*m*z^2
+ 155648*a^7*b*c^8*d*m*z^2 + 98304*a^7*b*c^8*e*l*z^2 + 57344*a^7*b*c^8*f*k
*z^2 + 32768*a^7*b*c^8*g*j*z^2 + 57344*a^6*b*c^9*d*h*z^2 + 32768*a^6*b*c^9
*e*g*z^2 - 32*a*b^10*c^5*d*f*z^2 - 491520*a^8*b^2*c^6*k*m*z^2 + 358400*a^7
*b^4*c^5*k*m*z^2 - 129024*a^6*b^6*c^4*k*m*z^2 + 24768*a^5*b^8*c^3*k*m*z^2
- 2432*a^4*b^10*c^2*k*m*z^2 - 90112*a^7*b^3*c^6*j*l*z^2 + 30720*a^6*b^5*c^
5*j*l*z^2 - 4608*a^5*b^7*c^4*j*l*z^2 + 256*a^4*b^9*c^3*j*l*z^2 - 21504*a^6
*b^5*c^5*h*m*z^2 + 9216*a^5*b^7*c^4*h*m*z^2 + 8192*a^7*b^3*c^6*h*m*z^2 - 1
568*a^4*b^9*c^3*h*m*z^2 + 96*a^3*b^11*c^2*h*m*z^2 - 172032*a^7*b^2*c^7*f*m
*z^2 + 116736*a^6*b^4*c^6*f*m*z^2 - 49152*a^7*b^2*c^7*g*l*z^2 + 45056*a^6*
b^4*c^6*g*l*z^2 - 35840*a^5*b^6*c^5*f*m*z^2 + 24576*a^7*b^2*c^7*h*k*z^2 -
15360*a^5*b^6*c^5*g*l*z^2 + 5184*a^4*b^8*c^4*f*m*z^2 - 3072*a^5*b^6*c^5*h*
k*z^2 + 2304*a^4*b^8*c^4*g*l*z^2 + 2048*a^6*b^4*c^6*h*k*z^2 + 576*a^4*b^8*
c^4*h*k*z^2 - 288*a^3*b^10*c^3*f*m*z^2 - 128*a^3*b^10*c^3*g*l*z^2 - 32*a^3
*b^10*c^3*h*k*z^2 - 147456*a^6*b^3*c^7*d*m*z^2 - 90112*a^6*b^3*c^7*e*l*...
```

3.41. $\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^2}\,dx$

3.42 $\int \frac{d+ex}{(4-5x^2+x^4)^3}\,dx$

3.42.1 Optimal result

Integrand size = 18, antiderivative size = 143

$$\begin{aligned}\int \frac{d+ex}{(4-5x^2+x^4)^3}\,dx &= \frac{dx(17-5x^2)}{144\,(4-5x^2+x^4)^2}+\frac{e(5-2x^2)}{36\,(4-5x^2+x^4)^2}\\ &\quad -\frac{dx(59-35x^2)}{3456\,(4-5x^2+x^4)}-\frac{e(5-2x^2)}{54\,(4-5x^2+x^4)}-\frac{313d\operatorname{arctanh}\left(\frac{x}{2}\right)}{20736}\\ &\quad +\frac{13}{648}d\operatorname{arctanh}(x)-\frac{1}{81}e\log\left(1-x^2\right)+\frac{1}{81}e\log\left(4-x^2\right)\end{aligned}$$

output
```
1/144*d*x*(-5*x^2+17)/(x^4-5*x^2+4)^2+1/36*e*(-2*x^2+5)/(x^4-5*x^2+4)^2-1/
3456*d*x*(-35*x^2+59)/(x^4-5*x^2+4)-1/54*e*(-2*x^2+5)/(x^4-5*x^2+4)-313/20
736*d*arctanh(1/2*x)+13/648*d*arctanh(x)-1/81*e*ln(-x^2+1)+1/81*e*ln(-x^2+
4)
```

3.42.2 Mathematica [A] (verified)

Time = 0.07 (sec) , antiderivative size = 128, normalized size of antiderivative = 0.90

$$\int \frac{d+ex}{(4-5x^2+x^4)^3}\,dx = \frac{\frac{288(e(20-8x^2)+dx(17-5x^2))}{(4-5x^2+x^4)^2}+\frac{12(64e(-5+2x^2)+dx(-59+35x^2))}{4-5x^2+x^4}-32(13d+16e)\log(1-x)+(313d+512e)\log(2-}{41472}$$

input
```
Integrate[(d + e*x)/(4 - 5*x^2 + x^4)^3,x]
```

output
```
((288*(e*(20 - 8*x^2) + d*x*(17 - 5*x^2)))/(4 - 5*x^2 + x^4)^2 + (12*(64*e
*(-5 + 2*x^2) + d*x*(-59 + 35*x^2)))/(4 - 5*x^2 + x^4) - 32*(13*d + 16*e)*
Log[1 - x] + (313*d + 512*e)*Log[2 - x] + 32*(13*d - 16*e)*Log[1 + x] + (-
313*d + 512*e)*Log[2 + x])/41472
```

3.42.3 Rubi [A] (verified)

Time = 0.40 (sec) , antiderivative size = 160, normalized size of antiderivative = 1.12, number of steps used = 12, number of rules used = 11, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.611, Rules used = {2202, 27, 1405, 25, 1432, 1084, 1492, 27, 1480, 220, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex}{(x^4-5x^2+4)^3}dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{d}{(x^4-5x^2+4)^3}dx+\int \frac{ex}{(x^4-5x^2+4)^3}dx$$

$$\Big\downarrow \text{27}$$

$$d\int \frac{1}{(x^4-5x^2+4)^3}dx+e\int \frac{x}{(x^4-5x^2+4)^3}dx$$

$$\Big\downarrow \text{1405}$$

$$d\left(\frac{x(17-5x^2)}{144(x^4-5x^2+4)^2}-\frac{1}{144}\int -\frac{19-25x^2}{(x^4-5x^2+4)^2}dx\right)+e\int \frac{x}{(x^4-5x^2+4)^3}dx$$

$$\Big\downarrow \text{25}$$

$$d\left(\frac{1}{144}\int \frac{19-25x^2}{(x^4-5x^2+4)^2}dx+\frac{x(17-5x^2)}{144(x^4-5x^2+4)^2}\right)+e\int \frac{x}{(x^4-5x^2+4)^3}dx$$

$$\Big\downarrow \text{1432}$$

$$d\left(\frac{1}{144}\int \frac{19-25x^2}{(x^4-5x^2+4)^2}dx+\frac{x(17-5x^2)}{144(x^4-5x^2+4)^2}\right)+\frac{1}{2}e\int \frac{1}{(x^4-5x^2+4)^3}dx^2$$

$$\Big\downarrow \text{1084}$$

$$d\left(\frac{1}{144}\int \frac{19-25x^2}{(x^4-5x^2+4)^2}dx+\frac{x(17-5x^2)}{144(x^4-5x^2+4)^2}\right)+$$
$$\frac{1}{2}e\int \left(-\frac{2}{81(4-x^2)}-\frac{1}{27(4-x^2)^2}-\frac{1}{27(4-x^2)^3}+\frac{2}{81(1-x^2)}-\frac{1}{27(1-x^2)^2}+\frac{1}{27(1-x^2)^3}\right)dx^2$$

$$\Big\downarrow \; 1492$$

$$d\left(\frac{1}{144}\left(-\frac{1}{72}\int -\frac{3(35x^2+173)}{x^4-5x^2+4}dx-\frac{x(59-35x^2)}{24(x^4-5x^2+4)}\right)+\frac{x(17-5x^2)}{144(x^4-5x^2+4)^2}\right)+$$
$$\frac{1}{2}e\int \left(-\frac{2}{81(4-x^2)}-\frac{1}{27(4-x^2)^2}-\frac{1}{27(4-x^2)^3}+\frac{2}{81(1-x^2)}-\frac{1}{27(1-x^2)^2}+\frac{1}{27(1-x^2)^3}\right)dx^2$$

$$\Big\downarrow \; 27$$

$$d\left(\frac{1}{144}\left(\frac{1}{24}\int \frac{35x^2+173}{x^4-5x^2+4}dx-\frac{x(59-35x^2)}{24(x^4-5x^2+4)}\right)+\frac{x(17-5x^2)}{144(x^4-5x^2+4)^2}\right)+$$
$$\frac{1}{2}e\int \left(-\frac{2}{81(4-x^2)}-\frac{1}{27(4-x^2)^2}-\frac{1}{27(4-x^2)^3}+\frac{2}{81(1-x^2)}-\frac{1}{27(1-x^2)^2}+\frac{1}{27(1-x^2)^3}\right)dx^2$$

$$\Big\downarrow \; 1480$$

$$d\left(\frac{1}{144}\left(\frac{1}{24}\left(\frac{313}{3}\int \frac{1}{x^2-4}dx-\frac{208}{3}\int \frac{1}{x^2-1}dx\right)-\frac{x(59-35x^2)}{24(x^4-5x^2+4)}\right)+\frac{x(17-5x^2)}{144(x^4-5x^2+4)^2}\right)+$$
$$\frac{1}{2}e\int \left(-\frac{2}{81(4-x^2)}-\frac{1}{27(4-x^2)^2}-\frac{1}{27(4-x^2)^3}+\frac{2}{81(1-x^2)}-\frac{1}{27(1-x^2)^2}+\frac{1}{27(1-x^2)^3}\right)dx^2$$

$$\Big\downarrow \; 220$$

$$\frac{1}{2}e\int \left(-\frac{2}{81(4-x^2)}-\frac{1}{27(4-x^2)^2}-\frac{1}{27(4-x^2)^3}+\frac{2}{81(1-x^2)}-\frac{1}{27(1-x^2)^2}+\frac{1}{27(1-x^2)^3}\right)dx^2+$$
$$d\left(\frac{1}{144}\left(\frac{1}{24}\left(\frac{208\text{arctanh}(x)}{3}-\frac{313}{6}\text{arctanh}\left(\frac{x}{2}\right)\right)-\frac{x(59-35x^2)}{24(x^4-5x^2+4)}\right)+\frac{x(17-5x^2)}{144(x^4-5x^2+4)^2}\right)$$

$$\Big\downarrow \; 2009$$

$$d\left(\frac{1}{144}\left(\frac{1}{24}\left(\frac{208\text{arctanh}(x)}{3}-\frac{313}{6}\text{arctanh}\left(\frac{x}{2}\right)\right)-\frac{x(59-35x^2)}{24(x^4-5x^2+4)}\right)+\frac{x(17-5x^2)}{144(x^4-5x^2+4)^2}\right)+$$
$$\frac{1}{2}e\left(-\frac{1}{27(1-x^2)}-\frac{1}{27(4-x^2)}+\frac{1}{54(1-x^2)^2}-\frac{1}{54(4-x^2)^2}-\frac{2}{81}\log(1-x^2)+\frac{2}{81}\log(4-x^2)\right)$$

input
```
Int[(d + e*x)/(4 - 5*x^2 + x^4)^3,x]
```

3.42. $\int \frac{d+ex}{(4-5x^2+x^4)^3}\,dx$

output
```
d*((x*(17 - 5*x^2))/(144*(4 - 5*x^2 + x^4)^2) + (-1/24*(x*(59 - 35*x^2))/(
4 - 5*x^2 + x^4) + ((-313*ArcTanh[x/2])/6 + (208*ArcTanh[x])/3)/24)/144) +
 (e*(1/(54*(1 - x^2)^2) - 1/(27*(1 - x^2)) - 1/(54*(4 - x^2)^2) - 1/(27*(4
 - x^2)) - (2*Log[1 - x^2])/81 + (2*Log[4 - x^2])/81))/2
```

3.42.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 220
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[b, 2])^(-
1))*ArcTanh[Rt[b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] &&
 (LtQ[a, 0] || GtQ[b, 0])
```

rule 1084
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> With[{q = Rt[b^2
- 4*a*c, 2]}, Simp[1/c^p   Int[ExpandIntegrand[(b/2 - q/2 + c*x)^p*(b/2 + q
/2 + c*x)^p, x], x], x] /;  !FractionalPowerFactorQ[q]] /; FreeQ[{a, b, c},
 x] && IntegerQ[p] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 1405
```
Int[((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[(-x)*(b^2
- 2*a*c + b*c*x^2)*((a + b*x^2 + c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))
), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(b^2 - 2*a*c + 2*(p + 1)*(
b^2 - 4*a*c) + b*c*(4*p + 7)*x^2)*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; Fr
eeQ[{a, b, c}, x] && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1] && IntegerQ[2*p]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.42.4 Maple [A] (verified)

Time = 0.09 (sec) , antiderivative size = 107, normalized size of antiderivative = 0.75

method	result
norman	$\frac{\frac{5}{9}e\,x^{2}+\frac{1}{27}e\,x^{6}-\frac{5}{18}e\,x^{4}+\frac{43}{864}dx+\frac{35}{3456}d\,x^{7}+\frac{35}{384}x^{3}d-\frac{13}{192}x^{5}d-\frac{25}{108}e}{\left(x^{4}-5x^{2}+4\right)^{2}}+\left(-\frac{313d}{41472}+\frac{e}{81}\right)\ln\left(x+2\right)+\left(-\frac{13d}{1296}-\frac{e}{81}\right)$
risch	$\frac{\frac{5}{9}e\,x^{2}+\frac{1}{27}e\,x^{6}-\frac{5}{18}e\,x^{4}+\frac{43}{864}dx+\frac{35}{3456}d\,x^{7}+\frac{35}{384}x^{3}d-\frac{13}{192}x^{5}d-\frac{25}{108}e}{\left(x^{4}-5x^{2}+4\right)^{2}}+\frac{13\ln(x+1)d}{1296}-\frac{\ln(x+1)e}{81}-\frac{313\ln(x+2)d}{41472}+\frac{\ln(x+}{81}$
default	$\left(-\frac{313d}{41472}+\frac{e}{81}\right)\ln\left(x+2\right)-\frac{-\frac{19d}{6912}+\frac{17e}{3456}}{x+2}-\frac{-\frac{d}{1728}+\frac{e}{864}}{2(x+2)^{2}}-\frac{-\frac{d}{432}+\frac{e}{144}}{x+1}-\frac{\frac{d}{216}-\frac{e}{216}}{2(x+1)^{2}}+\left(\frac{13d}{1296}-\frac{e}{81}\right)\ln\left(x-\right.$
parallelrisch	$1536e\,x^{6}-11520e\,x^{4}-9600e+420d\,x^{7}+2064dx+5008\ln(x-2)d+8192\ln(x-2)e-6656\ln(x-1)d-8192\ln(x-1)e-4160\ln(x+1$

input
```
int((e*x+d)/(x^4-5*x^2+4)^3,x,method=_RETURNVERBOSE)
```

output
```
(5/9*e*x^2+1/27*e*x^6-5/18*e*x^4+43/864*d*x+35/3456*d*x^7+35/384*x^3*d-13/
192*x^5*d-25/108*e)/(x^4-5*x^2+4)^2+(-313/41472*d+1/81*e)*ln(x+2)+(-13/129
6*d-1/81*e)*ln(x-1)+(13/1296*d-1/81*e)*ln(x+1)+(313/41472*d+1/81*e)*ln(x-2
)
```

3.42.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 307 vs. $2(125) = 250$.

Time = 0.29 (sec) , antiderivative size = 307, normalized size of antiderivative = 2.15

$$\int \frac{d+ex}{(4-5x^2+x^4)^3}\,dx$$
$$= \frac{420\,dx^7+1536\,ex^6-2808\,dx^5-11520\,ex^4+3780\,dx^3+23040\,ex^2+2064\,dx-((313\,d-512\,e)x^8-10}{}$$

input
```
integrate((e*x+d)/(x^4-5*x^2+4)^3,x, algorithm="fricas")
```

output
```
1/41472*(420*d*x^7 + 1536*e*x^6 - 2808*d*x^5 - 11520*e*x^4 + 3780*d*x^3 +
23040*e*x^2 + 2064*d*x - ((313*d - 512*e)*x^8 - 10*(313*d - 512*e)*x^6 + 3
3*(313*d - 512*e)*x^4 - 40*(313*d - 512*e)*x^2 + 5008*d - 8192*e)*log(x +
2) + 32*((13*d - 16*e)*x^8 - 10*(13*d - 16*e)*x^6 + 33*(13*d - 16*e)*x^4 -
 40*(13*d - 16*e)*x^2 + 208*d - 256*e)*log(x + 1) - 32*((13*d + 16*e)*x^8
- 10*(13*d + 16*e)*x^6 + 33*(13*d + 16*e)*x^4 - 40*(13*d + 16*e)*x^2 + 208
*d + 256*e)*log(x - 1) + ((313*d + 512*e)*x^8 - 10*(313*d + 512*e)*x^6 + 3
3*(313*d + 512*e)*x^4 - 40*(313*d + 512*e)*x^2 + 5008*d + 8192*e)*log(x -
2) - 9600*e)/(x^8 - 10*x^6 + 33*x^4 - 40*x^2 + 16)
```

3.42.6 Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 668 vs. $2(126) = 252$.

Time = 2.23 (sec) , antiderivative size = 668, normalized size of antiderivative = 4.67

$$\int \frac{d+ex}{(4-5x^2+x^4)^3}\,dx$$
$$= \frac{(13d-16e)\log\left(x+\frac{-1106258459719280d^4e-13113710954343d^4\cdot(13d-16e)-817263343042560d^2e^3+153628968222720d^2e^2\cdot(13d-16e}{2}\right.}{}$$
$$-\frac{(13d+16e)\log\left(x+\frac{-1106258459719280d^4e+13113710954343d^4\cdot(13d+16e)-817263343042560d^2e^3-153628968222720d^2e^2\cdot(13d+}{}\right.}{}$$
$$-\frac{(313d-512e)\log\left(x+\frac{-1106258459719280d^4e+\frac{13113710954343d^4\cdot(313d-512e)}{32}-817263343042560d^2e^3-4800905256960d^2e^2\cdot(313d-}{}\right.}{}$$
$$+\frac{(313d+512e)\log\left(x+\frac{-1106258459719280d^4e-\frac{13113710954343d^4\cdot(313d+512e)}{32}-817263343042560d^2e^3+4800905256960d^2e^2\cdot(313d+}{}\right.}{}$$
$$+\frac{35dx^7-234dx^5+315dx^3+172dx+128ex^6-960ex^4+1920ex^2-800e}{3456x^8-34560x^6+114048x^4-138240x^2+55296}$$

input
```
integrate((e*x+d)/(x**4-5*x**2+4)**3,x)
```

output
```
(13*d - 16*e)*log(x + (-1106258459719280*d**4*e - 13113710954343*d**4*(13*
d - 16*e) - 817263343042560*d**2*e**3 + 153628968222720*d**2*e**2*(13*d -
16*e) + 9530197557248*d**2*e*(13*d - 16*e)**2 + 88038005760*d**2*(13*d - 1
6*e)**3 + 5035763255214080*e**5 + 142661633703936*e**4*(13*d - 16*e) - 196
70950215680*e**3*(13*d - 16*e)**2 - 557272006656*e**2*(13*d - 16*e)**3)/(2
2941256248261*d**5 - 2312740746035200*d**3*e**2 + 4473912813420544*d*e**4)
)/1296 - (13*d + 16*e)*log(x + (-1106258459719280*d**4*e + 13113710954343*
d**4*(13*d + 16*e) - 817263343042560*d**2*e**3 - 153628968222720*d**2*e**2
*(13*d + 16*e) + 9530197557248*d**2*e*(13*d + 16*e)**2 - 88038005760*d**2*
(13*d + 16*e)**3 + 5035763255214080*e**5 - 142661633703936*e**4*(13*d + 16
*e) - 19670950215680*e**3*(13*d + 16*e)**2 + 557272006656*e**2*(13*d + 16*
e)**3)/(22941256248261*d**5 - 2312740746035200*d**3*e**2 + 447391281342054
4*d*e**4))/1296 - (313*d - 512*e)*log(x + (-1106258459719280*d**4*e + 1311
3710954343*d**4*(313*d - 512*e)/32 - 817263343042560*d**2*e**3 - 480090525
6960*d**2*e**2*(313*d - 512*e) + 9306833552*d**2*e*(313*d - 512*e)**2 - 85
974615*d**2*(313*d - 512*e)**3/32 + 5035763255214080*e**5 - 4458176053248*
e**4*(313*d - 512*e) - 19209912320*e**3*(313*d - 512*e)**2 + 17006592*e**2
*(313*d - 512*e)**3)/(22941256248261*d**5 - 2312740746035200*d**3*e**2 + 4
473912813420544*d*e**4))/41472 + (313*d + 512*e)*log(x + (-110625845971928
0*d**4*e - 13113710954343*d**4*(313*d + 512*e)/32 - 817263343042560*d**...
```

3.42.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 121, normalized size of antiderivative = 0.85

$$\begin{aligned}\int \frac{d+ex}{(4-5x^2+x^4)^3}\,dx &= -\frac{1}{41472}\,(313\,d-512\,e)\log\left(x+2\right)+\frac{1}{1296}\,(13\,d-16\,e)\log\left(x+1\right)\\&\quad-\frac{1}{1296}\,(13\,d+16\,e)\log\left(x-1\right)+\frac{1}{41472}\,(313\,d+512\,e)\log\left(x-2\right)\\&\quad+\frac{35\,dx^7+128\,ex^6-234\,dx^5-960\,ex^4+315\,dx^3+1920\,ex^2+172\,dx-800\,e}{3456\,(x^8-10\,x^6+33\,x^4-40\,x^2+16)}\end{aligned}$$

input
```
integrate((e*x+d)/(x^4-5*x^2+4)^3,x, algorithm="maxima")
```

output
```
-1/41472*(313*d - 512*e)*log(x + 2) + 1/1296*(13*d - 16*e)*log(x + 1) - 1/
1296*(13*d + 16*e)*log(x - 1) + 1/41472*(313*d + 512*e)*log(x - 2) + 1/345
6*(35*d*x^7 + 128*e*x^6 - 234*d*x^5 - 960*e*x^4 + 315*d*x^3 + 1920*e*x^2 +
 172*d*x - 800*e)/(x^8 - 10*x^6 + 33*x^4 - 40*x^2 + 16)
```

3.42.8 Giac [A] (verification not implemented)

Time = 0.31 (sec) , antiderivative size = 115, normalized size of antiderivative = 0.80

$$\begin{aligned}\int \frac{d+ex}{(4-5x^2+x^4)^3}\,dx &= -\frac{1}{41472}\,(313\,d-512\,e)\log\left(|x+2|\right)+\frac{1}{1296}\,(13\,d-16\,e)\log\left(|x+1|\right)\\ &\quad-\frac{1}{1296}\,(13\,d+16\,e)\log\left(|x-1|\right)+\frac{1}{41472}\,(313\,d+512\,e)\log\left(|x-2|\right)\\ &\quad+\frac{35\,dx^7+128\,ex^6-234\,dx^5-960\,ex^4+315\,dx^3+1920\,ex^2+172\,dx-800\,e}{3456\,(x^4-5\,x^2+4)^2}\end{aligned}$$

input
```
integrate((e*x+d)/(x^4-5*x^2+4)^3,x, algorithm="giac")
```

output
```
-1/41472*(313*d - 512*e)*log(abs(x + 2)) + 1/1296*(13*d - 16*e)*log(abs(x
+ 1)) - 1/1296*(13*d + 16*e)*log(abs(x - 1)) + 1/41472*(313*d + 512*e)*log
(abs(x - 2)) + 1/3456*(35*d*x^7 + 128*e*x^6 - 234*d*x^5 - 960*e*x^4 + 315*
d*x^3 + 1920*e*x^2 + 172*d*x - 800*e)/(x^4 - 5*x^2 + 4)^2
```

3.42.9 Mupad [B] (verification not implemented)

Time = 0.05 (sec) , antiderivative size = 118, normalized size of antiderivative = 0.83

$$\begin{aligned}\int \frac{d+ex}{(4-5x^2+x^4)^3}\,dx &= \ln(x+1)\left(\frac{13\,d}{1296}-\frac{e}{81}\right)-\ln(x-1)\left(\frac{13\,d}{1296}+\frac{e}{81}\right)\\ &\quad+\ln(x-2)\left(\frac{313\,d}{41472}+\frac{e}{81}\right)-\ln(x+2)\left(\frac{313\,d}{41472}-\frac{e}{81}\right)\\ &\quad+\frac{\frac{35\,d\,x^7}{3456}+\frac{e\,x^6}{27}-\frac{13\,d\,x^5}{192}-\frac{5\,e\,x^4}{18}+\frac{35\,d\,x^3}{384}+\frac{5\,e\,x^2}{9}+\frac{43\,d\,x}{864}-\frac{25\,e}{108}}{x^8-10\,x^6+33\,x^4-40\,x^2+16}\end{aligned}$$

input
```
int((d + e*x)/(x^4 - 5*x^2 + 4)^3,x)
```

output
```
log(x + 1)*((13*d)/1296 - e/81) - log(x - 1)*((13*d)/1296 + e/81) + log(x
- 2)*((313*d)/41472 + e/81) - log(x + 2)*((313*d)/41472 - e/81) + ((43*d*x
)/864 - (25*e)/108 + (35*d*x^3)/384 - (13*d*x^5)/192 + (35*d*x^7)/3456 + (
5*e*x^2)/9 - (5*e*x^4)/18 + (e*x^6)/27)/(33*x^4 - 40*x^2 - 10*x^6 + x^8 +
16)
```

3.42. $\int \frac{d+ex}{(4-5x^2+x^4)^3}\,dx$

3.43 $\int \frac{d+ex+fx^2}{(4-5x^2+x^4)^3}\,dx$

3.43.1 Optimal result

Integrand size = 23, antiderivative size = 175

$$\begin{aligned}\int \frac{d+ex+fx^2}{(4-5x^2+x^4)^3}\,dx &= \frac{e(5-2x^2)}{36\,(4-5x^2+x^4)^2}+\frac{x(17d+20f-(5d+8f)x^2)}{144\,(4-5x^2+x^4)^2} \\ &\quad -\frac{e(5-2x^2)}{54\,(4-5x^2+x^4)}-\frac{x(59d+380f-35(d+4f)x^2)}{3456\,(4-5x^2+x^4)} \\ &\quad -\frac{(313d+820f)\operatorname{arctanh}\left(\frac{x}{2}\right)}{20736}+\frac{1}{648}(13d+25f)\operatorname{arctanh}(x) \\ &\quad -\frac{1}{81}e\log\left(1-x^2\right)+\frac{1}{81}e\log\left(4-x^2\right)\end{aligned}$$

output
```
1/36*e*(-2*x^2+5)/(x^4-5*x^2+4)^2+1/144*x*(17*d+20*f-(5*d+8*f)*x^2)/(x^4-5
*x^2+4)^2-1/54*e*(-2*x^2+5)/(x^4-5*x^2+4)-1/3456*x*(59*d+380*f-35*(d+4*f)*
x^2)/(x^4-5*x^2+4)-1/20736*(313*d+820*f)*arctanh(1/2*x)+1/648*(13*d+25*f)*
arctanh(x)-1/81*e*ln(-x^2+1)+1/81*e*ln(-x^2+4)
```

3.43.2 Mathematica [A] (verified)

Time = 0.09 (sec) , antiderivative size = 161, normalized size of antiderivative = 0.92

$$\int \frac{d+ex+fx^2}{(4-5x^2+x^4)^3}\,dx$$
$$= \frac{\frac{288(17dx+20fx-5dx^3-8fx^3+e(20-8x^2))}{(4-5x^2+x^4)^2} + \frac{12(64e(-5+2x^2)+20fx(-19+7x^2)+dx(-59+35x^2))}{4-5x^2+x^4} - 32(13d+16e+25f)\log(1}{}$$

input
```
Integrate[(d + e*x + f*x^2)/(4 - 5*x^2 + x^4)^3,x]
```

output
```
((288*(17*d*x + 20*f*x - 5*d*x^3 - 8*f*x^3 + e*(20 - 8*x^2)))/(4 - 5*x^2 +
 x^4)^2 + (12*(64*e*(-5 + 2*x^2) + 20*f*x*(-19 + 7*x^2) + d*x*(-59 + 35*x^
2)))/(4 - 5*x^2 + x^4) - 32*(13*d + 16*e + 25*f)*Log[1 - x] + (313*d + 512
*e + 820*f)*Log[2 - x] + 32*(13*d - 16*e + 25*f)*Log[1 + x] + (-313*d + 51
2*e - 820*f)*Log[2 + x])/41472
```

3.43.3 Rubi [A] (verified)

Time = 0.50 (sec) , antiderivative size = 193, normalized size of antiderivative = 1.10, number of steps used = 12, number of rules used = 11, $\frac{\text{number of rules}}{\text{integrand size}} = 0.478$, Rules used = {2202, 27, 1432, 1084, 1492, 25, 1492, 27, 1480, 220, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2}{(x^4-5x^2+4)^3}\,dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{fx^2+d}{(x^4-5x^2+4)^3}dx + \int \frac{ex}{(x^4-5x^2+4)^3}dx$$

$$\Big\downarrow \text{27}$$

$$\int \frac{fx^2+d}{(x^4-5x^2+4)^3}dx + e\int \frac{x}{(x^4-5x^2+4)^3}dx$$

$$\Big\downarrow \text{1432}$$

$$\int \frac{fx^2+d}{(x^4-5x^2+4)^3}dx + \frac{1}{2}e\int \frac{1}{(x^4-5x^2+4)^3}dx^2$$

$$\Big\downarrow \; 1084$$

$$\int \frac{fx^2+d}{(x^4-5x^2+4)^3}dx+$$
$$\frac{1}{2}e\int \left(-\frac{2}{81\,(4-x^2)}-\frac{1}{27\,(4-x^2)^2}-\frac{1}{27\,(4-x^2)^3}+\frac{2}{81\,(1-x^2)}-\frac{1}{27\,(1-x^2)^2}+\frac{1}{27\,(1-x^2)^3}\right)dx^2$$

$$\Big\downarrow \; 1492$$

$$-\frac{1}{144}\int -\frac{-5(5d+8f)x^2+19d-20f}{(x^4-5x^2+4)^2}dx+$$
$$\frac{1}{2}e\int \left(-\frac{2}{81\,(4-x^2)}-\frac{1}{27\,(4-x^2)^2}-\frac{1}{27\,(4-x^2)^3}+\frac{2}{81\,(1-x^2)}-\frac{1}{27\,(1-x^2)^2}+\frac{1}{27\,(1-x^2)^3}\right)dx^2+$$
$$\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{144\,(x^4-5x^2+4)^2}$$

$$\Big\downarrow \; 25$$

$$\frac{1}{144}\int \frac{-5(5d+8f)x^2+19d-20f}{(x^4-5x^2+4)^2}dx+$$
$$\frac{1}{2}e\int \left(-\frac{2}{81\,(4-x^2)}-\frac{1}{27\,(4-x^2)^2}-\frac{1}{27\,(4-x^2)^3}+\frac{2}{81\,(1-x^2)}-\frac{1}{27\,(1-x^2)^2}+\frac{1}{27\,(1-x^2)^3}\right)dx^2+$$
$$\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{144\,(x^4-5x^2+4)^2}$$

$$\Big\downarrow \; 1492$$

$$\frac{1}{144}\left(-\frac{1}{72}\int -\frac{3\left(35(d+4f)x^2+173d+260f\right)}{x^4-5x^2+4}dx-\frac{x\left(-35x^2(d+4f)+59d+380f\right)}{24\,(x^4-5x^2+4)}\right)+$$
$$\frac{1}{2}e\int \left(-\frac{2}{81\,(4-x^2)}-\frac{1}{27\,(4-x^2)^2}-\frac{1}{27\,(4-x^2)^3}+\frac{2}{81\,(1-x^2)}-\frac{1}{27\,(1-x^2)^2}+\frac{1}{27\,(1-x^2)^3}\right)dx^2+$$
$$\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{144\,(x^4-5x^2+4)^2}$$

$$\Big\downarrow \; 27$$

$$\frac{1}{144}\left(\frac{1}{24}\int \frac{35(d+4f)x^2+173d+260f}{x^4-5x^2+4}dx-\frac{x\left(-35x^2(d+4f)+59d+380f\right)}{24\,(x^4-5x^2+4)}\right)+$$
$$\frac{1}{2}e\int \left(-\frac{2}{81\,(4-x^2)}-\frac{1}{27\,(4-x^2)^2}-\frac{1}{27\,(4-x^2)^3}+\frac{2}{81\,(1-x^2)}-\frac{1}{27\,(1-x^2)^2}+\frac{1}{27\,(1-x^2)^3}\right)dx^2+$$
$$\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{144\,(x^4-5x^2+4)^2}$$

$$\Big\downarrow \; 1480$$

3.43. $\int \frac{d+ex+fx^2}{(4-5x^2+x^4)^3}\,dx$

$$\frac{1}{144}\left(\frac{1}{24}\left(\frac{1}{3}(313d+820f)\int\frac{1}{x^2-4}dx-\frac{16}{3}(13d+25f)\int\frac{1}{x^2-1}dx\right)-\frac{x\left(-35x^2(d+4f)+59d+380f\right)}{24\left(x^4-5x^2+4\right)}\right)+$$
$$\frac{1}{2}e\int\left(-\frac{2}{81\left(4-x^2\right)}-\frac{1}{27\left(4-x^2\right)^2}-\frac{1}{27\left(4-x^2\right)^3}+\frac{2}{81\left(1-x^2\right)}-\frac{1}{27\left(1-x^2\right)^2}+\frac{1}{27\left(1-x^2\right)^3}\right)dx^2+$$
$$\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{144\left(x^4-5x^2+4\right)^2}$$

$\downarrow$ 220

$$\frac{1}{2}e\int\left(-\frac{2}{81\left(4-x^2\right)}-\frac{1}{27\left(4-x^2\right)^2}-\frac{1}{27\left(4-x^2\right)^3}+\frac{2}{81\left(1-x^2\right)}-\frac{1}{27\left(1-x^2\right)^2}+\frac{1}{27\left(1-x^2\right)^3}\right)dx^2+$$
$$\frac{1}{144}\left(\frac{1}{24}\left(\frac{16}{3}\text{arctanh}(x)(13d+25f)-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(313d+820f)\right)-\frac{x\left(-35x^2(d+4f)+59d+380f\right)}{24\left(x^4-5x^2+4\right)}\right)+$$
$$\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{144\left(x^4-5x^2+4\right)^2}$$

$\downarrow$ 2009

$$\frac{1}{144}\left(\frac{1}{24}\left(\frac{16}{3}\text{arctanh}(x)(13d+25f)-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(313d+820f)\right)-\frac{x\left(-35x^2(d+4f)+59d+380f\right)}{24\left(x^4-5x^2+4\right)}\right)+$$
$$\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{144\left(x^4-5x^2+4\right)^2}+$$
$$\frac{1}{2}e\left(-\frac{1}{27\left(1-x^2\right)}-\frac{1}{27\left(4-x^2\right)}+\frac{1}{54\left(1-x^2\right)^2}-\frac{1}{54\left(4-x^2\right)^2}-\frac{2}{81}\log\left(1-x^2\right)+\frac{2}{81}\log\left(4-x^2\right)\right)$$

input
```
Int[(d + e*x + f*x^2)/(4 - 5*x^2 + x^4)^3,x]
```

output
```
(x*(17*d + 20*f - (5*d + 8*f)*x^2))/(144*(4 - 5*x^2 + x^4)^2) + (-1/24*(x*
(59*d + 380*f - 35*(d + 4*f)*x^2))/(4 - 5*x^2 + x^4) + (-1/6*((313*d + 820
*f)*ArcTanh[x/2]) + (16*(13*d + 25*f)*ArcTanh[x])/3)/24)/144 + (e*(1/(54*(
1 - x^2)^2) - 1/(27*(1 - x^2)) - 1/(54*(4 - x^2)^2) - 1/(27*(4 - x^2)) - (
2*Log[1 - x^2])/81 + (2*Log[4 - x^2])/81))/2
```

3.43. $\int\frac{d+ex+fx^2}{\left(4-5x^2+x^4\right)^3}\,dx$

3.43.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 220
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[b, 2])^(-
1))*ArcTanh[Rt[b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] &&
 (LtQ[a, 0] || GtQ[b, 0])
```

rule 1084
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> With[{q = Rt[b^2
- 4*a*c, 2]}, Simp[1/c^p   Int[ExpandIntegrand[(b/2 - q/2 + c*x)^p*(b/2 + q
/2 + c*x)^p, x], x], x] /;  !FractionalPowerFactorQ[q]] /; FreeQ[{a, b, c},
 x] && IntegerQ[p] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

3.43. $\int \frac{d+ex+fx^2}{(4-5x^2+x^4)^3}\,dx$

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.43.4 Maple [A] (verified)

Time = 0.12 (sec) , antiderivative size = 139, normalized size of antiderivative = 0.79

method	result
norman	$\frac{\left(-\frac{13d}{192}-\frac{5f}{16}\right)x^5+\left(\frac{35d}{384}+\frac{21f}{32}\right)x^3+\left(\frac{35d}{3456}+\frac{35f}{864}\right)x^7+\left(\frac{43d}{864}-\frac{65f}{216}\right)x+\frac{5e\,x^2}{9}+\frac{e\,x^6}{27}-\frac{5e\,x^4}{18}-\frac{25e}{108}}{\left(x^4-5x^2+4\right)^2}+\left(-\frac{313d}{41472}+\frac{e}{81}-\frac{205f}{10368}\right)$
risch	$\frac{\left(-\frac{13d}{192}-\frac{5f}{16}\right)x^5+\left(\frac{35d}{384}+\frac{21f}{32}\right)x^3+\left(\frac{35d}{3456}+\frac{35f}{864}\right)x^7+\left(\frac{43d}{864}-\frac{65f}{216}\right)x+\frac{5e\,x^2}{9}+\frac{e\,x^6}{27}-\frac{5e\,x^4}{18}-\frac{25e}{108}}{\left(x^4-5x^2+4\right)^2}+\frac{313\ln(2-x)d}{41472}+\frac{\ln(2-x)e}{81}+$
default	$\left(-\frac{313d}{41472}+\frac{e}{81}-\frac{205f}{10368}\right)\ln\left(x+2\right)-\frac{-\frac{19d}{6912}+\frac{17e}{3456}-\frac{5f}{576}}{x+2}-\frac{-\frac{d}{1728}+\frac{e}{864}-\frac{f}{432}}{2(x+2)^2}-\frac{-\frac{d}{432}+\frac{e}{144}-\frac{5f}{432}}{x+1}-\frac{\frac{d}{216}-\frac{e}{216}+}{2(x+1)}$
parallelrisch	$-12960f\,x^5+1536e\,x^6-11520e\,x^4-9600e+27216f\,x^3+420d\,x^7+1680f\,x^7+2064dx+5008\ln(x-2)d+8192\ln(x-2)e-6656\ln($

input
```
int((f*x^2+e*x+d)/(x^4-5*x^2+4)^3,x,method=_RETURNVERBOSE)
```

output
```
((-13/192*d-5/16*f)*x^5+(35/384*d+21/32*f)*x^3+(35/3456*d+35/864*f)*x^7+(4
3/864*d-65/216*f)*x+5/9*e*x^2+1/27*e*x^6-5/18*e*x^4-25/108*e)/(x^4-5*x^2+4
)^2+(-313/41472*d+1/81*e-205/10368*f)*ln(x+2)+(-13/1296*d-1/81*e-25/1296*f
)*ln(x-1)+(13/1296*d-1/81*e+25/1296*f)*ln(x+1)+(313/41472*d+1/81*e+205/103
68*f)*ln(x-2)
```

3.43.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 389 vs. 2(156) = 312.

Time = 0.32 (sec) , antiderivative size = 389, normalized size of antiderivative = 2.22

$$\int \frac{d+ex+fx^2}{\left(4-5x^2+x^4\right)^3}\,dx$$
$$= \frac{420\,(d+4\,f)x^7+1536\,ex^6-216\,(13\,d+60\,f)x^5-11520\,ex^4+756\,(5\,d+36\,f)x^3+23040\,ex^2+48\,(43\,d}{}$$

input
```
integrate((f*x^2+e*x+d)/(x^4-5*x^2+4)^3,x, algorithm="fricas")
```

output
```
1/41472*(420*(d + 4*f)*x^7 + 1536*e*x^6 - 216*(13*d + 60*f)*x^5 - 11520*e*
x^4 + 756*(5*d + 36*f)*x^3 + 23040*e*x^2 + 48*(43*d - 260*f)*x - ((313*d -
 512*e + 820*f)*x^8 - 10*(313*d - 512*e + 820*f)*x^6 + 33*(313*d - 512*e +
 820*f)*x^4 - 40*(313*d - 512*e + 820*f)*x^2 + 5008*d - 8192*e + 13120*f)*
log(x + 2) + 32*((13*d - 16*e + 25*f)*x^8 - 10*(13*d - 16*e + 25*f)*x^6 +
33*(13*d - 16*e + 25*f)*x^4 - 40*(13*d - 16*e + 25*f)*x^2 + 208*d - 256*e
+ 400*f)*log(x + 1) - 32*((13*d + 16*e + 25*f)*x^8 - 10*(13*d + 16*e + 25*
f)*x^6 + 33*(13*d + 16*e + 25*f)*x^4 - 40*(13*d + 16*e + 25*f)*x^2 + 208*d
 + 256*e + 400*f)*log(x - 1) + ((313*d + 512*e + 820*f)*x^8 - 10*(313*d +
512*e + 820*f)*x^6 + 33*(313*d + 512*e + 820*f)*x^4 - 40*(313*d + 512*e +
820*f)*x^2 + 5008*d + 8192*e + 13120*f)*log(x - 2) - 9600*e)/(x^8 - 10*x^6
 + 33*x^4 - 40*x^2 + 16)
```

3.43.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2}{(4-5x^2+x^4)^3}\,dx = \text{Timed out}$$

input
```
integrate((f*x**2+e*x+d)/(x**4-5*x**2+4)**3,x)
```

output
```
Timed out
```

3.43.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 155, normalized size of antiderivative = 0.89

$$\begin{aligned}
&\int \frac{d+ex+fx^2}{(4-5x^2+x^4)^3}\,dx \\
&= -\frac{1}{41472}(313\,d-512\,e+820\,f)\log(x+2)+\frac{1}{1296}(13\,d-16\,e+25\,f)\log(x+1) \\
&\quad -\frac{1}{1296}(13\,d+16\,e+25\,f)\log(x-1)+\frac{1}{41472}(313\,d+512\,e+820\,f)\log(x-2) \\
&\quad +\frac{35\,(d+4\,f)x^7+128\,ex^6-18\,(13\,d+60\,f)x^5-960\,ex^4+63\,(5\,d+36\,f)x^3+1920\,ex^2+4\,(43\,d-260}{3456\,(x^8-10\,x^6+33\,x^4-40\,x^2+16)}
\end{aligned}$$

input
```
integrate((f*x^2+e*x+d)/(x^4-5*x^2+4)^3,x, algorithm="maxima")
```

output
```
-1/41472*(313*d - 512*e + 820*f)*log(x + 2) + 1/1296*(13*d - 16*e + 25*f)*
log(x + 1) - 1/1296*(13*d + 16*e + 25*f)*log(x - 1) + 1/41472*(313*d + 512
*e + 820*f)*log(x - 2) + 1/3456*(35*(d + 4*f)*x^7 + 128*e*x^6 - 18*(13*d +
 60*f)*x^5 - 960*e*x^4 + 63*(5*d + 36*f)*x^3 + 1920*e*x^2 + 4*(43*d - 260*
f)*x - 800*e)/(x^8 - 10*x^6 + 33*x^4 - 40*x^2 + 16)
```

3.43.8 Giac [A] (verification not implemented)

Time = 0.32 (sec) , antiderivative size = 149, normalized size of antiderivative = 0.85

$$\int \frac{d+ex+fx^2}{(4-5x^2+x^4)^3}\,dx$$
$$= -\frac{1}{41472}\,(313\,d-512\,e+820\,f)\log\left(|x+2|\right)+\frac{1}{1296}\,(13\,d-16\,e+25\,f)\log\left(|x+1|\right)$$
$$-\frac{1}{1296}\,(13\,d+16\,e+25\,f)\log\left(|x-1|\right)+\frac{1}{41472}\,(313\,d+512\,e+820\,f)\log\left(|x-2|\right)$$
$$+\frac{35\,dx^7+140\,fx^7+128\,ex^6-234\,dx^5-1080\,fx^5-960\,ex^4+315\,dx^3+2268\,fx^3+1920\,ex^2+172\,dx}{3456\,(x^4-5\,x^2+4)^2}$$

input
```
integrate((f*x^2+e*x+d)/(x^4-5*x^2+4)^3,x, algorithm="giac")
```

output
```
-1/41472*(313*d - 512*e + 820*f)*log(abs(x + 2)) + 1/1296*(13*d - 16*e + 2
5*f)*log(abs(x + 1)) - 1/1296*(13*d + 16*e + 25*f)*log(abs(x - 1)) + 1/414
72*(313*d + 512*e + 820*f)*log(abs(x - 2)) + 1/3456*(35*d*x^7 + 140*f*x^7
+ 128*e*x^6 - 234*d*x^5 - 1080*f*x^5 - 960*e*x^4 + 315*d*x^3 + 2268*f*x^3
+ 1920*e*x^2 + 172*d*x - 1040*f*x - 800*e)/(x^4 - 5*x^2 + 4)^2
```

3.43.9 Mupad [B] (verification not implemented)

Time = 0.06 (sec) , antiderivative size = 151, normalized size of antiderivative = 0.86

$$\int \frac{d+ex+fx^2}{(4-5x^2+x^4)^3}\,dx = \ln(x+1)\,\left(\frac{13\,d}{1296}-\frac{e}{81}+\frac{25\,f}{1296}\right)-\ln(x-1)\,\left(\frac{13\,d}{1296}+\frac{e}{81}+\frac{25\,f}{1296}\right)$$
$$+\ln(x-2)\,\left(\frac{313\,d}{41472}+\frac{e}{81}+\frac{205\,f}{10368}\right)-\ln(x+2)\,\left(\frac{313\,d}{41472}-\frac{e}{81}+\frac{205\,f}{10368}\right)$$
$$+\frac{\left(\frac{35\,d}{3456}+\frac{35\,f}{864}\right)\,x^7+\frac{e\,x^6}{27}+\left(-\frac{13\,d}{192}-\frac{5\,f}{16}\right)\,x^5-\frac{5\,e\,x^4}{18}+\left(\frac{35\,d}{384}+\frac{21\,f}{32}\right)\,x^3+\frac{5\,e\,x^2}{9}+\left(\frac{43\,d}{864}-\frac{65\,f}{216}\right)\,x-\frac{25\,e}{108}}{x^8-10\,x^6+33\,x^4-40\,x^2+16}$$

input
```
int((d + e*x + f*x^2)/(x^4 - 5*x^2 + 4)^3,x)
```

output
```
log(x + 1)*((13*d)/1296 - e/81 + (25*f)/1296) - log(x - 1)*((13*d)/1296 +
e/81 + (25*f)/1296) + log(x - 2)*((313*d)/41472 + e/81 + (205*f)/10368) -
log(x + 2)*((313*d)/41472 - e/81 + (205*f)/10368) + (x^3*((35*d)/384 + (21
*f)/32) - x^5*((13*d)/192 + (5*f)/16) - (25*e)/108 + x^7*((35*d)/3456 + (3
5*f)/864) + (5*e*x^2)/9 - (5*e*x^4)/18 + (e*x^6)/27 + x*((43*d)/864 - (65*
f)/216))/(33*x^4 - 40*x^2 - 10*x^6 + x^8 + 16)
```

3.43. $\int \frac{d+ex+fx^2}{(4-5x^2+x^4)^3}\,dx$

3.44 $\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^3}\,dx$

3.44.1 Optimal result

Integrand size = 28, antiderivative size = 204

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^3}\,dx &= \frac{x(17d+20f-(5d+8f)x^2)}{144(4-5x^2+x^4)^2}+\frac{5e+8g-(2e+5g)x^2}{36(4-5x^2+x^4)^2}\\ &\quad-\frac{(2e+5g)(5-2x^2)}{108(4-5x^2+x^4)}-\frac{x(59d+380f-35(d+4f)x^2)}{3456(4-5x^2+x^4)}\\ &\quad-\frac{(313d+820f)\operatorname{arctanh}\left(\frac{x}{2}\right)}{20736}+\frac{1}{648}(13d+25f)\operatorname{arctanh}(x)\\ &\quad-\frac{1}{162}(2e+5g)\log\left(1-x^2\right)+\frac{1}{162}(2e+5g)\log\left(4-x^2\right)\end{aligned}$$

output
```
1/144*x*(17*d+20*f-(5*d+8*f)*x^2)/(x^4-5*x^2+4)^2+1/36*(5*e+8*g-(2*e+5*g)*
x^2)/(x^4-5*x^2+4)^2-1/108*(2*e+5*g)*(-2*x^2+5)/(x^4-5*x^2+4)-1/3456*x*(59
*d+380*f-35*(d+4*f)*x^2)/(x^4-5*x^2+4)-1/20736*(313*d+820*f)*arctanh(1/2*x
)+1/648*(13*d+25*f)*arctanh(x)-1/162*(2*e+5*g)*ln(-x^2+1)+1/162*(2*e+5*g)*
ln(-x^2+4)
```

3.44.2 Mathematica [A] (verified)

Time = 0.06 (sec) , antiderivative size = 193, normalized size of antiderivative = 0.95

$$\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^3}\,dx = \frac{\frac{288(17dx+20fx-5dx^3-8fx^3+e(20-8x^2)-4g(-8+5x^2))}{(4-5x^2+x^4)^2}+\frac{12(64e(-5+2x^2)+160g(-5+2x^2)+20fx(-19+7x^2)+dx(-59+35x^2))}{4-5x^2+x^4}-32(}{}$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3)/(4 - 5*x^2 + x^4)^3,x]
```

output
```
((288*(17*d*x + 20*f*x - 5*d*x^3 - 8*f*x^3 + e*(20 - 8*x^2) - 4*g*(-8 + 5*
x^2)))/(4 - 5*x^2 + x^4)^2 + (12*(64*e*(-5 + 2*x^2) + 160*g*(-5 + 2*x^2) +
 20*f*x*(-19 + 7*x^2) + d*x*(-59 + 35*x^2)))/(4 - 5*x^2 + x^4) - 32*(13*d
+ 16*e + 25*f + 40*g)*Log[1 - x] + (313*d + 512*e + 820*f + 1280*g)*Log[2
- x] + 32*(13*d - 16*e + 25*f - 40*g)*Log[1 + x] + (-313*d + 512*e - 820*f
 + 1280*g)*Log[2 + x])/41472
```

3.44.3 Rubi [A] (verified)

Time = 0.57 (sec) , antiderivative size = 224, normalized size of antiderivative = 1.10, number of steps used = 11, number of rules used = 10, $\frac{\text{number of rules}}{\text{integrand size}} = 0.357$, Rules used = {2202, 1492, 25, 1492, 27, 1480, 220, 1576, 1141, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3}{(x^4-5x^2+4)^3}\,dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{fx^2+d}{(x^4-5x^2+4)^3}dx+\int \frac{x(gx^2+e)}{(x^4-5x^2+4)^3}dx$$

$$\Big\downarrow \text{1492}$$

$$-\frac{1}{144}\int -\frac{-5(5d+8f)x^2+19d-20f}{(x^4-5x^2+4)^2}dx+\int \frac{x(gx^2+e)}{(x^4-5x^2+4)^3}dx+\frac{x(-(x^2(5d+8f))+17d+20f)}{144(x^4-5x^2+4)^2}$$

$$\Big\downarrow \text{25}$$

$$\frac{1}{144}\int \frac{-5(5d+8f)x^2+19d-20f}{\left(x^4-5x^2+4\right)^2}dx+\int \frac{x\left(gx^2+e\right)}{\left(x^4-5x^2+4\right)^3}dx+\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{144\left(x^4-5x^2+4\right)^2}$$

$$\Big\downarrow \text{1492}$$

$$\frac{1}{144}\left(-\frac{1}{72}\int -\frac{3\left(35(d+4f)x^2+173d+260f\right)}{x^4-5x^2+4}dx-\frac{x\left(-35x^2(d+4f)+59d+380f\right)}{24\left(x^4-5x^2+4\right)}\right)+$$
$$\int \frac{x\left(gx^2+e\right)}{\left(x^4-5x^2+4\right)^3}dx+\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{144\left(x^4-5x^2+4\right)^2}$$

$$\Big\downarrow \text{27}$$

$$\frac{1}{144}\left(\frac{1}{24}\int \frac{35(d+4f)x^2+173d+260f}{x^4-5x^2+4}dx-\frac{x\left(-35x^2(d+4f)+59d+380f\right)}{24\left(x^4-5x^2+4\right)}\right)+$$
$$\int \frac{x\left(gx^2+e\right)}{\left(x^4-5x^2+4\right)^3}dx+\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{144\left(x^4-5x^2+4\right)^2}$$

$$\Big\downarrow \text{1480}$$

$$\frac{1}{144}\left(\frac{1}{24}\left(\frac{1}{3}(313d+820f)\int \frac{1}{x^2-4}dx-\frac{16}{3}(13d+25f)\int \frac{1}{x^2-1}dx\right)-\frac{x\left(-35x^2(d+4f)+59d+380f\right)}{24\left(x^4-5x^2+4\right)}\right)+$$
$$\int \frac{x\left(gx^2+e\right)}{\left(x^4-5x^2+4\right)^3}dx+\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{144\left(x^4-5x^2+4\right)^2}$$

$$\Big\downarrow \text{220}$$

$$\int \frac{x\left(gx^2+e\right)}{\left(x^4-5x^2+4\right)^3}dx+$$
$$\frac{1}{144}\left(\frac{1}{24}\left(\frac{16}{3}\text{arctanh}(x)(13d+25f)-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(313d+820f)\right)-\frac{x\left(-35x^2(d+4f)+59d+380f\right)}{24\left(x^4-5x^2+4\right)}\right)+$$
$$\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{144\left(x^4-5x^2+4\right)^2}$$

$$\Big\downarrow \text{1576}$$

$$\frac{1}{2}\int \frac{gx^2+e}{\left(x^4-5x^2+4\right)^3}dx^2+$$
$$\frac{1}{144}\left(\frac{1}{24}\left(\frac{16}{3}\text{arctanh}(x)(13d+25f)-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(313d+820f)\right)-\frac{x\left(-35x^2(d+4f)+59d+380f\right)}{24\left(x^4-5x^2+4\right)}\right)+$$
$$\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{144\left(x^4-5x^2+4\right)^2}$$

$$\Big\downarrow \text{1141}$$

3.44. $\int \frac{d+ex+fx^2+gx^3}{\left(4-5x^2+x^4\right)^3}\,dx$

$$\frac{1}{2}\int \left(\frac{e+g}{27\left(1-x^2\right)^3}+\frac{2e+5g}{81\left(1-x^2\right)}-\frac{2e+5g}{81\left(4-x^2\right)}-\frac{e+2g}{27\left(1-x^2\right)^2}-\frac{e+3g}{27\left(4-x^2\right)^2}-\frac{e+4g}{27\left(4-x^2\right)^3}\right)dx^2+$$
$$\frac{1}{144}\left(\frac{1}{24}\left(\frac{16}{3}\text{arctanh}(x)(13d+25f)-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(313d+820f)\right)-\frac{x\left(-35x^2(d+4f)+59d+380f\right)}{24\left(x^4-5x^2+4\right)}\right)+$$
$$\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{144\left(x^4-5x^2+4\right)^2}$$

$$\Big\downarrow \text{2009}$$

$$\frac{1}{144}\left(\frac{1}{24}\left(\frac{16}{3}\text{arctanh}(x)(13d+25f)-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(313d+820f)\right)-\frac{x\left(-35x^2(d+4f)+59d+380f\right)}{24\left(x^4-5x^2+4\right)}\right)+$$
$$\frac{x\left(-\left(x^2(5d+8f)\right)+17d+20f\right)}{144\left(x^4-5x^2+4\right)^2}+$$
$$\frac{1}{2}\left(\frac{e+g}{54\left(1-x^2\right)^2}-\frac{e+2g}{27\left(1-x^2\right)}-\frac{e+3g}{27\left(4-x^2\right)}-\frac{e+4g}{54\left(4-x^2\right)^2}-\frac{1}{81}(2e+5g)\log\left(1-x^2\right)+\frac{1}{81}(2e+5g)\log\left(4-x^2\right.\right.$$

input
```
Int[(d + e*x + f*x^2 + g*x^3)/(4 - 5*x^2 + x^4)^3,x]
```

output
```
(x*(17*d + 20*f - (5*d + 8*f)*x^2))/(144*(4 - 5*x^2 + x^4)^2) + (-1/24*(x*
(59*d + 380*f - 35*(d + 4*f)*x^2))/(4 - 5*x^2 + x^4) + (-1/6*((313*d + 820
*f)*ArcTanh[x/2]) + (16*(13*d + 25*f)*ArcTanh[x])/3)/24)/144 + ((e + g)/(5
4*(1 - x^2)^2) - (e + 2*g)/(27*(1 - x^2)) - (e + 4*g)/(54*(4 - x^2)^2) - (
e + 3*g)/(27*(4 - x^2)) - ((2*e + 5*g)*Log[1 - x^2])/81 + ((2*e + 5*g)*Log
[4 - x^2])/81)/2
```

3.44.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 220
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[b, 2])^(-
1))*ArcTanh[Rt[b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] &&
 (LtQ[a, 0] || GtQ[b, 0])
```

rule 1141
```
Int[((d_.) + (e_.)*(x_))^(m_.)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_
Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[1/c^p   Int[ExpandIntegrand[
(d + e*x)^m*(b/2 - q/2 + c*x)^p*(b/2 + q/2 + c*x)^p, x], x], x] /; EqQ[p, -
1] ||  !FractionalPowerFactorQ[q]] /; FreeQ[{a, b, c, d, e}, x] && ILtQ[p,
0] && IntegerQ[m] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.44. $\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^3}\,dx$

3.44.4 Maple [A] (verified)

Time = 0.11 (sec) , antiderivative size = 169, normalized size of antiderivative = 0.83

method	result
norman	$\frac{\left(-\frac{13d}{192}-\frac{5f}{16}\right)x^5+\left(\frac{35d}{384}+\frac{21f}{32}\right)x^3+\left(\frac{35d}{3456}+\frac{35f}{864}\right)x^7+\left(\frac{43d}{864}-\frac{65f}{216}\right)x+\left(-\frac{5e}{18}-\frac{25g}{36}\right)x^4+\left(\frac{5e}{9}+\frac{25g}{18}\right)x^2+\left(\frac{e}{27}+\frac{5g}{54}\right)x^6-\frac{25e}{108}-\frac{19g}{27}}{(x^4-5x^2+4)^2}$
risch	$\frac{\left(-\frac{13d}{192}-\frac{5f}{16}\right)x^5+\left(\frac{35d}{384}+\frac{21f}{32}\right)x^3+\left(\frac{35d}{3456}+\frac{35f}{864}\right)x^7+\left(\frac{43d}{864}-\frac{65f}{216}\right)x+\left(-\frac{5e}{18}-\frac{25g}{36}\right)x^4+\left(\frac{5e}{9}+\frac{25g}{18}\right)x^2+\left(\frac{e}{27}+\frac{5g}{54}\right)x^6-\frac{25e}{108}-\frac{19g}{27}}{(x^4-5x^2+4)^2}$
default	$\left(-\frac{313d}{41472}+\frac{e}{81}-\frac{205f}{10368}+\frac{5g}{162}\right)\ln(x+2)-\frac{-\frac{19d}{6912}+\frac{17e}{3456}-\frac{5f}{576}+\frac{13g}{864}}{x+2}-\frac{-\frac{d}{1728}+\frac{e}{864}-\frac{f}{432}+\frac{g}{216}}{2(x+2)^2}-\frac{-\frac{d}{432}+\frac{e}{144}}{x+1}$
parallelrisch	Expression too large to display

input
```
int((g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^3,x,method=_RETURNVERBOSE)
```

output
```
((-13/192*d-5/16*f)*x^5+(35/384*d+21/32*f)*x^3+(35/3456*d+35/864*f)*x^7+(4
3/864*d-65/216*f)*x+(-5/18*e-25/36*g)*x^4+(5/9*e+25/18*g)*x^2+(1/27*e+5/54
*g)*x^6-25/108*e-19/27*g)/(x^4-5*x^2+4)^2+(-313/41472*d+1/81*e-205/10368*f
+5/162*g)*ln(x+2)+(-13/1296*d-1/81*e-25/1296*f-5/162*g)*ln(x-1)+(13/1296*d
-1/81*e+25/1296*f-5/162*g)*ln(x+1)+(313/41472*d+1/81*e+205/10368*f+5/162*g
)*ln(x-2)
```

3.44.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 470 vs. 2(184) = 368.

Time = 0.50 (sec) , antiderivative size = 470, normalized size of antiderivative = 2.30

$$\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^3}\,dx$$
$$=\frac{420\,(d+4\,f)x^7+768\,(2\,e+5\,g)x^6-216\,(13\,d+60\,f)x^5-5760\,(2\,e+5\,g)x^4+756\,(5\,d+36\,f)x^3+1152}{}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^3,x, algorithm="fricas")
```

output
```
1/41472*(420*(d + 4*f)*x^7 + 768*(2*e + 5*g)*x^6 - 216*(13*d + 60*f)*x^5 -
 5760*(2*e + 5*g)*x^4 + 756*(5*d + 36*f)*x^3 + 11520*(2*e + 5*g)*x^2 + 48*
(43*d - 260*f)*x - ((313*d - 512*e + 820*f - 1280*g)*x^8 - 10*(313*d - 512
*e + 820*f - 1280*g)*x^6 + 33*(313*d - 512*e + 820*f - 1280*g)*x^4 - 40*(3
13*d - 512*e + 820*f - 1280*g)*x^2 + 5008*d - 8192*e + 13120*f - 20480*g)*
log(x + 2) + 32*((13*d - 16*e + 25*f - 40*g)*x^8 - 10*(13*d - 16*e + 25*f
- 40*g)*x^6 + 33*(13*d - 16*e + 25*f - 40*g)*x^4 - 40*(13*d - 16*e + 25*f
- 40*g)*x^2 + 208*d - 256*e + 400*f - 640*g)*log(x + 1) - 32*((13*d + 16*e
 + 25*f + 40*g)*x^8 - 10*(13*d + 16*e + 25*f + 40*g)*x^6 + 33*(13*d + 16*e
 + 25*f + 40*g)*x^4 - 40*(13*d + 16*e + 25*f + 40*g)*x^2 + 208*d + 256*e +
 400*f + 640*g)*log(x - 1) + ((313*d + 512*e + 820*f + 1280*g)*x^8 - 10*(3
13*d + 512*e + 820*f + 1280*g)*x^6 + 33*(313*d + 512*e + 820*f + 1280*g)*x
^4 - 40*(313*d + 512*e + 820*f + 1280*g)*x^2 + 5008*d + 8192*e + 13120*f +
 20480*g)*log(x - 2) - 9600*e - 29184*g)/(x^8 - 10*x^6 + 33*x^4 - 40*x^2 +
 16)
```

3.44.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^3}\,dx = \text{Timed out}$$

input
```
integrate((g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4)**3,x)
```

output
```
Timed out
```

3.44.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 188, normalized size of antiderivative = 0.92

$$\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^3}\,dx = -\frac{1}{41472}\,(313\,d-512\,e+820\,f-1280\,g)\log{(x+2)} + \frac{1}{1296}\,(13\,d-16\,e+25\,f-40\,g)\log{(x+1)} - \frac{1}{1296}\,(13\,d+16\,e+25\,f+40\,g)\log{(x-1)} + \frac{1}{41472}\,(313\,d+512\,e+820\,f+1280\,g)\log{(x-2)} + \frac{35\,(d+4\,f)x^7+64\,(2\,e+5\,g)x^6-18\,(13\,d+60\,f)x^5-480\,(2\,e+5\,g)x^4+63\,(5\,d+36\,f)x^3+960\,(2\,e}{3456\,(x^8-10\,x^6+33\,x^4-40\,x^2+16)}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^3,x, algorithm="maxima")
```

output
```
-1/41472*(313*d - 512*e + 820*f - 1280*g)*log(x + 2) + 1/1296*(13*d - 16*e
 + 25*f - 40*g)*log(x + 1) - 1/1296*(13*d + 16*e + 25*f + 40*g)*log(x - 1)
 + 1/41472*(313*d + 512*e + 820*f + 1280*g)*log(x - 2) + 1/3456*(35*(d + 4
*f)*x^7 + 64*(2*e + 5*g)*x^6 - 18*(13*d + 60*f)*x^5 - 480*(2*e + 5*g)*x^4
+ 63*(5*d + 36*f)*x^3 + 960*(2*e + 5*g)*x^2 + 4*(43*d - 260*f)*x - 800*e -
 2432*g)/(x^8 - 10*x^6 + 33*x^4 - 40*x^2 + 16)
```

3.44.8 Giac [A] (verification not implemented)

Time = 0.31 (sec) , antiderivative size = 182, normalized size of antiderivative = 0.89

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^3}\,dx &= -\frac{1}{41472}(313\,d-512\,e+820\,f-1280\,g)\log(|x+2|)\\ &+\frac{1}{1296}(13\,d-16\,e+25\,f-40\,g)\log(|x+1|)\\ &-\frac{1}{1296}(13\,d+16\,e+25\,f+40\,g)\log(|x-1|)\\ &+\frac{1}{41472}(313\,d+512\,e+820\,f+1280\,g)\log(|x-2|)\\ &+\frac{35\,dx^7+140\,fx^7+128\,ex^6+320\,gx^6-234\,dx^5-1080\,fx^5-960\,ex^4-2400\,gx^4+315\,dx^3+2268\,fx}{3456\,(x^4-5\,x^2+4)^2}\end{aligned}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^3,x, algorithm="giac")
```

output
```
-1/41472*(313*d - 512*e + 820*f - 1280*g)*log(abs(x + 2)) + 1/1296*(13*d -
 16*e + 25*f - 40*g)*log(abs(x + 1)) - 1/1296*(13*d + 16*e + 25*f + 40*g)*
log(abs(x - 1)) + 1/41472*(313*d + 512*e + 820*f + 1280*g)*log(abs(x - 2))
 + 1/3456*(35*d*x^7 + 140*f*x^7 + 128*e*x^6 + 320*g*x^6 - 234*d*x^5 - 1080
*f*x^5 - 960*e*x^4 - 2400*g*x^4 + 315*d*x^3 + 2268*f*x^3 + 1920*e*x^2 + 48
00*g*x^2 + 172*d*x - 1040*f*x - 800*e - 2432*g)/(x^4 - 5*x^2 + 4)^2
```

3.44.9 Mupad [B] (verification not implemented)

Time = 7.86 (sec) , antiderivative size = 182, normalized size of antiderivative = 0.89

$$\int \frac{d+ex+fx^2+gx^3}{(4-5x^2+x^4)^3}\,dx$$
$$= \frac{\left(\frac{35\,d}{3456}+\frac{35\,f}{864}\right)\,x^7+\left(\frac{e}{27}+\frac{5\,g}{54}\right)\,x^6+\left(-\frac{13\,d}{192}-\frac{5\,f}{16}\right)\,x^5+\left(-\frac{5\,e}{18}-\frac{25\,g}{36}\right)\,x^4+\left(\frac{35\,d}{384}+\frac{21\,f}{32}\right)\,x^3+\left(\frac{5\,e}{9}+\frac{25\,g}{18}\right)\,x^2 \cdots}{x^8-10\,x^6+33\,x^4-40\,x^2+16}$$
$$-\ln\left(x-1\right)\,\left(\frac{13\,d}{1296}+\frac{e}{81}+\frac{25\,f}{1296}+\frac{5\,g}{162}\right)+\ln\left(x+1\right)\,\left(\frac{13\,d}{1296}-\frac{e}{81}+\frac{25\,f}{1296}-\frac{5\,g}{162}\right)$$
$$+\ln\left(x-2\right)\,\left(\frac{313\,d}{41472}+\frac{e}{81}+\frac{205\,f}{10368}+\frac{5\,g}{162}\right)-\ln\left(x+2\right)\,\left(\frac{313\,d}{41472}-\frac{e}{81}+\frac{205\,f}{10368}-\frac{5\,g}{162}\right)$$

input
```
int((d + e*x + f*x^2 + g*x^3)/(x^4 - 5*x^2 + 4)^3,x)
```

output
```
(x^3*((35*d)/384 + (21*f)/32) - (19*g)/27 - x^5*((13*d)/192 + (5*f)/16) -
(25*e)/108 + x^7*((35*d)/3456 + (35*f)/864) + x^2*((5*e)/9 + (25*g)/18) -
x^4*((5*e)/18 + (25*g)/36) + x^6*(e/27 + (5*g)/54) + x*((43*d)/864 - (65*f
)/216))/(33*x^4 - 40*x^2 - 10*x^6 + x^8 + 16) - log(x - 1)*((13*d)/1296 +
e/81 + (25*f)/1296 + (5*g)/162) + log(x + 1)*((13*d)/1296 - e/81 + (25*f)/
1296 - (5*g)/162) + log(x - 2)*((313*d)/41472 + e/81 + (205*f)/10368 + (5*
g)/162) - log(x + 2)*((313*d)/41472 - e/81 + (205*f)/10368 - (5*g)/162)
```

3.45 $\int \frac{d+ex+fx^2+gx^3+hx^4}{(4-5x^2+x^4)^3}\,dx$

3.45.1 Optimal result

Integrand size = 33, antiderivative size = 224

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3+hx^4}{(4-5x^2+x^4)^3}\,dx &= \frac{5e+8g-(2e+5g)x^2}{36(4-5x^2+x^4)^2}\\ &+\frac{x(17d+20f+32h-(5d+8f+20h)x^2)}{144(4-5x^2+x^4)^2}\\ &-\frac{(2e+5g)(5-2x^2)}{108(4-5x^2+x^4)}\\ &-\frac{x(59d+380f+848h-5(7d+28f+64h)x^2)}{3456(4-5x^2+x^4)}\\ &-\frac{(313d+820f+1936h)\operatorname{arctanh}\left(\frac{x}{2}\right)}{20736}\\ &+\frac{1}{648}(13d+25f+61h)\operatorname{arctanh}(x)\\ &-\frac{1}{162}(2e+5g)\log\left(1-x^2\right)+\frac{1}{162}(2e+5g)\log\left(4-x^2\right)\end{aligned}$$

output
```
1/36*(5*e+8*g-(2*e+5*g)*x^2)/(x^4-5*x^2+4)^2+1/144*x*(17*d+20*f+32*h-(5*d+
8*f+20*h)*x^2)/(x^4-5*x^2+4)^2-1/108*(2*e+5*g)*(-2*x^2+5)/(x^4-5*x^2+4)-1/
3456*x*(59*d+380*f+848*h-5*(7*d+28*f+64*h)*x^2)/(x^4-5*x^2+4)-1/20736*(313
*d+820*f+1936*h)*arctanh(1/2*x)+1/648*(13*d+25*f+61*h)*arctanh(x)-1/162*(2
*e+5*g)*ln(-x^2+1)+1/162*(2*e+5*g)*ln(-x^2+4)
```

3.45.2 Mathematica [A] (verified)

Time = 0.09 (sec) , antiderivative size = 231, normalized size of antiderivative = 1.03

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(4-5x^2+x^4)^3}\,dx = \frac{20e+32g+17dx+20fx+32hx-8ex^2-20gx^2-5dx^3-8fx^3-20hx^3}{144(4-5x^2+x^4)^2} + \frac{-320e-800g-59dx-380fx-848hx+128ex^2+320gx^2+35dx^3+140fx^3+320hx^3}{3456(4-5x^2+x^4)} + \frac{(-13d-16e-25f-40g-61h)\log(1-x)}{1296} + \frac{(313d+512e+820f+1280g+1936h)\log(2-x)}{41472} + \frac{(13d-16e+25f-40g+61h)\log(1+x)}{1296} + \frac{(-313d+512e-820f+1280g-1936h)\log(2+x)}{41472}$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4)/(4 - 5*x^2 + x^4)^3,x]
```

output
```
(20*e + 32*g + 17*d*x + 20*f*x + 32*h*x - 8*e*x^2 - 20*g*x^2 - 5*d*x^3 - 8
*f*x^3 - 20*h*x^3)/(144*(4 - 5*x^2 + x^4)^2) + (-320*e - 800*g - 59*d*x -
380*f*x - 848*h*x + 128*e*x^2 + 320*g*x^2 + 35*d*x^3 + 140*f*x^3 + 320*h*x
^3)/(3456*(4 - 5*x^2 + x^4)) + ((-13*d - 16*e - 25*f - 40*g - 61*h)*Log[1
- x])/1296 + ((313*d + 512*e + 820*f + 1280*g + 1936*h)*Log[2 - x])/41472
+ ((13*d - 16*e + 25*f - 40*g + 61*h)*Log[1 + x])/1296 + ((-313*d + 512*e
- 820*f + 1280*g - 1936*h)*Log[2 + x])/41472
```

3.45.3 Rubi [A] (verified)

Time = 0.63 (sec) , antiderivative size = 244, normalized size of antiderivative = 1.09, number of steps used = 11, number of rules used = 10, $\frac{\text{number of rules}}{\text{integrand size}} = 0.303$, Rules used = {2202, 1576, 1141, 2009, 2206, 25, 1492, 27, 1480, 220}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(x^4-5x^2+4)^3}\,dx$$

$$\Big\downarrow \text{ 2202}$$

$$\int \frac{hx^4+fx^2+d}{(x^4-5x^2+4)^3}dx+\int \frac{x(gx^2+e)}{(x^4-5x^2+4)^3}dx$$

$$\Big\downarrow \text{ 1576}$$

$$\int \frac{hx^4+fx^2+d}{(x^4-5x^2+4)^3}dx+\frac{1}{2}\int \frac{gx^2+e}{(x^4-5x^2+4)^3}dx^2$$

$$\Big\downarrow \text{ 1141}$$

$$\int \frac{hx^4+fx^2+d}{(x^4-5x^2+4)^3}dx+$$
$$\frac{1}{2}\int \left(\frac{e+g}{27(1-x^2)^3}+\frac{2e+5g}{81(1-x^2)}-\frac{2e+5g}{81(4-x^2)}-\frac{e+2g}{27(1-x^2)^2}-\frac{e+3g}{27(4-x^2)^2}-\frac{e+4g}{27(4-x^2)^3}\right)dx^2$$

$$\Big\downarrow \text{ 2009}$$

$$\int \frac{hx^4+fx^2+d}{(x^4-5x^2+4)^3}dx+$$
$$\frac{1}{2}\left(\frac{e+g}{54(1-x^2)^2}-\frac{e+2g}{27(1-x^2)}-\frac{e+3g}{27(4-x^2)}-\frac{e+4g}{54(4-x^2)^2}-\frac{1}{81}(2e+5g)\log\left(1-x^2\right)+\frac{1}{81}(2e+5g)\log\left(4-x^2\right.\right.$$

$$\Big\downarrow \text{ 2206}$$

$$-\frac{1}{144}\int -\frac{-5(5d+8f+20h)x^2+19d-20f-32h}{(x^4-5x^2+4)^2}dx+$$
$$\frac{x\left(-\left(x^2(5d+8f+20h)\right)+17d+20f+32h\right)}{144(x^4-5x^2+4)^2}+$$
$$\frac{1}{2}\left(\frac{e+g}{54(1-x^2)^2}-\frac{e+2g}{27(1-x^2)}-\frac{e+3g}{27(4-x^2)}-\frac{e+4g}{54(4-x^2)^2}-\frac{1}{81}(2e+5g)\log\left(1-x^2\right)+\frac{1}{81}(2e+5g)\log\left(4-x^2\right.\right.$$

$$\Big\downarrow \text{ 25}$$

$$\frac{1}{144}\int \frac{-5(5d+8f+20h)x^2+19d-20f-32h}{(x^4-5x^2+4)^2}dx+$$
$$\frac{x\left(-\left(x^2(5d+8f+20h)\right)+17d+20f+32h\right)}{144(x^4-5x^2+4)^2}+$$
$$\frac{1}{2}\left(\frac{e+g}{54(1-x^2)^2}-\frac{e+2g}{27(1-x^2)}-\frac{e+3g}{27(4-x^2)}-\frac{e+4g}{54(4-x^2)^2}-\frac{1}{81}(2e+5g)\log\left(1-x^2\right)+\frac{1}{81}(2e+5g)\log\left(4-x^2\right.\right.$$

$$\Big\downarrow \text{ 1492}$$

3.45. $\int \frac{d+ex+fx^2+gx^3+hx^4}{(4-5x^2+x^4)^3}\,dx$

$$\frac{1}{144}\left(-\frac{1}{72}\int -\frac{3\left(5(7d+28f+64h)x^2+173d+260f+656h\right)}{x^4-5x^2+4}dx-\frac{x\left(-5x^2(7d+28f+64h)+59d+380f+84\right.}{24\left(x^4-5x^2+4\right)}\right.$$
$$\frac{x\left(-\left(x^2(5d+8f+20h)\right)+17d+20f+32h\right)}{144\left(x^4-5x^2+4\right)^2}+$$
$$\frac{1}{2}\left(\frac{e+g}{54\left(1-x^2\right)^2}-\frac{e+2g}{27\left(1-x^2\right)}-\frac{e+3g}{27\left(4-x^2\right)}-\frac{e+4g}{54\left(4-x^2\right)^2}-\frac{1}{81}(2e+5g)\log\left(1-x^2\right)+\frac{1}{81}(2e+5g)\log\left(4-x^2\right.\right.$$

$$\Big\downarrow \; 27$$

$$\frac{1}{144}\left(\frac{1}{24}\int \frac{5(7d+28f+64h)x^2+173d+260f+656h}{x^4-5x^2+4}dx-\frac{x\left(-5x^2(7d+28f+64h)+59d+380f+848h\right)}{24\left(x^4-5x^2+4\right)}\right)-$$
$$\frac{x\left(-\left(x^2(5d+8f+20h)\right)+17d+20f+32h\right)}{144\left(x^4-5x^2+4\right)^2}+$$
$$\frac{1}{2}\left(\frac{e+g}{54\left(1-x^2\right)^2}-\frac{e+2g}{27\left(1-x^2\right)}-\frac{e+3g}{27\left(4-x^2\right)}-\frac{e+4g}{54\left(4-x^2\right)^2}-\frac{1}{81}(2e+5g)\log\left(1-x^2\right)+\frac{1}{81}(2e+5g)\log\left(4-x^2\right.\right.$$

$$\Big\downarrow \; 1480$$

$$\frac{1}{144}\left(\frac{1}{24}\left(\frac{1}{3}(313d+820f+1936h)\int \frac{1}{x^2-4}dx-\frac{16}{3}(13d+25f+61h)\int \frac{1}{x^2-1}dx\right)-\frac{x\left(-5x^2(7d+28f+64h\right.}{24\left(x^4-\right.}\right.$$
$$\frac{x\left(-\left(x^2(5d+8f+20h)\right)+17d+20f+32h\right)}{144\left(x^4-5x^2+4\right)^2}+$$
$$\frac{1}{2}\left(\frac{e+g}{54\left(1-x^2\right)^2}-\frac{e+2g}{27\left(1-x^2\right)}-\frac{e+3g}{27\left(4-x^2\right)}-\frac{e+4g}{54\left(4-x^2\right)^2}-\frac{1}{81}(2e+5g)\log\left(1-x^2\right)+\frac{1}{81}(2e+5g)\log\left(4-x^2\right.\right.$$

$$\Big\downarrow \; 220$$

$$\frac{1}{144}\left(\frac{1}{24}\left(\frac{16}{3}\text{arctanh}(x)(13d+25f+61h)-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(313d+820f+1936h)\right)-\frac{x\left(-5x^2(7d+28f+64h\right)}{24\left(x^4-5\right.}\right.$$
$$\frac{x\left(-\left(x^2(5d+8f+20h)\right)+17d+20f+32h\right)}{144\left(x^4-5x^2+4\right)^2}+$$
$$\frac{1}{2}\left(\frac{e+g}{54\left(1-x^2\right)^2}-\frac{e+2g}{27\left(1-x^2\right)}-\frac{e+3g}{27\left(4-x^2\right)}-\frac{e+4g}{54\left(4-x^2\right)^2}-\frac{1}{81}(2e+5g)\log\left(1-x^2\right)+\frac{1}{81}(2e+5g)\log\left(4-x^2\right.\right.$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4)/(4 - 5*x^2 + x^4)^3,x]
```

3.45. $\int \frac{d+ex+fx^2+gx^3+hx^4}{\left(4-5x^2+x^4\right)^3}\,dx$

output
```
(x*(17*d + 20*f + 32*h - (5*d + 8*f + 20*h)*x^2))/(144*(4 - 5*x^2 + x^4)^2
) + (-1/24*(x*(59*d + 380*f + 848*h - 5*(7*d + 28*f + 64*h)*x^2))/(4 - 5*x
^2 + x^4) + (-1/6*((313*d + 820*f + 1936*h)*ArcTanh[x/2]) + (16*(13*d + 25
*f + 61*h)*ArcTanh[x])/3)/24)/144 + ((e + g)/(54*(1 - x^2)^2) - (e + 2*g)/
(27*(1 - x^2)) - (e + 4*g)/(54*(4 - x^2)^2) - (e + 3*g)/(27*(4 - x^2)) - (
(2*e + 5*g)*Log[1 - x^2])/81 + ((2*e + 5*g)*Log[4 - x^2])/81)/2
```

3.45.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 220
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[b, 2])^(-
1))*ArcTanh[Rt[b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] &&
 (LtQ[a, 0] || GtQ[b, 0])
```

rule 1141
```
Int[((d_.) + (e_.)*(x_))^(m_.)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_
Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[1/c^p   Int[ExpandIntegrand[
(d + e*x)^m*(b/2 - q/2 + c*x)^p*(b/2 + q/2 + c*x)^p, x], x], x] /; EqQ[p, -
1] ||  !FractionalPowerFactorQ[q]] /; FreeQ[{a, b, c, d, e}, x] && ILtQ[p,
0] && IntegerQ[m] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2206
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> With[{d =
 Coeff[PolynomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 0], e = Coeff[Poly
nomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 2]}, Simp[x*(a + b*x^2 + c*x^
4)^(p + 1)*((a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)/(2*a*(p + 1)*(b
^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(a + b*x^2 + c
*x^4)^(p + 1)*ExpandToSum[2*a*(p + 1)*(b^2 - 4*a*c)*PolynomialQuotient[Px,
a + b*x^2 + c*x^4, x] + b^2*d*(2*p + 3) - 2*a*c*d*(4*p + 5) - a*b*e + c*(4*
p + 7)*(b*d - 2*a*e)*x^2, x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x
^2] && Expon[Px, x^2] > 1 && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1]
```

3.45. $\int \frac{d+ex+fx^2+gx^3+hx^4}{(4-5x^2+x^4)^3}\,dx$

3.45.4 Maple [A] (verified)

Time = 0.12 (sec) , antiderivative size = 193, normalized size of antiderivative = 0.86

method	result
norman	$\frac{\left(-\frac{13d}{192}-\frac{5f}{16}-\frac{17h}{24}\right)x^5+\left(\frac{35d}{384}+\frac{21f}{32}+\frac{35h}{24}\right)x^3+\left(\frac{35d}{3456}+\frac{35f}{864}+\frac{5h}{54}\right)x^7+\left(\frac{43d}{864}-\frac{65f}{216}-\frac{41h}{54}\right)x+\left(-\frac{5e}{18}-\frac{25g}{36}\right)x^4+\left(\frac{5e}{9}+\frac{25g}{18}\right)x^2+\left(\frac{e}{27}\right.}{\left(x^4-5x^2+4\right)^2}$
risch	$\frac{\left(-\frac{13d}{192}-\frac{5f}{16}-\frac{17h}{24}\right)x^5+\left(\frac{35d}{384}+\frac{21f}{32}+\frac{35h}{24}\right)x^3+\left(\frac{35d}{3456}+\frac{35f}{864}+\frac{5h}{54}\right)x^7+\left(\frac{43d}{864}-\frac{65f}{216}-\frac{41h}{54}\right)x+\left(-\frac{5e}{18}-\frac{25g}{36}\right)x^4+\left(\frac{5e}{9}+\frac{25g}{18}\right)x^2+\left(\frac{e}{27}\right.}{\left(x^4-5x^2+4\right)^2}$
default	$\left(-\frac{313d}{41472}+\frac{e}{81}-\frac{205f}{10368}+\frac{5g}{162}-\frac{121h}{2592}\right)\ln\left(x+2\right)-\frac{-\frac{19d}{6912}+\frac{17e}{3456}-\frac{5f}{576}+\frac{13g}{864}-\frac{11h}{432}}{x+2}-\frac{-\frac{d}{1728}+\frac{e}{864}-\frac{f}{432}+\frac{g}{216}-}{2(x+2)^2}$
parallelrisch	Expression too large to display

input
```
int((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^3,x,method=_RETURNVERBOSE)
```

output
```
((-13/192*d-5/16*f-17/24*h)*x^5+(35/384*d+21/32*f+35/24*h)*x^3+(35/3456*d+
35/864*f+5/54*h)*x^7+(43/864*d-65/216*f-41/54*h)*x+(-5/18*e-25/36*g)*x^4+(
5/9*e+25/18*g)*x^2+(1/27*e+5/54*g)*x^6-25/108*e-19/27*g)/(x^4-5*x^2+4)^2+(
-313/41472*d+1/81*e-205/10368*f+5/162*g-121/2592*h)*ln(x+2)+(-13/1296*d-1/
81*e-25/1296*f-5/162*g-61/1296*h)*ln(x-1)+(13/1296*d-1/81*e+25/1296*f-5/16
2*g+61/1296*h)*ln(x+1)+(313/41472*d+1/81*e+205/10368*f+5/162*g+121/2592*h)
*ln(x-2)
```

3.45.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 544 vs. 2(204) = 408.

Time = 1.45 (sec) , antiderivative size = 544, normalized size of antiderivative = 2.43

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{\left(4-5x^2+x^4\right)^3}\,dx$$
$$= \frac{60\,(7\,d+28\,f+64\,h)x^7+768\,(2\,e+5\,g)x^6-216\,(13\,d+60\,f+136\,h)x^5-5760\,(2\,e+5\,g)x^4+756\,(5\,d}{}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^3,x, algorithm="fricas")
```

output
```
1/41472*(60*(7*d + 28*f + 64*h)*x^7 + 768*(2*e + 5*g)*x^6 - 216*(13*d + 60
*f + 136*h)*x^5 - 5760*(2*e + 5*g)*x^4 + 756*(5*d + 36*f + 80*h)*x^3 + 115
20*(2*e + 5*g)*x^2 + 48*(43*d - 260*f - 656*h)*x - ((313*d - 512*e + 820*f
 - 1280*g + 1936*h)*x^8 - 10*(313*d - 512*e + 820*f - 1280*g + 1936*h)*x^6
 + 33*(313*d - 512*e + 820*f - 1280*g + 1936*h)*x^4 - 40*(313*d - 512*e +
820*f - 1280*g + 1936*h)*x^2 + 5008*d - 8192*e + 13120*f - 20480*g + 30976
*h)*log(x + 2) + 32*((13*d - 16*e + 25*f - 40*g + 61*h)*x^8 - 10*(13*d - 1
6*e + 25*f - 40*g + 61*h)*x^6 + 33*(13*d - 16*e + 25*f - 40*g + 61*h)*x^4
- 40*(13*d - 16*e + 25*f - 40*g + 61*h)*x^2 + 208*d - 256*e + 400*f - 640*
g + 976*h)*log(x + 1) - 32*((13*d + 16*e + 25*f + 40*g + 61*h)*x^8 - 10*(1
3*d + 16*e + 25*f + 40*g + 61*h)*x^6 + 33*(13*d + 16*e + 25*f + 40*g + 61*
h)*x^4 - 40*(13*d + 16*e + 25*f + 40*g + 61*h)*x^2 + 208*d + 256*e + 400*f
 + 640*g + 976*h)*log(x - 1) + ((313*d + 512*e + 820*f + 1280*g + 1936*h)*
x^8 - 10*(313*d + 512*e + 820*f + 1280*g + 1936*h)*x^6 + 33*(313*d + 512*e
 + 820*f + 1280*g + 1936*h)*x^4 - 40*(313*d + 512*e + 820*f + 1280*g + 193
6*h)*x^2 + 5008*d + 8192*e + 13120*f + 20480*g + 30976*h)*log(x - 2) - 960
0*e - 29184*g)/(x^8 - 10*x^6 + 33*x^4 - 40*x^2 + 16)
```

3.45.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4}{(4 - 5x^2 + x^4)^3} \, dx = \text{Timed out}$$

input
```
integrate((h*x**4+g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4)**3,x)
```

output
```
Timed out
```

3.45.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 214, normalized size of antiderivative = 0.96

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(4-5x^2+x^4)^3}\,dx$$
$$= -\frac{1}{41472}(313\,d-512\,e+820\,f-1280\,g+1936\,h)\log(x+2)$$
$$+\frac{1}{1296}(13\,d-16\,e+25\,f-40\,g+61\,h)\log(x+1)$$
$$-\frac{1}{1296}(13\,d+16\,e+25\,f+40\,g+61\,h)\log(x-1)$$
$$+\frac{1}{41472}(313\,d+512\,e+820\,f+1280\,g+1936\,h)\log(x-2)$$
$$+\frac{5\,(7\,d+28\,f+64\,h)x^7+64\,(2\,e+5\,g)x^6-18\,(13\,d+60\,f+136\,h)x^5-480\,(2\,e+5\,g)x^4+63\,(5\,d+3}{3456\,(x^8-10\,x^6+33\,x^4-40\,x^2}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^3,x, algorithm="maxima")
```

output
```
-1/41472*(313*d - 512*e + 820*f - 1280*g + 1936*h)*log(x + 2) + 1/1296*(13
*d - 16*e + 25*f - 40*g + 61*h)*log(x + 1) - 1/1296*(13*d + 16*e + 25*f +
40*g + 61*h)*log(x - 1) + 1/41472*(313*d + 512*e + 820*f + 1280*g + 1936*h
)*log(x - 2) + 1/3456*(5*(7*d + 28*f + 64*h)*x^7 + 64*(2*e + 5*g)*x^6 - 18
*(13*d + 60*f + 136*h)*x^5 - 480*(2*e + 5*g)*x^4 + 63*(5*d + 36*f + 80*h)*
x^3 + 960*(2*e + 5*g)*x^2 + 4*(43*d - 260*f - 656*h)*x - 800*e - 2432*g)/(
x^8 - 10*x^6 + 33*x^4 - 40*x^2 + 16)
```

3.45.8 Giac [A] (verification not implemented)

Time = 0.36 (sec) , antiderivative size = 216, normalized size of antiderivative = 0.96

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(4-5x^2+x^4)^3}\,dx$$
$$= -\frac{1}{41472}(313\,d-512\,e+820\,f-1280\,g+1936\,h)\log(|x+2|)$$
$$+\frac{1}{1296}(13\,d-16\,e+25\,f-40\,g+61\,h)\log(|x+1|)$$
$$-\frac{1}{1296}(13\,d+16\,e+25\,f+40\,g+61\,h)\log(|x-1|)$$
$$+\frac{1}{41472}(313\,d+512\,e+820\,f+1280\,g+1936\,h)\log(|x-2|)$$
$$+\frac{35\,dx^7+140\,fx^7+320\,hx^7+128\,ex^6+320\,gx^6-234\,dx^5-1080\,fx^5-2448\,hx^5-960\,ex^4-2400\,gx}{3456\,(x^4}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^3,x, algorithm="giac")
```

output
```
-1/41472*(313*d - 512*e + 820*f - 1280*g + 1936*h)*log(abs(x + 2)) + 1/129
6*(13*d - 16*e + 25*f - 40*g + 61*h)*log(abs(x + 1)) - 1/1296*(13*d + 16*e
 + 25*f + 40*g + 61*h)*log(abs(x - 1)) + 1/41472*(313*d + 512*e + 820*f +
1280*g + 1936*h)*log(abs(x - 2)) + 1/3456*(35*d*x^7 + 140*f*x^7 + 320*h*x^
7 + 128*e*x^6 + 320*g*x^6 - 234*d*x^5 - 1080*f*x^5 - 2448*h*x^5 - 960*e*x^
4 - 2400*g*x^4 + 315*d*x^3 + 2268*f*x^3 + 5040*h*x^3 + 1920*e*x^2 + 4800*g
*x^2 + 172*d*x - 1040*f*x - 2624*h*x - 800*e - 2432*g)/(x^4 - 5*x^2 + 4)^2
```

3.45.9 Mupad [B] (verification not implemented)

Time = 0.15 (sec) , antiderivative size = 209, normalized size of antiderivative = 0.93

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(4-5x^2+x^4)^3}\,dx = \ln(x+1)\left(\frac{13\,d}{1296}-\frac{e}{81}+\frac{25\,f}{1296}-\frac{5\,g}{162}+\frac{61\,h}{1296}\right)$$
$$-\ln(x-1)\left(\frac{13\,d}{1296}+\frac{e}{81}+\frac{25\,f}{1296}+\frac{5\,g}{162}+\frac{61\,h}{1296}\right)$$
$$-\frac{\left(-\frac{35\,d}{3456}-\frac{35\,f}{864}-\frac{5\,h}{54}\right)x^7+\left(-\frac{e}{27}-\frac{5\,g}{54}\right)x^6+\left(\frac{13\,d}{192}+\frac{5\,f}{16}+\frac{17\,h}{24}\right)x^5+\left(\frac{5\,e}{18}+\frac{25\,g}{36}\right)x^4+\left(-\frac{35\,d}{384}-\frac{21\,f}{32}-\frac{35\,h}{24}\right.}{x^8-10\,x^6+33\,x^4-40\,x^2+16}$$
$$+\ln(x-2)\left(\frac{313\,d}{41472}+\frac{e}{81}+\frac{205\,f}{10368}+\frac{5\,g}{162}+\frac{121\,h}{2592}\right)$$
$$-\ln(x+2)\left(\frac{313\,d}{41472}-\frac{e}{81}+\frac{205\,f}{10368}-\frac{5\,g}{162}+\frac{121\,h}{2592}\right)$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4)/(x^4 - 5*x^2 + 4)^3,x)
```

output
```
log(x + 1)*((13*d)/1296 - e/81 + (25*f)/1296 - (5*g)/162 + (61*h)/1296) -
log(x - 1)*((13*d)/1296 + e/81 + (25*f)/1296 + (5*g)/162 + (61*h)/1296) -
((25*e)/108 + (19*g)/27 - x^2*((5*e)/9 + (25*g)/18) + x^4*((5*e)/18 + (25*
g)/36) - x^6*(e/27 + (5*g)/54) + x*((65*f)/216 - (43*d)/864 + (41*h)/54) +
 x^5*((13*d)/192 + (5*f)/16 + (17*h)/24) - x^3*((35*d)/384 + (21*f)/32 + (
35*h)/24) - x^7*((35*d)/3456 + (35*f)/864 + (5*h)/54))/(33*x^4 - 40*x^2 -
10*x^6 + x^8 + 16) + log(x - 2)*((313*d)/41472 + e/81 + (205*f)/10368 + (5
*g)/162 + (121*h)/2592) - log(x + 2)*((313*d)/41472 - e/81 + (205*f)/10368
 - (5*g)/162 + (121*h)/2592)
```

3.46 $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(4-5x^2+x^4)^3}\,dx$

3.46.1 Optimal result

Integrand size = 38, antiderivative size = 239

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(4-5x^2+x^4)^3}\,dx &= \frac{x(17d+20f+32h-(5d+8f+20h)x^2)}{144\,(4-5x^2+x^4)^2} \\ &+ \frac{5e+8g+20i-(2e+5g+17i)x^2}{36\,(4-5x^2+x^4)^2} \\ &- \frac{(2e+5g+11i)\,(5-2x^2)}{108\,(4-5x^2+x^4)} \\ &- \frac{x(59d+380f+848h-5(7d+28f+64h)x^2)}{3456\,(4-5x^2+x^4)} \\ &- \frac{(313d+820f+1936h)\text{arctanh}\left(\frac{x}{2}\right)}{20736} \\ &+ \frac{1}{648}(13d+25f+61h)\text{arctanh}(x) \\ &- \frac{1}{162}(2e+5g+11i)\log\left(1-x^2\right) \\ &+ \frac{1}{162}(2e+5g+11i)\log\left(4-x^2\right)\end{aligned}$$

output
```
1/144*x*(17*d+20*f+32*h-(5*d+8*f+20*h)*x^2)/(x^4-5*x^2+4)^2+1/36*(5*e+8*g+
20*i-(2*e+5*g+17*i)*x^2)/(x^4-5*x^2+4)^2-1/108*(2*e+5*g+11*i)*(-2*x^2+5)/(
x^4-5*x^2+4)-1/3456*x*(59*d+380*f+848*h-5*(7*d+28*f+64*h)*x^2)/(x^4-5*x^2+
4)-1/20736*(313*d+820*f+1936*h)*arctanh(1/2*x)+1/648*(13*d+25*f+61*h)*arct
anh(x)-1/162*(2*e+5*g+11*i)*ln(-x^2+1)+1/162*(2*e+5*g+11*i)*ln(-x^2+4)
```

3.46.2 Mathematica [A] (verified)

Time = 0.09 (sec) , antiderivative size = 261, normalized size of antiderivative = 1.09

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(4-5x^2+x^4)^3}\,dx$$
$$= \frac{20e+32g+80i+17dx+20fx+32hx-8ex^2-20gx^2-68ix^2-5dx^3-8fx^3-20hx^3}{144(4-5x^2+x^4)^2}$$
$$+\frac{-320e-800g-1760i-59dx-380fx-848hx+128ex^2+320gx^2+704ix^2+35dx^3+140fx^3+32}{3456(4-5x^2+x^4)}$$
$$+\frac{(-13d-16e-25f-40g-61h-88i)\log(1-x)}{1296}$$
$$+\frac{(313d+512e+820f+1280g+1936h+2816i)\log(2-x)}{41472}$$
$$+\frac{(13d-16e+25f-40g+61h-88i)\log(1+x)}{1296}$$
$$+\frac{(-313d+512e-820f+1280g-1936h+2816i)\log(2+x)}{41472}$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(4 - 5*x^2 + x^4)^3,x]
```

output
```
(20*e + 32*g + 80*i + 17*d*x + 20*f*x + 32*h*x - 8*e*x^2 - 20*g*x^2 - 68*i
*x^2 - 5*d*x^3 - 8*f*x^3 - 20*h*x^3)/(144*(4 - 5*x^2 + x^4)^2) + (-320*e -
 800*g - 1760*i - 59*d*x - 380*f*x - 848*h*x + 128*e*x^2 + 320*g*x^2 + 704
*i*x^2 + 35*d*x^3 + 140*f*x^3 + 320*h*x^3)/(3456*(4 - 5*x^2 + x^4)) + ((-1
3*d - 16*e - 25*f - 40*g - 61*h - 88*i)*Log[1 - x])/1296 + ((313*d + 512*e
 + 820*f + 1280*g + 1936*h + 2816*i)*Log[2 - x])/41472 + ((13*d - 16*e + 2
5*f - 40*g + 61*h - 88*i)*Log[1 + x])/1296 + ((-313*d + 512*e - 820*f + 12
80*g - 1936*h + 2816*i)*Log[2 + x])/41472
```

3.46.3 Rubi [A] (verified)

Time = 0.68 (sec) , antiderivative size = 242, normalized size of antiderivative = 1.01, number of steps used = 13, number of rules used = 12, $\frac{\text{number of rules}}{\text{integrand size}} = 0.316$, Rules used = {2202, 2194, 2191, 27, 1084, 2009, 2206, 25, 1492, 27, 1480, 220}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(x^4-5x^2+4)^3}\,dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{hx^4+fx^2+d}{(x^4-5x^2+4)^3}dx+\int \frac{x\left(ix^4+gx^2+e\right)}{(x^4-5x^2+4)^3}dx$$

$$\Big\downarrow \text{2194}$$

$$\int \frac{hx^4+fx^2+d}{(x^4-5x^2+4)^3}dx+\frac{1}{2}\int \frac{ix^4+gx^2+e}{(x^4-5x^2+4)^3}dx^2$$

$$\Big\downarrow \text{2191}$$

$$\int \frac{hx^4+fx^2+d}{(x^4-5x^2+4)^3}dx+\frac{1}{2}\left(\frac{-\left(x^2(2e+5g+17i)\right)+5e+8g+20i}{18\left(x^4-5x^2+4\right)^2}-\frac{1}{18}\int \frac{3(2e+5g+11i)}{(x^4-5x^2+4)^2}dx^2\right)$$

$$\Big\downarrow \text{27}$$

$$\int \frac{hx^4+fx^2+d}{(x^4-5x^2+4)^3}dx+$$
$$\frac{1}{2}\left(\frac{-\left(x^2(2e+5g+17i)\right)+5e+8g+20i}{18\left(x^4-5x^2+4\right)^2}-\frac{1}{6}(2e+5g+11i)\int \frac{1}{(x^4-5x^2+4)^2}dx^2\right)$$

$$\Big\downarrow \text{1084}$$

$$\int \frac{hx^4+fx^2+d}{(x^4-5x^2+4)^3}dx+$$
$$\frac{1}{2}\left(\frac{-\left(x^2(2e+5g+17i)\right)+5e+8g+20i}{18\left(x^4-5x^2+4\right)^2}-\frac{1}{6}(2e+5g+11i)\int \left(\frac{2}{27\left(4-x^2\right)}+\frac{1}{9\left(4-x^2\right)^2}-\frac{2}{27\left(1-x^2\right)}+\frac{}{9(1}\right.\right.$$

$$\Big\downarrow \text{2009}$$

$$\int \frac{hx^4+fx^2+d}{(x^4-5x^2+4)^3}dx+$$
$$\frac{1}{2}\left(\frac{-\left(x^2(2e+5g+17i)\right)+5e+8g+20i}{18\left(x^4-5x^2+4\right)^2}-\frac{1}{6}\left(\frac{1}{9\left(1-x^2\right)}+\frac{1}{9\left(4-x^2\right)}+\frac{2}{27}\log\left(1-x^2\right)-\frac{2}{27}\log\left(4-x^2\right)\right)(2e\right.$$

$$\Big\downarrow \text{2206}$$

$$-\frac{1}{144}\int -\frac{-5(5d+8f+20h)x^2+19d-20f-32h}{(x^4-5x^2+4)^2}dx+$$
$$\frac{x\left(-\left(x^2(5d+8f+20h)\right)+17d+20f+32h\right)}{144\left(x^4-5x^2+4\right)^2}+$$
$$\frac{1}{2}\left(\frac{-\left(x^2(2e+5g+17i)\right)+5e+8g+20i}{18\left(x^4-5x^2+4\right)^2}-\frac{1}{6}\left(\frac{1}{9\left(1-x^2\right)}+\frac{1}{9\left(4-x^2\right)}+\frac{2}{27}\log\left(1-x^2\right)-\frac{2}{27}\log\left(4-x^2\right)\right)(2e\right.$$

3.46. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(4-5x^2+x^4)^3}\,dx$

$$\Big\downarrow 25$$

$$\frac{1}{144}\int \frac{-5(5d+8f+20h)x^2+19d-20f-32h}{\left(x^4-5x^2+4\right)^2}dx+\frac{x\left(-\left(x^2(5d+8f+20h)\right)+17d+20f+32h\right)}{144\left(x^4-5x^2+4\right)^2}+\frac{1}{2}\left(\frac{-\left(x^2(2e+5g+17i)\right)+5e+8g+20i}{18\left(x^4-5x^2+4\right)^2}-\frac{1}{6}\left(\frac{1}{9\left(1-x^2\right)}+\frac{1}{9\left(4-x^2\right)}+\frac{2}{27}\log\left(1-x^2\right)-\frac{2}{27}\log\left(4-x^2\right)\right)(2e$$

$$\Big\downarrow 1492$$

$$\frac{1}{144}\left(-\frac{1}{72}\int -\frac{3\left(5(7d+28f+64h)x^2+173d+260f+656h\right)}{x^4-5x^2+4}dx-\frac{x\left(-5x^2(7d+28f+64h)+59d+380f+84\right.}{24\left(x^4-5x^2+4\right)}\right.$$
$$\frac{x\left(-\left(x^2(5d+8f+20h)\right)+17d+20f+32h\right)}{144\left(x^4-5x^2+4\right)^2}+\frac{1}{2}\left(\frac{-\left(x^2(2e+5g+17i)\right)+5e+8g+20i}{18\left(x^4-5x^2+4\right)^2}-\frac{1}{6}\left(\frac{1}{9\left(1-x^2\right)}+\frac{1}{9\left(4-x^2\right)}+\frac{2}{27}\log\left(1-x^2\right)-\frac{2}{27}\log\left(4-x^2\right)\right)(2e$$

$$\Big\downarrow 27$$

$$\frac{1}{144}\left(\frac{1}{24}\int \frac{5(7d+28f+64h)x^2+173d+260f+656h}{x^4-5x^2+4}dx-\frac{x\left(-5x^2(7d+28f+64h)+59d+380f+848h\right)}{24\left(x^4-5x^2+4\right)}\right)-$$
$$\frac{x\left(-\left(x^2(5d+8f+20h)\right)+17d+20f+32h\right)}{144\left(x^4-5x^2+4\right)^2}+\frac{1}{2}\left(\frac{-\left(x^2(2e+5g+17i)\right)+5e+8g+20i}{18\left(x^4-5x^2+4\right)^2}-\frac{1}{6}\left(\frac{1}{9\left(1-x^2\right)}+\frac{1}{9\left(4-x^2\right)}+\frac{2}{27}\log\left(1-x^2\right)-\frac{2}{27}\log\left(4-x^2\right)\right)(2e$$

$$\Big\downarrow 1480$$

$$\frac{1}{144}\left(\frac{1}{24}\left(\frac{1}{3}(313d+820f+1936h)\int \frac{1}{x^2-4}dx-\frac{16}{3}(13d+25f+61h)\int \frac{1}{x^2-1}dx\right)-\frac{x\left(-5x^2(7d+28f+64\right.}{24\left(x^4-\right.}\right.$$
$$\frac{x\left(-\left(x^2(5d+8f+20h)\right)+17d+20f+32h\right)}{144\left(x^4-5x^2+4\right)^2}+\frac{1}{2}\left(\frac{-\left(x^2(2e+5g+17i)\right)+5e+8g+20i}{18\left(x^4-5x^2+4\right)^2}-\frac{1}{6}\left(\frac{1}{9\left(1-x^2\right)}+\frac{1}{9\left(4-x^2\right)}+\frac{2}{27}\log\left(1-x^2\right)-\frac{2}{27}\log\left(4-x^2\right)\right)(2e$$

$$\Big\downarrow 220$$

3.46. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{\left(4-5x^2+x^4\right)^3}\,dx$

$$\frac{1}{144}\left(\frac{1}{24}\left(\frac{16}{3}\text{arctanh}(x)(13d+25f+61h)-\frac{1}{6}\text{arctanh}\left(\frac{x}{2}\right)(313d+820f+1936h)\right)-\frac{x(-5x^2(7d+28f+64h)}{24(x^4-5}\right.$$

$$\frac{x\left(-\left(x^2(5d+8f+20h)\right)+17d+20f+32h\right)}{144\left(x^4-5x^2+4\right)^2}+$$

$$\frac{1}{2}\left(\frac{-\left(x^2(2e+5g+17i)\right)+5e+8g+20i}{18\left(x^4-5x^2+4\right)^2}-\frac{1}{6}\left(\frac{1}{9(1-x^2)}+\frac{1}{9(4-x^2)}+\frac{2}{27}\log\left(1-x^2\right)-\frac{2}{27}\log\left(4-x^2\right)\right)(2e\right.$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(4 - 5*x^2 + x^4)^3,x]
```

output
```
(x*(17*d + 20*f + 32*h - (5*d + 8*f + 20*h)*x^2))/(144*(4 - 5*x^2 + x^4)^2
) + (-1/24*(x*(59*d + 380*f + 848*h - 5*(7*d + 28*f + 64*h)*x^2))/(4 - 5*x
^2 + x^4) + (-1/6*((313*d + 820*f + 1936*h)*ArcTanh[x/2]) + (16*(13*d + 25
*f + 61*h)*ArcTanh[x])/3)/24)/144 + ((5*e + 8*g + 20*i - (2*e + 5*g + 17*i
)*x^2)/(18*(4 - 5*x^2 + x^4)^2) - ((2*e + 5*g + 11*i)*(1/(9*(1 - x^2)) + 1
/(9*(4 - x^2)) + (2*Log[1 - x^2])/27 - (2*Log[4 - x^2])/27))/6)/2
```

3.46.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 220
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[b, 2])^(-
1))*ArcTanh[Rt[b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] &&
 (LtQ[a, 0] || GtQ[b, 0])
```

rule 1084
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> With[{q = Rt[b^2
- 4*a*c, 2]}, Simp[1/c^p   Int[ExpandIntegrand[(b/2 - q/2 + c*x)^p*(b/2 + q
/2 + c*x)^p, x], x], x] /;  !FractionalPowerFactorQ[q]] /; FreeQ[{a, b, c},
 x] && IntegerQ[p] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2191
```
Int[(Pq_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> With[{Q =
PolynomialQuotient[Pq, a + b*x + c*x^2, x], f = Coeff[PolynomialRemainder[P
q, a + b*x + c*x^2, x], x, 0], g = Coeff[PolynomialRemainder[Pq, a + b*x +
c*x^2, x], x, 1]}, Simp[(b*f - 2*a*g + (2*c*f - b*g)*x)*((a + b*x + c*x^2)^
(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] + Simp[1/((p + 1)*(b^2 - 4*a*c))   Int
[(a + b*x + c*x^2)^(p + 1)*ExpandToSum[(p + 1)*(b^2 - 4*a*c)*Q - (2*p + 3)*
(2*c*f - b*g), x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Pq, x] && NeQ[b^
2 - 4*a*c, 0] && LtQ[p, -1]
```

rule 2194
```
Int[(Pq_)*(x_)^(m_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :
> Simp[1/2   Subst[Int[x^((m - 1)/2)*SubstFor[x^2, Pq, x]*(a + b*x + c*x^2)
^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pq, x^2] && IntegerQ
[(m - 1)/2]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.46. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(4-5x^2+x^4)^3}\,dx$

rule 2206
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> With[{d =
 Coeff[PolynomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 0], e = Coeff[Poly
nomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 2]}, Simp[x*(a + b*x^2 + c*x^
4)^(p + 1)*((a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)/(2*a*(p + 1)*(b
^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(a + b*x^2 + c
*x^4)^(p + 1)*ExpandToSum[2*a*(p + 1)*(b^2 - 4*a*c)*PolynomialQuotient[Px,
a + b*x^2 + c*x^4, x] + b^2*d*(2*p + 3) - 2*a*c*d*(4*p + 5) - a*b*e + c*(4*
p + 7)*(b*d - 2*a*e)*x^2, x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x
^2] && Expon[Px, x^2] > 1 && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1]
```

3.46.4 Maple [A] (verified)

Time = 0.13 (sec) , antiderivative size = 217, normalized size of antiderivative = 0.91

method	result
norman	$\frac{\left(-\frac{13d}{192}-\frac{5f}{16}-\frac{17h}{24}\right)x^5+\left(\frac{35d}{384}+\frac{21f}{32}+\frac{35h}{24}\right)x^3+\left(\frac{35d}{3456}+\frac{35f}{864}+\frac{5h}{54}\right)x^7+\left(\frac{43d}{864}-\frac{65f}{216}-\frac{41h}{54}\right)x+\left(-\frac{5e}{18}-\frac{25g}{36}-\frac{55i}{36}\right)x^4+\left(\frac{5e}{9}+\frac{25g}{18}+\frac{26}{9}\right.}{\left(x^4-5x^2+4\right)^2}$
default	$\left(-\frac{313d}{41472}+\frac{e}{81}-\frac{205f}{10368}+\frac{5g}{162}-\frac{121h}{2592}+\frac{11i}{162}\right)\ln\left(x+2\right)-\frac{-\frac{19d}{6912}+\frac{17e}{3456}-\frac{5f}{576}+\frac{13g}{864}-\frac{11h}{432}+\frac{i}{24}}{x+2}-\frac{-\frac{d}{1728}+\frac{e}{864}}{}$
risch	$-\frac{11\ln(x+1)i}{162}-\frac{205\ln(x+2)f}{10368}-\frac{25\ln(1-x)f}{1296}+\frac{25\ln(x+1)f}{1296}+\frac{205\ln(2-x)f}{10368}+\frac{313\ln(2-x)d}{41472}+\frac{\ln(2-x)e}{81}-\frac{313\ln}{4}$
parallelrisch	Expression too large to display

input
```
int((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^3,x,method=_RETURNVERBOS
E)
```

output
```
((-13/192*d-5/16*f-17/24*h)*x^5+(35/384*d+21/32*f+35/24*h)*x^3+(35/3456*d+
35/864*f+5/54*h)*x^7+(43/864*d-65/216*f-41/54*h)*x+(-5/18*e-25/36*g-55/36*
i)*x^4+(5/9*e+25/18*g+26/9*i)*x^2+(1/27*e+5/54*g+11/54*i)*x^6-25/108*e-19/
27*g-40/27*i)/(x^4-5*x^2+4)^2+(-313/41472*d+1/81*e-205/10368*f+5/162*g-121
/2592*h+11/162*i)*ln(x+2)+(-13/1296*d-1/81*e-25/1296*f-5/162*g-61/1296*h-1
1/162*i)*ln(x-1)+(13/1296*d-1/81*e+25/1296*f-5/162*g+61/1296*h-11/162*i)*l
n(x+1)+(313/41472*d+1/81*e+205/10368*f+5/162*g+121/2592*h+11/162*i)*ln(x-2
)
```

3.46.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 616 vs. $2(219) = 438$.

Time = 6.60 (sec) , antiderivative size = 616, normalized size of antiderivative = 2.58

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(4-5x^2+x^4)^3}\,dx$$
$$= \frac{60\,(7\,d+28\,f+64\,h)x^7+768\,(2\,e+5\,g+11\,i)x^6-216\,(13\,d+60\,f+136\,h)x^5-5760\,(2\,e+5\,g+11\,i)x}{}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^3,x, algorithm="fr
icas")
```

output
```
1/41472*(60*(7*d + 28*f + 64*h)*x^7 + 768*(2*e + 5*g + 11*i)*x^6 - 216*(13
*d + 60*f + 136*h)*x^5 - 5760*(2*e + 5*g + 11*i)*x^4 + 756*(5*d + 36*f + 8
0*h)*x^3 + 2304*(10*e + 25*g + 52*i)*x^2 + 48*(43*d - 260*f - 656*h)*x - (
(313*d - 512*e + 820*f - 1280*g + 1936*h - 2816*i)*x^8 - 10*(313*d - 512*e
 + 820*f - 1280*g + 1936*h - 2816*i)*x^6 + 33*(313*d - 512*e + 820*f - 128
0*g + 1936*h - 2816*i)*x^4 - 40*(313*d - 512*e + 820*f - 1280*g + 1936*h -
 2816*i)*x^2 + 5008*d - 8192*e + 13120*f - 20480*g + 30976*h - 45056*i)*lo
g(x + 2) + 32*((13*d - 16*e + 25*f - 40*g + 61*h - 88*i)*x^8 - 10*(13*d -
16*e + 25*f - 40*g + 61*h - 88*i)*x^6 + 33*(13*d - 16*e + 25*f - 40*g + 61
*h - 88*i)*x^4 - 40*(13*d - 16*e + 25*f - 40*g + 61*h - 88*i)*x^2 + 208*d
- 256*e + 400*f - 640*g + 976*h - 1408*i)*log(x + 1) - 32*((13*d + 16*e +
25*f + 40*g + 61*h + 88*i)*x^8 - 10*(13*d + 16*e + 25*f + 40*g + 61*h + 88
*i)*x^6 + 33*(13*d + 16*e + 25*f + 40*g + 61*h + 88*i)*x^4 - 40*(13*d + 16
*e + 25*f + 40*g + 61*h + 88*i)*x^2 + 208*d + 256*e + 400*f + 640*g + 976*
h + 1408*i)*log(x - 1) + ((313*d + 512*e + 820*f + 1280*g + 1936*h + 2816*
i)*x^8 - 10*(313*d + 512*e + 820*f + 1280*g + 1936*h + 2816*i)*x^6 + 33*(3
13*d + 512*e + 820*f + 1280*g + 1936*h + 2816*i)*x^4 - 40*(313*d + 512*e +
 820*f + 1280*g + 1936*h + 2816*i)*x^2 + 5008*d + 8192*e + 13120*f + 20480
*g + 30976*h + 45056*i)*log(x - 2) - 9600*e - 29184*g - 61440*i)/(x^8 - 10
*x^6 + 33*x^4 - 40*x^2 + 16)
```

3.46.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(4-5x^2+x^4)^3}\,dx=\text{Timed out}$$

input
```
integrate((i*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4)**3,x)
```

output
```
Timed out
```

3.46.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 238, normalized size of antiderivative = 1.00

$$\begin{aligned}
&\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(4-5x^2+x^4)^3}\,dx \\
&= -\frac{1}{41472}\,(313\,d-512\,e+820\,f-1280\,g+1936\,h-2816\,i)\log\left(x+2\right) \\
&\quad+\frac{1}{1296}\,(13\,d-16\,e+25\,f-40\,g+61\,h-88\,i)\log\left(x+1\right) \\
&\quad-\frac{1}{1296}\,(13\,d+16\,e+25\,f+40\,g+61\,h+88\,i)\log\left(x-1\right) \\
&\quad+\frac{1}{41472}\,(313\,d+512\,e+820\,f+1280\,g+1936\,h+2816\,i)\log\left(x-2\right) \\
&\quad+\frac{5\,(7\,d+28\,f+64\,h)x^7+64\,(2\,e+5\,g+11\,i)x^6-18\,(13\,d+60\,f+136\,h)x^5-480\,(2\,e+5\,g+11\,i)x^4}{3456\,(x^8-10\,x^6}
\end{aligned}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^3,x, algorithm="ma
xima")
```

output
```
-1/41472*(313*d - 512*e + 820*f - 1280*g + 1936*h - 2816*i)*log(x + 2) + 1
/1296*(13*d - 16*e + 25*f - 40*g + 61*h - 88*i)*log(x + 1) - 1/1296*(13*d
+ 16*e + 25*f + 40*g + 61*h + 88*i)*log(x - 1) + 1/41472*(313*d + 512*e +
820*f + 1280*g + 1936*h + 2816*i)*log(x - 2) + 1/3456*(5*(7*d + 28*f + 64*
h)*x^7 + 64*(2*e + 5*g + 11*i)*x^6 - 18*(13*d + 60*f + 136*h)*x^5 - 480*(2
*e + 5*g + 11*i)*x^4 + 63*(5*d + 36*f + 80*h)*x^3 + 192*(10*e + 25*g + 52*
i)*x^2 + 4*(43*d - 260*f - 656*h)*x - 800*e - 2432*g - 5120*i)/(x^8 - 10*x
^6 + 33*x^4 - 40*x^2 + 16)
```

3.46.8 Giac [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 249, normalized size of antiderivative = 1.04

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(4-5x^2+x^4)^3}\,dx$$
$$= -\frac{1}{41472}(313\,d-512\,e+820\,f-1280\,g+1936\,h-2816\,i)\log(|x+2|)$$
$$+\frac{1}{1296}(13\,d-16\,e+25\,f-40\,g+61\,h-88\,i)\log(|x+1|)$$
$$-\frac{1}{1296}(13\,d+16\,e+25\,f+40\,g+61\,h+88\,i)\log(|x-1|)$$
$$+\frac{1}{41472}(313\,d+512\,e+820\,f+1280\,g+1936\,h+2816\,i)\log(|x-2|)$$
$$+\frac{35\,dx^7+140\,fx^7+320\,hx^7+128\,ex^6+320\,gx^6+704\,ix^6-234\,dx^5-1080\,fx^5-2448\,hx^5-960\,ex^4}{}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^3,x, algorithm="gi
ac")
```

output
```
-1/41472*(313*d - 512*e + 820*f - 1280*g + 1936*h - 2816*i)*log(abs(x + 2)
) + 1/1296*(13*d - 16*e + 25*f - 40*g + 61*h - 88*i)*log(abs(x + 1)) - 1/1
296*(13*d + 16*e + 25*f + 40*g + 61*h + 88*i)*log(abs(x - 1)) + 1/41472*(3
13*d + 512*e + 820*f + 1280*g + 1936*h + 2816*i)*log(abs(x - 2)) + 1/3456*
(35*d*x^7 + 140*f*x^7 + 320*h*x^7 + 128*e*x^6 + 320*g*x^6 + 704*i*x^6 - 23
4*d*x^5 - 1080*f*x^5 - 2448*h*x^5 - 960*e*x^4 - 2400*g*x^4 - 5280*i*x^4 +
315*d*x^3 + 2268*f*x^3 + 5040*h*x^3 + 1920*e*x^2 + 4800*g*x^2 + 9984*i*x^2
 + 172*d*x - 1040*f*x - 2624*h*x - 800*e - 2432*g - 5120*i)/(x^4 - 5*x^2 +
 4)^2
```

3.46.9 Mupad [B] (verification not implemented)

Time = 0.37 (sec) , antiderivative size = 233, normalized size of antiderivative = 0.97

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(4-5x^2+x^4)^3}\,dx = \ln(x+1)\left(\frac{13\,d}{1296}-\frac{e}{81}+\frac{25\,f}{1296}-\frac{5\,g}{162}+\frac{61\,h}{1296}-\frac{11\,i}{162}\right)$$
$$-\ln(x-1)\left(\frac{13\,d}{1296}+\frac{e}{81}+\frac{25\,f}{1296}+\frac{5\,g}{162}+\frac{61\,h}{1296}+\frac{11\,i}{162}\right)$$
$$-\frac{\left(-\frac{35\,d}{3456}-\frac{35\,f}{864}-\frac{5\,h}{54}\right)x^7+\left(-\frac{e}{27}-\frac{5\,g}{54}-\frac{11\,i}{54}\right)x^6+\left(\frac{13\,d}{192}+\frac{5\,f}{16}+\frac{17\,h}{24}\right)x^5+\left(\frac{5\,e}{18}+\frac{25\,g}{36}+\frac{55\,i}{36}\right)x^4+\left(-\frac{35\,d}{384}\right.}{x^8-10\,x^6+33\,x^4-40\,x}$$
$$+\ln(x-2)\left(\frac{313\,d}{41472}+\frac{e}{81}+\frac{205\,f}{10368}+\frac{5\,g}{162}+\frac{121\,h}{2592}+\frac{11\,i}{162}\right)$$
$$-\ln(x+2)\left(\frac{313\,d}{41472}-\frac{e}{81}+\frac{205\,f}{10368}-\frac{5\,g}{162}+\frac{121\,h}{2592}-\frac{11\,i}{162}\right)$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(x^4 - 5*x^2 + 4)^3,x)
```

output
```
log(x + 1)*((13*d)/1296 - e/81 + (25*f)/1296 - (5*g)/162 + (61*h)/1296 - (
11*i)/162) - log(x - 1)*((13*d)/1296 + e/81 + (25*f)/1296 + (5*g)/162 + (6
1*h)/1296 + (11*i)/162) - ((25*e)/108 + (19*g)/27 + (40*i)/27 + x*((65*f)/
216 - (43*d)/864 + (41*h)/54) + x^5*((13*d)/192 + (5*f)/16 + (17*h)/24) -
x^3*((35*d)/384 + (21*f)/32 + (35*h)/24) - x^7*((35*d)/3456 + (35*f)/864 +
 (5*h)/54) - x^2*((5*e)/9 + (25*g)/18 + (26*i)/9) - x^6*(e/27 + (5*g)/54 +
 (11*i)/54) + x^4*((5*e)/18 + (25*g)/36 + (55*i)/36))/(33*x^4 - 40*x^2 - 1
0*x^6 + x^8 + 16) + log(x - 2)*((313*d)/41472 + e/81 + (205*f)/10368 + (5*
g)/162 + (121*h)/2592 + (11*i)/162) - log(x + 2)*((313*d)/41472 - e/81 + (
205*f)/10368 - (5*g)/162 + (121*h)/2592 - (11*i)/162)
```

3.47 $\int \frac{d+ex}{(1+x^2+x^4)^3} dx$

3.47.1 Optimal result

Integrand size = 16, antiderivative size = 185

$$\begin{aligned}\int \frac{d+ex}{(1+x^2+x^4)^3} dx = & \frac{dx(1-x^2)}{12(1+x^2+x^4)^2} + \frac{e(1+2x^2)}{12(1+x^2+x^4)^2} + \frac{dx(2-7x^2)}{24(1+x^2+x^4)} \\ & + \frac{e(1+2x^2)}{6(1+x^2+x^4)} - \frac{13d\arctan\left(\frac{1-2x}{\sqrt{3}}\right)}{48\sqrt{3}} + \frac{13d\arctan\left(\frac{1+2x}{\sqrt{3}}\right)}{48\sqrt{3}} \\ & + \frac{2e\arctan\left(\frac{1+2x^2}{\sqrt{3}}\right)}{3\sqrt{3}} - \frac{9}{32}d\log\left(1-x+x^2\right) + \frac{9}{32}d\log\left(1+x+x^2\right)\end{aligned}$$

output
```
1/12*d*x*(-x^2+1)/(x^4+x^2+1)^2+1/12*e*(2*x^2+1)/(x^4+x^2+1)^2+1/24*d*x*(-
7*x^2+2)/(x^4+x^2+1)+1/6*e*(2*x^2+1)/(x^4+x^2+1)-9/32*d*ln(x^2-x+1)+9/32*d
*ln(x^2+x+1)-13/144*d*arctan(1/3*(1-2*x)*3^(1/2))*3^(1/2)+13/144*d*arctan(
1/3*(1+2*x)*3^(1/2))*3^(1/2)+2/9*e*arctan(1/3*(2*x^2+1)*3^(1/2))*3^(1/2)
```

3.47.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.53 (sec) , antiderivative size = 186, normalized size of antiderivative = 1.01

$$\int \frac{d+ex}{(1+x^2+x^4)^3}\,dx = \frac{1}{144}\left(\frac{12(e+dx+2ex^2-dx^3)}{(1+x^2+x^4)^2}+\frac{6(dx(2-7x^2)+e(4+8x^2))}{1+x^2+x^4}-\frac{\left(-47i+7\sqrt{3}\right)d\arctan\left(\frac{1}{2}\left(-i+\sqrt{3}\right)x\right)}{\sqrt{\frac{1}{6}\left(1+i\sqrt{3}\right)}}-\frac{\left(47i+7\sqrt{3}\right)d\arctan\left(\frac{1}{2}\left(i+\sqrt{3}\right)x\right)}{\sqrt{\frac{1}{6}\left(1-i\sqrt{3}\right)}}-32\sqrt{3}e\arctan\left(\frac{\sqrt{3}}{1+2x^2}\right)\right)$$

input
```
Integrate[(d + e*x)/(1 + x^2 + x^4)^3,x]
```

output
```
((12*(e + d*x + 2*e*x^2 - d*x^3))/(1 + x^2 + x^4)^2 + (6*(d*x*(2 - 7*x^2)
+ e*(4 + 8*x^2)))/(1 + x^2 + x^4) - ((-47*I + 7*Sqrt[3])*d*ArcTan[((-I + S
qrt[3])*x)/2])/Sqrt[(1 + I*Sqrt[3])/6] - ((47*I + 7*Sqrt[3])*d*ArcTan[((I
+ Sqrt[3])*x)/2])/Sqrt[(1 - I*Sqrt[3])/6] - 32*Sqrt[3]*e*ArcTan[Sqrt[3]/(1
 + 2*x^2)])/144
```

3.47.3 Rubi [A] (verified)

Time = 0.48 (sec) , antiderivative size = 201, normalized size of antiderivative = 1.09, number of steps used = 17, number of rules used = 16, $\frac{\text{number of rules}}{\text{integrand size}} = 1.000$, Rules used = {2202, 27, 1405, 1432, 1086, 1086, 1083, 217, 1492, 27, 1483, 1142, 25, 1083, 217, 1103}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex}{(x^4+x^2+1)^3}dx$$

$$\Big\downarrow \; 2202$$

$$\int \frac{d}{(x^4+x^2+1)^3}dx+\int \frac{ex}{(x^4+x^2+1)^3}dx$$

$$\Big\downarrow \; 27$$

$$d\int \frac{1}{(x^4+x^2+1)^3}dx+e\int \frac{x}{(x^4+x^2+1)^3}dx$$

$$\Big\downarrow \text{1405}$$

$$d\left(\frac{1}{12}\int \frac{11-5x^2}{\left(x^4+x^2+1\right)^2}dx+\frac{x\left(1-x^2\right)}{12\left(x^4+x^2+1\right)^2}\right)+e\int \frac{x}{\left(x^4+x^2+1\right)^3}dx$$

$$\Big\downarrow \text{1432}$$

$$d\left(\frac{1}{12}\int \frac{11-5x^2}{\left(x^4+x^2+1\right)^2}dx+\frac{x\left(1-x^2\right)}{12\left(x^4+x^2+1\right)^2}\right)+\frac{1}{2}e\int \frac{1}{\left(x^4+x^2+1\right)^3}dx^2$$

$$\Big\downarrow \text{1086}$$

$$d\left(\frac{1}{12}\int \frac{11-5x^2}{\left(x^4+x^2+1\right)^2}dx+\frac{x\left(1-x^2\right)}{12\left(x^4+x^2+1\right)^2}\right)+\frac{1}{2}e\left(\int \frac{1}{\left(x^4+x^2+1\right)^2}dx^2+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)$$

$$\Big\downarrow \text{1086}$$

$$d\left(\frac{1}{12}\int \frac{11-5x^2}{\left(x^4+x^2+1\right)^2}dx+\frac{x\left(1-x^2\right)}{12\left(x^4+x^2+1\right)^2}\right)+$$
$$\frac{1}{2}e\left(\frac{2}{3}\int \frac{1}{x^4+x^2+1}dx^2+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)$$

$$\Big\downarrow \text{1083}$$

$$d\left(\frac{1}{12}\int \frac{11-5x^2}{\left(x^4+x^2+1\right)^2}dx+\frac{x\left(1-x^2\right)}{12\left(x^4+x^2+1\right)^2}\right)+$$
$$\frac{1}{2}e\left(-\frac{4}{3}\int \frac{1}{-x^4-3}d\left(2x^2+1\right)+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)$$

$$\Big\downarrow \text{217}$$

$$d\left(\frac{1}{12}\int \frac{11-5x^2}{\left(x^4+x^2+1\right)^2}dx+\frac{x\left(1-x^2\right)}{12\left(x^4+x^2+1\right)^2}\right)+$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)$$

$$\Big\downarrow \text{1492}$$

$$d\left(\frac{1}{12}\left(\frac{1}{6}\int \frac{3\left(20-7x^2\right)}{x^4+x^2+1}dx+\frac{x\left(2-7x^2\right)}{2\left(x^4+x^2+1\right)}\right)+\frac{x\left(1-x^2\right)}{12\left(x^4+x^2+1\right)^2}\right)+$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)$$

$$\Big\downarrow \text{27}$$

3.47. $\int \frac{d+ex}{(1+x^2+x^4)^3}\,dx$

$$d\left(\frac{1}{12}\left(\frac{1}{2}\int \frac{20-7x^2}{x^4+x^2+1}dx+\frac{x(2-7x^2)}{2(x^4+x^2+1)}\right)+\frac{x(1-x^2)}{12(x^4+x^2+1)^2}\right)+$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3(x^4+x^2+1)}+\frac{2x^2+1}{6(x^4+x^2+1)^2}\right)$$

$$\Big\downarrow \text{ 1483}$$

$$d\left(\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\int \frac{20-27x}{x^2-x+1}dx+\frac{1}{2}\int \frac{27x+20}{x^2+x+1}dx\right)+\frac{x(2-7x^2)}{2(x^4+x^2+1)}\right)+\frac{x(1-x^2)}{12(x^4+x^2+1)^2}\right)+$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3(x^4+x^2+1)}+\frac{2x^2+1}{6(x^4+x^2+1)^2}\right)$$

$$\Big\downarrow \text{ 1142}$$

$$d\left(\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{13}{2}\int \frac{1}{x^2-x+1}dx-\frac{27}{2}\int -\frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{13}{2}\int \frac{1}{x^2+x+1}dx+\frac{27}{2}\int \frac{2x+1}{x^2+x+1}dx\right)\right)\right.$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3(x^4+x^2+1)}+\frac{2x^2+1}{6(x^4+x^2+1)^2}\right)$$

$$\Big\downarrow \text{ 25}$$

$$d\left(\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{13}{2}\int \frac{1}{x^2-x+1}dx+\frac{27}{2}\int \frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{13}{2}\int \frac{1}{x^2+x+1}dx+\frac{27}{2}\int \frac{2x+1}{x^2+x+1}dx\right)\right)+\right.$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3(x^4+x^2+1)}+\frac{2x^2+1}{6(x^4+x^2+1)^2}\right)$$

$$\Big\downarrow \text{ 1083}$$

$$d\left(\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{27}{2}\int \frac{1-2x}{x^2-x+1}dx-13\int \frac{1}{-(2x-1)^2-3}d(2x-1)\right)+\frac{1}{2}\left(\frac{27}{2}\int \frac{2x+1}{x^2+x+1}dx-13\int \frac{1}{-(2x+1}\right.\right.\right.\right.$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3(x^4+x^2+1)}+\frac{2x^2+1}{6(x^4+x^2+1)^2}\right)$$

$$\Big\downarrow \text{ 217}$$

3.47. $\int \frac{d+ex}{(1+x^2+x^4)^3}\,dx$

$$d\left(\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{27}{2}\int\frac{1-2x}{x^2-x+1}dx+\frac{13\arctan\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}\right)+\frac{1}{2}\left(\frac{27}{2}\int\frac{2x+1}{x^2+x+1}dx+\frac{13\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}\right)\right)+\frac{}{2}\right.\right.$$

$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)$$

$\Big\downarrow$ 1103

$$d\left(\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{13\arctan\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}}-\frac{27}{2}\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{13\arctan\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}}+\frac{27}{2}\log\left(x^2+x+1\right)\right)\right)+\frac{}{2}\right.\right.$$

$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)$$

input
```
Int[(d + e*x)/(1 + x^2 + x^4)^3,x]
```

output
```
(e*((1 + 2*x^2)/(6*(1 + x^2 + x^4)^2) + (1 + 2*x^2)/(3*(1 + x^2 + x^4)) +
(4*ArcTan[(1 + 2*x^2)/Sqrt[3]])/(3*Sqrt[3])))/2 + d*((x*(1 - x^2))/(12*(1
+ x^2 + x^4)^2) + ((x*(2 - 7*x^2))/(2*(1 + x^2 + x^4)) + (((13*ArcTan[(-1
+ 2*x)/Sqrt[3]])/Sqrt[3] - (27*Log[1 - x + x^2])/2)/2 + ((13*ArcTan[(1 + 2
*x)/Sqrt[3]])/Sqrt[3] + (27*Log[1 + x + x^2])/2)/2)/2)/12)
```

3.47.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 217
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1086
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] - Simp[2*c*((2*p +
 3)/((p + 1)*(b^2 - 4*a*c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; Fre
eQ[{a, b, c}, x] && ILtQ[p, -1]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1405
```
Int[((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[(-x)*(b^2
- 2*a*c + b*c*x^2)*((a + b*x^2 + c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))
), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(b^2 - 2*a*c + 2*(p + 1)*(
b^2 - 4*a*c) + b*c*(4*p + 7)*x^2)*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; Fr
eeQ[{a, b, c}, x] && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1] && IntegerQ[2*p]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 1483
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[a/c, 2]}, With[{r = Rt[2*q - b/c, 2]}, Simp[1/(2*c*q*r)   In
t[(d*r - (d - e*q)*x)/(q - r*x + x^2), x], x] + Simp[1/(2*c*q*r)   Int[(d*r
 + (d - e*q)*x)/(q + r*x + x^2), x], x]]] /; FreeQ[{a, b, c, d, e}, x] && N
eQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NegQ[b^2 - 4*a*c]
```

3.47. $\int \frac{d+ex}{(1+x^2+x^4)^3} \, dx$

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.47.4 Maple [A] (verified)

Time = 0.18 (sec) , antiderivative size = 158, normalized size of antiderivative = 0.85

method	result
default	$-\frac{\left(\frac{7d}{3}-\frac{4e}{3}\right)x^{3}-6d\,x^{2}+\left(\frac{20d}{3}+\frac{e}{3}\right)x-4d-2e}{16\left(x^{2}-x+1\right)^{2}}-\frac{9d\ln\left(x^{2}-x+1\right)}{32}-\frac{\left(-\frac{13d}{2}-16e\right)\sqrt{3}\,\arctan\left(\frac{(2x-1)\sqrt{3}}{3}\right)}{72}+\frac{\left(-\frac{7d}{3}-\frac{4e}{3}\right)x^{3}-6d\,x^{2}\ldots}{16(x^{2}\ldots}$
risch	$-\frac{9d\ln\left(15457716d^{2}x^{2}+7237632e^{2}x^{2}-15457716d^{2}x-7237632e^{2}x+15457716d^{2}+7237632e^{2}\right)}{32}+\frac{13\sqrt{3}\,d\arctan\left(\frac{1458d^{2}x\sqrt{3}}{2187d^{2}+1024e^{2}}+\ldots\right.}{\ldots}$

input
```
int((e*x+d)/(x^4+x^2+1)^3,x,method=_RETURNVERBOSE)
```

output
```
-1/16*((7/3*d-4/3*e)*x^3-6*d*x^2+(20/3*d+1/3*e)*x-4*d-2*e)/(x^2-x+1)^2-9/3
2*d*ln(x^2-x+1)-1/72*(-13/2*d-16*e)*3^(1/2)*arctan(1/3*(2*x-1)*3^(1/2))+1/
16*((-7/3*d-4/3*e)*x^3-6*d*x^2+(-20/3*d+1/3*e)*x-4*d+2*e)/(x^2+x+1)^2+9/32
*d*ln(x^2+x+1)+1/72*(13/2*d-16*e)*arctan(1/3*(1+2*x)*3^(1/2))*3^(1/2)
```

3.47.5 Fricas [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 278, normalized size of antiderivative = 1.50

$$\int \frac{d+ex}{(1+x^2+x^4)^3}\,dx =$$
$$-\frac{84\,dx^7-96\,ex^6+60\,dx^5-144\,ex^4+84\,dx^3-192\,ex^2-2\,\sqrt{3}((13\,d-32\,e)x^8+2\,(13\,d-32\,e)x^6+3\,(\ldots}{\ldots}$$

input
```
integrate((e*x+d)/(x^4+x^2+1)^3,x, algorithm="fricas")
```

output
```
-1/288*(84*d*x^7 - 96*e*x^6 + 60*d*x^5 - 144*e*x^4 + 84*d*x^3 - 192*e*x^2
- 2*sqrt(3)*((13*d - 32*e)*x^8 + 2*(13*d - 32*e)*x^6 + 3*(13*d - 32*e)*x^4
 + 2*(13*d - 32*e)*x^2 + 13*d - 32*e)*arctan(1/3*sqrt(3)*(2*x + 1)) - 2*sq
rt(3)*((13*d + 32*e)*x^8 + 2*(13*d + 32*e)*x^6 + 3*(13*d + 32*e)*x^4 + 2*(
13*d + 32*e)*x^2 + 13*d + 32*e)*arctan(1/3*sqrt(3)*(2*x - 1)) - 48*d*x - 8
1*(d*x^8 + 2*d*x^6 + 3*d*x^4 + 2*d*x^2 + d)*log(x^2 + x + 1) + 81*(d*x^8 +
 2*d*x^6 + 3*d*x^4 + 2*d*x^2 + d)*log(x^2 - x + 1) - 72*e)/(x^8 + 2*x^6 +
3*x^4 + 2*x^2 + 1)
```

3.47.6 Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 2.19 (sec) , antiderivative size = 1103, normalized size of antiderivative = 5.96

$$\int \frac{d+ex}{(1+x^2+x^4)^3}\,dx = \text{Too large to display}$$

input
```
integrate((e*x+d)/(x**4+x**2+1)**3,x)
```

output
```
(-9*d/32 - sqrt(3)*I*(13*d + 32*e)/288)*log(x + (-1025428432*d**4*e - 3347
52912*d**4*(-9*d/32 - sqrt(3)*I*(13*d + 32*e)/288) - 431308800*d**2*e**3 -
 3143688192*d**2*e**2*(-9*d/32 - sqrt(3)*I*(13*d + 32*e)/288) + 9917005824
*d**2*e*(-9*d/32 - sqrt(3)*I*(13*d + 32*e)/288)**2 + 11878244352*d**2*(-9*
d/32 - sqrt(3)*I*(13*d + 32*e)/288)**3 + 142606336*e**5 + 754974720*e**4*(
-9*d/32 - sqrt(3)*I*(13*d + 32*e)/288) + 3850371072*e**3*(-9*d/32 - sqrt(3
)*I*(13*d + 32*e)/288)**2 + 20384317440*e**2*(-9*d/32 - sqrt(3)*I*(13*d +
32*e)/288)**3)/(217696167*d**5 - 1217128448*d**3*e**2 - 617611264*d*e**4))
 + (-9*d/32 + sqrt(3)*I*(13*d + 32*e)/288)*log(x + (-1025428432*d**4*e - 3
34752912*d**4*(-9*d/32 + sqrt(3)*I*(13*d + 32*e)/288) - 431308800*d**2*e**
3 - 3143688192*d**2*e**2*(-9*d/32 + sqrt(3)*I*(13*d + 32*e)/288) + 9917005
824*d**2*e*(-9*d/32 + sqrt(3)*I*(13*d + 32*e)/288)**2 + 11878244352*d**2*(
-9*d/32 + sqrt(3)*I*(13*d + 32*e)/288)**3 + 142606336*e**5 + 754974720*e**
4*(-9*d/32 + sqrt(3)*I*(13*d + 32*e)/288) + 3850371072*e**3*(-9*d/32 + sqr
t(3)*I*(13*d + 32*e)/288)**2 + 20384317440*e**2*(-9*d/32 + sqrt(3)*I*(13*d
 + 32*e)/288)**3)/(217696167*d**5 - 1217128448*d**3*e**2 - 617611264*d*e**
4)) + (9*d/32 - sqrt(3)*I*(13*d - 32*e)/288)*log(x + (-1025428432*d**4*e -
 334752912*d**4*(9*d/32 - sqrt(3)*I*(13*d - 32*e)/288) - 431308800*d**2*e*
*3 - 3143688192*d**2*e**2*(9*d/32 - sqrt(3)*I*(13*d - 32*e)/288) + 9917005
824*d**2*e*(9*d/32 - sqrt(3)*I*(13*d - 32*e)/288)**2 + 11878244352*d**2...
```

3.47.7 Maxima [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 137, normalized size of antiderivative = 0.74

$$\int \frac{d+ex}{(1+x^2+x^4)^3}\,dx = \frac{1}{144}\sqrt{3}(13\,d-32\,e)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x+1)\right) + \frac{1}{144}\sqrt{3}(13\,d+32\,e)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x-1)\right) + \frac{9}{32}\,d\log\left(x^2+x+1\right) - \frac{9}{32}\,d\log\left(x^2-x+1\right) - \frac{7\,dx^7-8\,ex^6+5\,dx^5-12\,ex^4+7\,dx^3-16\,ex^2-4\,dx-6\,e}{24\,(x^8+2\,x^6+3\,x^4+2\,x^2+1)}$$

input
```
integrate((e*x+d)/(x^4+x^2+1)^3,x, algorithm="maxima")
```

output
```
1/144*sqrt(3)*(13*d - 32*e)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/144*sqrt(3)*
(13*d + 32*e)*arctan(1/3*sqrt(3)*(2*x - 1)) + 9/32*d*log(x^2 + x + 1) - 9/
32*d*log(x^2 - x + 1) - 1/24*(7*d*x^7 - 8*e*x^6 + 5*d*x^5 - 12*e*x^4 + 7*d
*x^3 - 16*e*x^2 - 4*d*x - 6*e)/(x^8 + 2*x^6 + 3*x^4 + 2*x^2 + 1)
```

3.47.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 125, normalized size of antiderivative = 0.68

$$\int \frac{d+ex}{(1+x^2+x^4)^3}\,dx = \frac{1}{144}\sqrt{3}(13\,d-32\,e)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x+1)\right) + \frac{1}{144}\sqrt{3}(13\,d+32\,e)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x-1)\right) + \frac{9}{32}\,d\log\left(x^2+x+1\right) - \frac{9}{32}\,d\log\left(x^2-x+1\right) - \frac{7\,dx^7-8\,ex^6+5\,dx^5-12\,ex^4+7\,dx^3-16\,ex^2-4\,dx-6\,e}{24\,\left(x^4+x^2+1\right)^2}$$

input
```
integrate((e*x+d)/(x^4+x^2+1)^3,x, algorithm="giac")
```

output
```
1/144*sqrt(3)*(13*d - 32*e)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/144*sqrt(3)*
(13*d + 32*e)*arctan(1/3*sqrt(3)*(2*x - 1)) + 9/32*d*log(x^2 + x + 1) - 9/
32*d*log(x^2 - x + 1) - 1/24*(7*d*x^7 - 8*e*x^6 + 5*d*x^5 - 12*e*x^4 + 7*d
*x^3 - 16*e*x^2 - 4*d*x - 6*e)/(x^4 + x^2 + 1)^2
```

3.47.9 Mupad [B] (verification not implemented)

Time = 0.15 (sec) , antiderivative size = 185, normalized size of antiderivative = 1.00

$$\int \frac{d+ex}{(1+x^2+x^4)^3}\,dx = \frac{-\frac{7\,d\,x^7}{24}+\frac{e\,x^6}{3}-\frac{5\,d\,x^5}{24}+\frac{e\,x^4}{2}-\frac{7\,d\,x^3}{24}+\frac{2\,e\,x^2}{3}+\frac{d\,x}{6}+\frac{e}{4}}{x^8+2\,x^6+3\,x^4+2\,x^2+1} - \ln\left(x-\frac{1}{2}-\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{9\,d}{32}+\frac{\sqrt{3}\,d\,13\mathrm{i}}{288}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}\right) + \ln\left(x+\frac{1}{2}-\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{9\,d}{32}-\frac{\sqrt{3}\,d\,13\mathrm{i}}{288}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}\right) + \ln\left(x-\frac{1}{2}+\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(-\frac{9\,d}{32}+\frac{\sqrt{3}\,d\,13\mathrm{i}}{288}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}\right) + \ln\left(x+\frac{1}{2}+\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{9\,d}{32}+\frac{\sqrt{3}\,d\,13\mathrm{i}}{288}-\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}\right)$$

input
```
int((d + e*x)/(x^2 + x^4 + 1)^3,x)
```

output
```
(e/4 + (d*x)/6 - (7*d*x^3)/24 - (5*d*x^5)/24 - (7*d*x^7)/24 + (2*e*x^2)/3
+ (e*x^4)/2 + (e*x^6)/3)/(2*x^2 + 3*x^4 + 2*x^6 + x^8 + 1) - log(x - (3^(1
/2)*1i)/2 - 1/2)*((9*d)/32 + (3^(1/2)*d*13i)/288 + (3^(1/2)*e*1i)/9) + log
(x - (3^(1/2)*1i)/2 + 1/2)*((9*d)/32 - (3^(1/2)*d*13i)/288 + (3^(1/2)*e*1i
)/9) + log(x + (3^(1/2)*1i)/2 - 1/2)*((3^(1/2)*d*13i)/288 - (9*d)/32 + (3^
(1/2)*e*1i)/9) + log(x + (3^(1/2)*1i)/2 + 1/2)*((9*d)/32 + (3^(1/2)*d*13i)
/288 - (3^(1/2)*e*1i)/9)
```

3.47. $\int \frac{d+ex}{(1+x^2+x^4)^3}\,dx$

3.48 $\int \frac{d+ex+fx^2}{\left(1+x^2+x^4\right)^3}\,dx$

3.48.1 Optimal result

Integrand size = 21, antiderivative size = 223

$$\int \frac{d+ex+fx^2}{\left(1+x^2+x^4\right)^3}\,dx = \frac{e(1+2x^2)}{12\left(1+x^2+x^4\right)^2}+\frac{x(d+f-(d-2f)x^2)}{12\left(1+x^2+x^4\right)^2}+\frac{e(1+2x^2)}{6\left(1+x^2+x^4\right)} + \frac{x(2d+3f-7(d-f)x^2)}{24\left(1+x^2+x^4\right)}-\frac{(13d+2f)\arctan\left(\frac{1-2x}{\sqrt{3}}\right)}{48\sqrt{3}} + \frac{(13d+2f)\arctan\left(\frac{1+2x}{\sqrt{3}}\right)}{48\sqrt{3}}+\frac{2e\arctan\left(\frac{1+2x^2}{\sqrt{3}}\right)}{3\sqrt{3}} - \frac{1}{32}(9d-4f)\log\left(1-x+x^2\right)+\frac{1}{32}(9d-4f)\log\left(1+x+x^2\right)$$

output
```
1/12*e*(2*x^2+1)/(x^4+x^2+1)^2+1/12*x*(d+f-(d-2*f)*x^2)/(x^4+x^2+1)^2+1/6*
e*(2*x^2+1)/(x^4+x^2+1)+1/24*x*(2*d+3*f-7*(d-f)*x^2)/(x^4+x^2+1)-1/32*(9*d
-4*f)*ln(x^2-x+1)+1/32*(9*d-4*f)*ln(x^2+x+1)-1/144*(13*d+2*f)*arctan(1/3*(
1-2*x)*3^(1/2))*3^(1/2)+1/144*(13*d+2*f)*arctan(1/3*(1+2*x)*3^(1/2))*3^(1/
2)+2/9*e*arctan(1/3*(2*x^2+1)*3^(1/2))*3^(1/2)
```

3.48.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.39 (sec) , antiderivative size = 235, normalized size of antiderivative = 1.05

$$\int \frac{d+ex+fx^2}{(1+x^2+x^4)^3}\,dx = \frac{1}{144}\left(\frac{6(2dx+3fx-7dx^3+7fx^3+e(4+8x^2))}{1+x^2+x^4} + \frac{12(e+2ex^2+x(d+f-dx^2+2fx^2))}{(1+x^2+x^4)^2} - \frac{\left(\left(-47i+7\sqrt{3}\right)d+\left(17i-7\sqrt{3}\right)f\right)\arctan\left(\frac{1}{2}\left(-i+\sqrt{3}\right)x\right)}{\sqrt{\frac{1}{6}\left(1+i\sqrt{3}\right)}} - \frac{\left(\left(47i+7\sqrt{3}\right)d-\left(17i+7\sqrt{3}\right)f\right)\arctan\left(\frac{1}{2}\left(i+\sqrt{3}\right)x\right)}{\sqrt{\frac{1}{6}\left(1-i\sqrt{3}\right)}} - 32\sqrt{3}e\arctan\left(\frac{\sqrt{3}}{1+2x^2}\right)\right)$$

input
```
Integrate[(d + e*x + f*x^2)/(1 + x^2 + x^4)^3,x]
```

output
```
((6*(2*d*x + 3*f*x - 7*d*x^3 + 7*f*x^3 + e*(4 + 8*x^2)))/(1 + x^2 + x^4) +
 (12*(e + 2*e*x^2 + x*(d + f - d*x^2 + 2*f*x^2)))/(1 + x^2 + x^4)^2 - (((-
47*I + 7*Sqrt[3])*d + (17*I - 7*Sqrt[3])*f)*ArcTan[((-I + Sqrt[3])*x)/2])/
Sqrt[(1 + I*Sqrt[3])/6] - (((47*I + 7*Sqrt[3])*d - (17*I + 7*Sqrt[3])*f)*A
rcTan[((I + Sqrt[3])*x)/2])/Sqrt[(1 - I*Sqrt[3])/6] - 32*Sqrt[3]*e*ArcTan[
Sqrt[3]/(1 + 2*x^2)])/144
```

3.48.3 Rubi [A] (verified)

Time = 0.56 (sec) , antiderivative size = 240, normalized size of antiderivative = 1.08, number of steps used = 17, number of rules used = 16, $\frac{\text{number of rules}}{\text{integrand size}} = 0.762$, Rules used = {2202, 27, 1432, 1086, 1086, 1083, 217, 1492, 1492, 27, 1483, 1142, 25, 1083, 217, 1103}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2}{\left(x^4+x^2+1\right)^3}\,dx$$

$$\Big\downarrow \ 2202$$

$$\int \frac{fx^2+d}{\left(x^4+x^2+1\right)^3}dx+\int \frac{ex}{\left(x^4+x^2+1\right)^3}dx$$

$$\Big\downarrow \ 27$$

$$\int \frac{fx^2+d}{\left(x^4+x^2+1\right)^3}dx+e\int \frac{x}{\left(x^4+x^2+1\right)^3}dx$$

$$\Big\downarrow \ 1432$$

$$\int \frac{fx^2+d}{\left(x^4+x^2+1\right)^3}dx+\frac{1}{2}e\int \frac{1}{\left(x^4+x^2+1\right)^3}dx^2$$

$$\Big\downarrow \ 1086$$

$$\int \frac{fx^2+d}{\left(x^4+x^2+1\right)^3}dx+\frac{1}{2}e\left(\int \frac{1}{\left(x^4+x^2+1\right)^2}dx^2+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)$$

$$\Big\downarrow \ 1086$$

$$\int \frac{fx^2+d}{\left(x^4+x^2+1\right)^3}dx+\frac{1}{2}e\left(\frac{2}{3}\int \frac{1}{x^4+x^2+1}dx^2+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)$$

$$\Big\downarrow \ 1083$$

$$\int \frac{fx^2+d}{\left(x^4+x^2+1\right)^3}dx+\frac{1}{2}e\left(-\frac{4}{3}\int \frac{1}{-x^4-3}d\left(2x^2+1\right)+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)$$

$$\Big\downarrow \ 217$$

$$\int \frac{fx^2+d}{\left(x^4+x^2+1\right)^3}dx+\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)$$

$$\Big\downarrow \ 1492$$

$$\frac{1}{12}\int \frac{-5(d-2f)x^2+11d-f}{\left(x^4+x^2+1\right)^2}dx+$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \ 1492$$

3.48. $\int \frac{d+ex+fx^2}{\left(1+x^2+x^4\right)^3}\,dx$

$$\frac{1}{12}\left(\frac{1}{6}\int \frac{3\left(5(4d-f)-7(d-f)x^2\right)}{x^4+x^2+1}dx+\frac{x\left(-7x^2(d-f)+2d+3f\right)}{2\left(x^4+x^2+1\right)}\right)+$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \text{27}$$

$$\frac{1}{12}\left(\frac{1}{2}\int \frac{5(4d-f)-7(d-f)x^2}{x^4+x^2+1}dx+\frac{x\left(-7x^2(d-f)+2d+3f\right)}{2\left(x^4+x^2+1\right)}\right)+$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \text{1483}$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\int \frac{5(4d-f)-3(9d-4f)x}{x^2-x+1}dx+\frac{1}{2}\int \frac{5(4d-f)+3(9d-4f)x}{x^2+x+1}dx\right)+\frac{x\left(-7x^2(d-f)+2d+3f\right)}{2\left(x^4+x^2+1\right)}\right)+$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \text{1142}$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}(13d+2f)\int \frac{1}{x^2-x+1}dx-\frac{3}{2}(9d-4f)\int -\frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(13d+2f)\int \frac{1}{x^2+x+1}dx+\right.\right.\right.$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \text{25}$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}(13d+2f)\int \frac{1}{x^2-x+1}dx+\frac{3}{2}(9d-4f)\int \frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(13d+2f)\int \frac{1}{x^2+x+1}dx+\frac{3}{2}\right.\right.\right.$$
$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \text{1083}$$

3.48. $\int \frac{d+ex+fx^2}{(1+x^2+x^4)^3}\,dx$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{3}{2}(9d-4f)\int\frac{1-2x}{x^2-x+1}dx-(13d+2f)\int\frac{1}{-(2x-1)^2-3}d(2x-1)\right)+\frac{1}{2}\left(\frac{3}{2}(9d-4f)\int\frac{2x+1}{x^2+x+\right.\right.\right.$$

$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \text{217}$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{3}{2}(9d-4f)\int\frac{1-2x}{x^2-x+1}dx+\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(13d+2f)}{\sqrt{3}}\right)+\frac{1}{2}\left(\frac{3}{2}(9d-4f)\int\frac{2x+1}{x^2+x+1}dx+\frac{\text{arct}}{}\right.\right.\right.$$

$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \text{1103}$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(13d+2f)}{\sqrt{3}}-\frac{3}{2}(9d-4f)\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(13d+2f)}{\sqrt{3}}+\frac{3}{2}(9d\right.\right.\right.$$

$$\frac{1}{2}e\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}+\frac{2x^2+1}{6\left(x^4+x^2+1\right)^2}\right)+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

input
```
Int[(d + e*x + f*x^2)/(1 + x^2 + x^4)^3,x]
```

output
```
(x*(d + f - (d - 2*f)*x^2))/(12*(1 + x^2 + x^4)^2) + (e*((1 + 2*x^2)/(6*(1
 + x^2 + x^4)^2) + (1 + 2*x^2)/(3*(1 + x^2 + x^4)) + (4*ArcTan[(1 + 2*x^2)
/Sqrt[3]])/(3*Sqrt[3])))/2 + ((x*(2*d + 3*f - 7*(d - f)*x^2))/(2*(1 + x^2
+ x^4)) + ((((13*d + 2*f)*ArcTan[(-1 + 2*x)/Sqrt[3]])/Sqrt[3] - (3*(9*d -
4*f)*Log[1 - x + x^2])/2)/2 + (((13*d + 2*f)*ArcTan[(1 + 2*x)/Sqrt[3]])/Sq
rt[3] + (3*(9*d - 4*f)*Log[1 + x + x^2])/2)/2)/2)/12
```

3.48. $\int \frac{d+ex+fx^2}{\left(1+x^2+x^4\right)^3}\,dx$

3.48.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 217
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1086
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] - Simp[2*c*((2*p +
 3)/((p + 1)*(b^2 - 4*a*c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; Fre
eQ[{a, b, c}, x] && ILtQ[p, -1]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 1483
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[a/c, 2]}, With[{r = Rt[2*q - b/c, 2]}, Simp[1/(2*c*q*r)   In
t[(d*r - (d - e*q)*x)/(q - r*x + x^2), x], x] + Simp[1/(2*c*q*r)   Int[(d*r
 + (d - e*q)*x)/(q + r*x + x^2), x], x]]] /; FreeQ[{a, b, c, d, e}, x] && N
eQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NegQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.48.4 Maple [A] (verified)

Time = 0.28 (sec) , antiderivative size = 202, normalized size of antiderivative = 0.91

method	result
default	$-\frac{\left(\frac{7d}{3}-\frac{7f}{3}-\frac{4e}{3}\right)x^3+(-6d+4f)x^2+\left(\frac{20d}{3}-\frac{13f}{3}+\frac{e}{3}\right)x-4d+\frac{4f}{3}-2e}{16(x^2-x+1)^2}-\frac{(27d-12f)\ln(x^2-x+1)}{96}-\frac{\left(-\frac{13d}{2}-16e-f\right)\sqrt{3}\arctan\left(\frac{(2x}{}\right)}{72}$
risch	Expression too large to display

input
```
int((f*x^2+e*x+d)/(x^4+x^2+1)^3,x,method=_RETURNVERBOSE)
```

output
```
-1/16*((7/3*d-7/3*f-4/3*e)*x^3+(-6*d+4*f)*x^2+(20/3*d-13/3*f+1/3*e)*x-4*d+
4/3*f-2*e)/(x^2-x+1)^2-1/96*(27*d-12*f)*ln(x^2-x+1)-1/72*(-13/2*d-16*e-f)*
3^(1/2)*arctan(1/3*(2*x-1)*3^(1/2))+1/16*((-7/3*d+7/3*f-4/3*e)*x^3+(-6*d+4
*f)*x^2+(-20/3*d+13/3*f+1/3*e)*x-4*d+4/3*f+2*e)/(x^2+x+1)^2+1/96*(27*d-12*
f)*ln(x^2+x+1)+1/72*(13/2*d-16*e+f)*arctan(1/3*(1+2*x)*3^(1/2))*3^(1/2)
```

3.48.5 Fricas [A] (verification not implemented)

Time = 0.31 (sec) , antiderivative size = 384, normalized size of antiderivative = 1.72

$$\int \frac{d+ex+fx^2}{(1+x^2+x^4)^3}\,dx =$$
$$-\frac{84\,(d-f)x^7-96\,ex^6+60\,(d-2\,f)x^5-144\,ex^4+84\,(d-2\,f)x^3-192\,ex^2-2\,\sqrt{3}((13\,d-32\,e+2\,f}{}$$

input
```
integrate((f*x^2+e*x+d)/(x^4+x^2+1)^3,x, algorithm="fricas")
```

output
```
-1/288*(84*(d - f)*x^7 - 96*e*x^6 + 60*(d - 2*f)*x^5 - 144*e*x^4 + 84*(d -
 2*f)*x^3 - 192*e*x^2 - 2*sqrt(3)*((13*d - 32*e + 2*f)*x^8 + 2*(13*d - 32*
e + 2*f)*x^6 + 3*(13*d - 32*e + 2*f)*x^4 + 2*(13*d - 32*e + 2*f)*x^2 + 13*
d - 32*e + 2*f)*arctan(1/3*sqrt(3)*(2*x + 1)) - 2*sqrt(3)*((13*d + 32*e +
2*f)*x^8 + 2*(13*d + 32*e + 2*f)*x^6 + 3*(13*d + 32*e + 2*f)*x^4 + 2*(13*d
 + 32*e + 2*f)*x^2 + 13*d + 32*e + 2*f)*arctan(1/3*sqrt(3)*(2*x - 1)) - 12
*(4*d + 5*f)*x - 9*((9*d - 4*f)*x^8 + 2*(9*d - 4*f)*x^6 + 3*(9*d - 4*f)*x^
4 + 2*(9*d - 4*f)*x^2 + 9*d - 4*f)*log(x^2 + x + 1) + 9*((9*d - 4*f)*x^8 +
 2*(9*d - 4*f)*x^6 + 3*(9*d - 4*f)*x^4 + 2*(9*d - 4*f)*x^2 + 9*d - 4*f)*lo
g(x^2 - x + 1) - 72*e)/(x^8 + 2*x^6 + 3*x^4 + 2*x^2 + 1)
```

3.48.6 Sympy [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 138.76 (sec) , antiderivative size = 4496, normalized size of antiderivative = 20.16

$$\int \frac{d+ex+fx^2}{(1+x^2+x^4)^3}\,dx = \text{Too large to display}$$

input
```
integrate((f*x**2+e*x+d)/(x**4+x**2+1)**3,x)
```

output
```
(-9*d/32 + f/8 - sqrt(3)*I*(13*d + 32*e + 2*f)/288)*log(x + (-1025428432*d
**5*e - 334752912*d**5*(-9*d/32 + f/8 - sqrt(3)*I*(13*d + 32*e + 2*f)/288)
 + 2008961360*d**4*e*f + 1151575920*d**4*f*(-9*d/32 + f/8 - sqrt(3)*I*(13*
d + 32*e + 2*f)/288) - 431308800*d**3*e**3 - 3143688192*d**3*e**2*(-9*d/32
 + f/8 - sqrt(3)*I*(13*d + 32*e + 2*f)/288) - 1598857120*d**3*e*f**2 + 991
7005824*d**3*e*(-9*d/32 + f/8 - sqrt(3)*I*(13*d + 32*e + 2*f)/288)**2 - 94
4300160*d**3*f**2*(-9*d/32 + f/8 - sqrt(3)*I*(13*d + 32*e + 2*f)/288) + 11
878244352*d**3*(-9*d/32 + f/8 - sqrt(3)*I*(13*d + 32*e + 2*f)/288)**3 + 23
3164800*d**2*e**3*f + 4409634816*d**2*e**2*f*(-9*d/32 + f/8 - sqrt(3)*I*(1
3*d + 32*e + 2*f)/288) + 662937520*d**2*e*f**3 - 13004623872*d**2*e*f*(-9*
d/32 + f/8 - sqrt(3)*I*(13*d + 32*e + 2*f)/288)**2 + 231796080*d**2*f**3*(
-9*d/32 + f/8 - sqrt(3)*I*(13*d + 32*e + 2*f)/288) - 10089639936*d**2*f*(-
9*d/32 + f/8 - sqrt(3)*I*(13*d + 32*e + 2*f)/288)**3 + 142606336*d*e**5 +
754974720*d*e**4*(-9*d/32 + f/8 - sqrt(3)*I*(13*d + 32*e + 2*f)/288) - 184
3200*d*e**3*f**2 + 3850371072*d*e**3*(-9*d/32 + f/8 - sqrt(3)*I*(13*d + 32
*e + 2*f)/288)**2 - 1926291456*d*e**2*f**2*(-9*d/32 + f/8 - sqrt(3)*I*(13*
d + 32*e + 2*f)/288) + 20384317440*d*e**2*(-9*d/32 + f/8 - sqrt(3)*I*(13*d
 + 32*e + 2*f)/288)**3 - 146756960*d*e*f**4 + 5813379072*d*e*f**2*(-9*d/32
 + f/8 - sqrt(3)*I*(13*d + 32*e + 2*f)/288)**2 + 12679200*d*f**4*(-9*d/32
+ f/8 - sqrt(3)*I*(13*d + 32*e + 2*f)/288) + 1116758016*d*f**2*(-9*d/32...
```

3.48.7 Maxima [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 173, normalized size of antiderivative = 0.78

$$\int \frac{d+ex+fx^2}{(1+x^2+x^4)^3}\,dx = \frac{1}{144}\sqrt{3}(13\,d-32\,e+2\,f)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x+1)\right) + \frac{1}{144}\sqrt{3}(13\,d+32\,e+2\,f)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x-1)\right) + \frac{1}{32}(9\,d-4\,f)\log\left(x^2+x+1\right) - \frac{1}{32}(9\,d-4\,f)\log\left(x^2-x+1\right) - \frac{7\,(d-f)x^7-8\,ex^6+5\,(d-2\,f)x^5-12\,ex^4+7\,(d-2\,f)x^3-16\,ex^2-(4\,d+5\,f)x-6\,e}{24\,(x^8+2\,x^6+3\,x^4+2\,x^2+1)}$$

input
```
integrate((f*x^2+e*x+d)/(x^4+x^2+1)^3,x, algorithm="maxima")
```

output
```
1/144*sqrt(3)*(13*d - 32*e + 2*f)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/144*sq
rt(3)*(13*d + 32*e + 2*f)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/32*(9*d - 4*f)
*log(x^2 + x + 1) - 1/32*(9*d - 4*f)*log(x^2 - x + 1) - 1/24*(7*(d - f)*x^
7 - 8*e*x^6 + 5*(d - 2*f)*x^5 - 12*e*x^4 + 7*(d - 2*f)*x^3 - 16*e*x^2 - (4
*d + 5*f)*x - 6*e)/(x^8 + 2*x^6 + 3*x^4 + 2*x^2 + 1)
```

3.48.8 Giac [A] (verification not implemented)

Time = 0.33 (sec) , antiderivative size = 165, normalized size of antiderivative = 0.74

$$\int \frac{d+ex+fx^2}{(1+x^2+x^4)^3}\,dx = \frac{1}{144}\sqrt{3}(13\,d-32\,e+2\,f)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x+1)\right) + \frac{1}{144}\sqrt{3}(13\,d+32\,e+2\,f)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x-1)\right) + \frac{1}{32}\,(9\,d-4\,f)\log\left(x^2+x+1\right) - \frac{1}{32}\,(9\,d-4\,f)\log\left(x^2-x+1\right) - \frac{7\,dx^7-7\,fx^7-8\,ex^6+5\,dx^5-10\,fx^5-12\,ex^4+7\,dx^3-14\,fx^3-16\,ex^2-4\,dx-5\,fx-6\,e}{24\,(x^4+x^2+1)^2}$$

input
```
integrate((f*x^2+e*x+d)/(x^4+x^2+1)^3,x, algorithm="giac")
```

output
```
1/144*sqrt(3)*(13*d - 32*e + 2*f)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1/144*sq
rt(3)*(13*d + 32*e + 2*f)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/32*(9*d - 4*f)
*log(x^2 + x + 1) - 1/32*(9*d - 4*f)*log(x^2 - x + 1) - 1/24*(7*d*x^7 - 7*
f*x^7 - 8*e*x^6 + 5*d*x^5 - 10*f*x^5 - 12*e*x^4 + 7*d*x^3 - 14*f*x^3 - 16*
e*x^2 - 4*d*x - 5*f*x - 6*e)/(x^4 + x^2 + 1)^2
```

3.48.9 Mupad [B] (verification not implemented)

Time = 7.97 (sec) , antiderivative size = 249, normalized size of antiderivative = 1.12

$$\int \frac{d+ex+fx^2}{(1+x^2+x^4)^3}\,dx$$
$$= \frac{\left(\frac{7f}{24}-\frac{7d}{24}\right)x^7+\frac{ex^6}{3}+\left(\frac{5f}{12}-\frac{5d}{24}\right)x^5+\frac{ex^4}{2}+\left(\frac{7f}{12}-\frac{7d}{24}\right)x^3+\frac{2ex^2}{3}+\left(\frac{d}{6}+\frac{5f}{24}\right)x+\frac{e}{4}}{x^8+2x^6+3x^4+2x^2+1}$$
$$-\ln\left(x-\frac{1}{2}-\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{9d}{32}-\frac{f}{8}+\frac{\sqrt{3}\,d\,13\mathrm{i}}{288}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{144}\right)$$
$$-\ln\left(x+\frac{1}{2}-\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{f}{8}-\frac{9d}{32}+\frac{\sqrt{3}\,d\,13\mathrm{i}}{288}-\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{144}\right)$$
$$+\ln\left(x-\frac{1}{2}+\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{f}{8}-\frac{9d}{32}+\frac{\sqrt{3}\,d\,13\mathrm{i}}{288}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{144}\right)$$
$$+\ln\left(x+\frac{1}{2}+\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\left(\frac{9d}{32}-\frac{f}{8}+\frac{\sqrt{3}\,d\,13\mathrm{i}}{288}-\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{144}\right)$$

input
```
int((d + e*x + f*x^2)/(x^2 + x^4 + 1)^3,x)
```

output
```
(e/4 - x^5*((5*d)/24 - (5*f)/12) - x^3*((7*d)/24 - (7*f)/12) - x^7*((7*d)/
24 - (7*f)/24) + (2*e*x^2)/3 + (e*x^4)/2 + (e*x^6)/3 + x*(d/6 + (5*f)/24))
/(2*x^2 + 3*x^4 + 2*x^6 + x^8 + 1) - log(x - (3^(1/2)*1i)/2 - 1/2)*((9*d)/
32 - f/8 + (3^(1/2)*d*13i)/288 + (3^(1/2)*e*1i)/9 + (3^(1/2)*f*1i)/144) -
log(x - (3^(1/2)*1i)/2 + 1/2)*(f/8 - (9*d)/32 + (3^(1/2)*d*13i)/288 - (3^(
1/2)*e*1i)/9 + (3^(1/2)*f*1i)/144) + log(x + (3^(1/2)*1i)/2 - 1/2)*(f/8 -
(9*d)/32 + (3^(1/2)*d*13i)/288 + (3^(1/2)*e*1i)/9 + (3^(1/2)*f*1i)/144) +
log(x + (3^(1/2)*1i)/2 + 1/2)*((9*d)/32 - f/8 + (3^(1/2)*d*13i)/288 - (3^(
1/2)*e*1i)/9 + (3^(1/2)*f*1i)/144)
```

3.49 $\int \frac{d+ex+fx^2+gx^3}{(1+x^2+x^4)^3} dx$

3.49.1 Optimal result

Integrand size = 26, antiderivative size = 243

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3}{(1+x^2+x^4)^3} dx &= \frac{x(d+f-(d-2f)x^2)}{12(1+x^2+x^4)^2}+\frac{e-2g+(2e-g)x^2}{12(1+x^2+x^4)^2}\\ &\quad+\frac{(2e-g)(1+2x^2)}{12(1+x^2+x^4)}+\frac{x(2d+3f-7(d-f)x^2)}{24(1+x^2+x^4)}\\ &\quad-\frac{(13d+2f)\arctan\left(\frac{1-2x}{\sqrt{3}}\right)}{48\sqrt{3}}+\frac{(13d+2f)\arctan\left(\frac{1+2x}{\sqrt{3}}\right)}{48\sqrt{3}}\\ &\quad+\frac{(2e-g)\arctan\left(\frac{1+2x^2}{\sqrt{3}}\right)}{3\sqrt{3}}-\frac{1}{32}(9d-4f)\log\left(1-x+x^2\right)\\ &\quad+\frac{1}{32}(9d-4f)\log\left(1+x+x^2\right)\end{aligned}$$

output
```
1/12*x*(d+f-(d-2*f)*x^2)/(x^4+x^2+1)^2+1/12*(e-2*g+(2*e-g)*x^2)/(x^4+x^2+1
)^2+1/12*(2*e-g)*(2*x^2+1)/(x^4+x^2+1)+1/24*x*(2*d+3*f-7*(d-f)*x^2)/(x^4+x
^2+1)-1/32*(9*d-4*f)*ln(x^2-x+1)+1/32*(9*d-4*f)*ln(x^2+x+1)-1/144*(13*d+2*
f)*arctan(1/3*(1-2*x)*3^(1/2))*3^(1/2)+1/144*(13*d+2*f)*arctan(1/3*(1+2*x)
*3^(1/2))*3^(1/2)+1/9*(2*e-g)*arctan(1/3*(2*x^2+1)*3^(1/2))*3^(1/2)
```

3.49.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.44 (sec) , antiderivative size = 259, normalized size of antiderivative = 1.07

$$\int \frac{d+ex+fx^2+gx^3}{(1+x^2+x^4)^3}\,dx = \frac{1}{144}\left(\frac{6(2dx+3fx-7dx^3+7fx^3-2g(1+2x^2)+e(4+8x^2))}{1+x^2+x^4} + \frac{12(e+2ex^2-g(2+x^2)+x(d+f-dx^2+2fx^2))}{(1+x^2+x^4)^2} - \frac{\left(\left(-47i+7\sqrt{3}\right)d+\left(17i-7\sqrt{3}\right)f\right)\arctan\left(\frac{1}{2}\left(-i+\sqrt{3}\right)x\right)}{\sqrt{\frac{1}{6}\left(1+i\sqrt{3}\right)}} - \frac{\left(\left(47i+7\sqrt{3}\right)d-\left(17i+7\sqrt{3}\right)f\right)\arctan\left(\frac{1}{2}\left(i+\sqrt{3}\right)x\right)}{\sqrt{\frac{1}{6}\left(1-i\sqrt{3}\right)}} - 16\sqrt{3}(2e-g)\arctan\left(\frac{\sqrt{3}}{1+2x^2}\right)\right)$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3)/(1 + x^2 + x^4)^3,x]
```

output
```
((6*(2*d*x + 3*f*x - 7*d*x^3 + 7*f*x^3 - 2*g*(1 + 2*x^2) + e*(4 + 8*x^2)))
/(1 + x^2 + x^4) + (12*(e + 2*e*x^2 - g*(2 + x^2) + x*(d + f - d*x^2 + 2*f
*x^2)))/(1 + x^2 + x^4)^2 - (((-47*I + 7*Sqrt[3])*d + (17*I - 7*Sqrt[3])*f
)*ArcTan[((-I + Sqrt[3])*x)/2])/Sqrt[(1 + I*Sqrt[3])/6] - (((47*I + 7*Sqrt
[3])*d - (17*I + 7*Sqrt[3])*f)*ArcTan[((I + Sqrt[3])*x)/2])/Sqrt[(1 - I*Sq
rt[3])/6] - 16*Sqrt[3]*(2*e - g)*ArcTan[Sqrt[3]/(1 + 2*x^2)])/144
```

3.49.3 Rubi [A] (verified)

Time = 0.61 (sec) , antiderivative size = 260, normalized size of antiderivative = 1.07, number of steps used = 16, number of rules used = 15, $\frac{\text{number of rules}}{\text{integrand size}} = 0.577$, Rules used = {2202, 1492, 1492, 27, 1483, 1142, 25, 1083, 217, 1103, 1576, 1159, 1086, 1083, 217}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3}{(x^4+x^2+1)^3}\,dx$$

$$\Big\downarrow \; 2202$$

$$\int \frac{fx^2+d}{\left(x^4+x^2+1\right)^3}dx+\int \frac{x\left(gx^2+e\right)}{\left(x^4+x^2+1\right)^3}dx$$

$$\Big\downarrow \; 1492$$

$$\frac{1}{12}\int \frac{-5(d-2f)x^2+11d-f}{\left(x^4+x^2+1\right)^2}dx+\int \frac{x\left(gx^2+e\right)}{\left(x^4+x^2+1\right)^3}dx+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \; 1492$$

$$\frac{1}{12}\left(\frac{1}{6}\int \frac{3\left(5(4d-f)-7(d-f)x^2\right)}{x^4+x^2+1}dx+\frac{x\left(-7x^2(d-f)+2d+3f\right)}{2\left(x^4+x^2+1\right)}\right)+\int \frac{x\left(gx^2+e\right)}{\left(x^4+x^2+1\right)^3}dx+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \; 27$$

$$\frac{1}{12}\left(\frac{1}{2}\int \frac{5(4d-f)-7(d-f)x^2}{x^4+x^2+1}dx+\frac{x\left(-7x^2(d-f)+2d+3f\right)}{2\left(x^4+x^2+1\right)}\right)+\int \frac{x\left(gx^2+e\right)}{\left(x^4+x^2+1\right)^3}dx+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \; 1483$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\int \frac{5(4d-f)-3(9d-4f)x}{x^2-x+1}dx+\frac{1}{2}\int \frac{5(4d-f)+3(9d-4f)x}{x^2+x+1}dx\right)+\frac{x\left(-7x^2(d-f)+2d+3f\right)}{2\left(x^4+x^2+1\right)}\right)+\int \frac{x\left(gx^2+e\right)}{\left(x^4+x^2+1\right)^3}dx+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \; 1142$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}(13d+2f)\int \frac{1}{x^2-x+1}dx-\frac{3}{2}(9d-4f)\int -\frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(13d+2f)\int \frac{1}{x^2+x+1}dx+\right.\right.\right.$$
$$\int \frac{x\left(gx^2+e\right)}{\left(x^4+x^2+1\right)^3}dx+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \; 25$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}(13d+2f)\int \frac{1}{x^2-x+1}dx+\frac{3}{2}(9d-4f)\int \frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(13d+2f)\int \frac{1}{x^2+x+1}dx+\frac{3}{2}\right.\right.\right.$$
$$\int \frac{x\left(gx^2+e\right)}{\left(x^4+x^2+1\right)^3}dx+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

3.49. $\int \frac{d+ex+fx^2+gx^3}{\left(1+x^2+x^4\right)^3}\,dx$

$$\Big\downarrow 1083$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{3}{2}(9d-4f)\int\frac{1-2x}{x^2-x+1}dx-(13d+2f)\int\frac{1}{-(2x-1)^2-3}d(2x-1)\right)+\frac{1}{2}\left(\frac{3}{2}(9d-4f)\int\frac{2x+1}{x^2+x+\right.\right.\right.$$
$$\int\frac{x\left(gx^2+e\right)}{\left(x^4+x^2+1\right)^3}dx+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow 217$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{3}{2}(9d-4f)\int\frac{1-2x}{x^2-x+1}dx+\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(13d+2f)}{\sqrt{3}}\right)+\frac{1}{2}\left(\frac{3}{2}(9d-4f)\int\frac{2x+1}{x^2+x+1}dx+\frac{\text{arct}}{}\right.\right.\right.$$
$$\int\frac{x\left(gx^2+e\right)}{\left(x^4+x^2+1\right)^3}dx+\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow 1103$$

$$\int\frac{x\left(gx^2+e\right)}{\left(x^4+x^2+1\right)^3}dx+$$
$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(13d+2f)}{\sqrt{3}}-\frac{3}{2}(9d-4f)\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(13d+2f)}{\sqrt{3}}+\frac{3}{2}(9d\right.\right.\right.$$
$$\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow 1576$$

$$\frac{1}{2}\int\frac{gx^2+e}{\left(x^4+x^2+1\right)^3}dx^2+$$
$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(13d+2f)}{\sqrt{3}}-\frac{3}{2}(9d-4f)\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(13d+2f)}{\sqrt{3}}+\frac{3}{2}(9d\right.\right.\right.$$
$$\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow 1159$$

$$\frac{1}{2}\left(\frac{1}{2}(2e-g)\int\frac{1}{\left(x^4+x^2+1\right)^2}dx^2+\frac{x^2(2e-g)+e-2g}{6\left(x^4+x^2+1\right)^2}\right)+$$
$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(13d+2f)}{\sqrt{3}}-\frac{3}{2}(9d-4f)\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(13d+2f)}{\sqrt{3}}+\frac{3}{2}(9d\right.\right.\right.$$
$$\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow 1086$$

3.49. $\int\frac{d+ex+fx^2+gx^3}{\left(1+x^2+x^4\right)^3}\,dx$

$$\frac{1}{2}\left(\frac{1}{2}(2e-g)\left(\frac{2}{3}\int\frac{1}{x^4+x^2+1}dx^2+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)+\frac{x^2(2e-g)+e-2g}{6\left(x^4+x^2+1\right)^2}\right)+$$
$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(13d+2f)}{\sqrt{3}}-\frac{3}{2}(9d-4f)\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(13d+2f)}{\sqrt{3}}+\frac{3}{2}(9d\right.\right.\right.$$
$$\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$\Big\downarrow$ 1083

$$\frac{1}{2}\left(\frac{1}{2}(2e-g)\left(\frac{2x^2+1}{3\left(x^4+x^2+1\right)}-\frac{4}{3}\int\frac{1}{-x^4-3}d\left(2x^2+1\right)\right)+\frac{x^2(2e-g)+e-2g}{6\left(x^4+x^2+1\right)^2}\right)+$$
$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(13d+2f)}{\sqrt{3}}-\frac{3}{2}(9d-4f)\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(13d+2f)}{\sqrt{3}}+\frac{3}{2}(9d\right.\right.\right.$$
$$\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

$\Big\downarrow$ 217

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(13d+2f)}{\sqrt{3}}-\frac{3}{2}(9d-4f)\log\left(x^2-x+1\right)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(13d+2f)}{\sqrt{3}}+\frac{3}{2}(9d\right.\right.\right.$$
$$\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g)+\frac{x^2(2e-g)+e-2g}{6\left(x^4+x^2+1\right)^2}\right)+$$
$$\frac{x\left(-\left(x^2(d-2f)\right)+d+f\right)}{12\left(x^4+x^2+1\right)^2}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3)/(1 + x^2 + x^4)^3,x]
```

output
```
(x*(d + f - (d - 2*f)*x^2))/(12*(1 + x^2 + x^4)^2) + ((e - 2*g + (2*e - g)
*x^2)/(6*(1 + x^2 + x^4)^2) + ((2*e - g)*((1 + 2*x^2)/(3*(1 + x^2 + x^4))
+ (4*ArcTan[(1 + 2*x^2)/Sqrt[3]])/(3*Sqrt[3])))/2)/2 + ((x*(2*d + 3*f - 7*
(d - f)*x^2))/(2*(1 + x^2 + x^4)) + ((((13*d + 2*f)*ArcTan[(-1 + 2*x)/Sqrt
[3]])/Sqrt[3] - (3*(9*d - 4*f)*Log[1 - x + x^2])/2)/2 + (((13*d + 2*f)*Arc
Tan[(1 + 2*x)/Sqrt[3]])/Sqrt[3] + (3*(9*d - 4*f)*Log[1 + x + x^2])/2)/2)/2
)/12
```

3.49. $\int\frac{d+ex+fx^2+gx^3}{\left(1+x^2+x^4\right)^3}\,dx$

3.49.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 217
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1086
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] - Simp[2*c*((2*p +
 3)/((p + 1)*(b^2 - 4*a*c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; Fre
eQ[{a, b, c}, x] && ILtQ[p, -1]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1159
```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[((b*d - 2*a*e + (2*c*d - b*e)*x)/((p + 1)*(b^2 - 4*a*c)))*(a + b*
x + c*x^2)^(p + 1), x] - Simp[(2*p + 3)*((2*c*d - b*e)/((p + 1)*(b^2 - 4*a*
c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; FreeQ[{a, b, c, d, e}, x] &
& LtQ[p, -1] && NeQ[p, -3/2]
```

3.49. $\int \frac{d+ex+fx^2+gx^3}{(1+x^2+x^4)^3}\,dx$

rule 1483
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[a/c, 2]}, With[{r = Rt[2*q - b/c, 2]}, Simp[1/(2*c*q*r)   In
t[(d*r - (d - e*q)*x)/(q - r*x + x^2), x], x] + Simp[1/(2*c*q*r)   Int[(d*r
 + (d - e*q)*x)/(q + r*x + x^2), x], x]]] /; FreeQ[{a, b, c, d, e}, x] && N
eQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NegQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.49.4 Maple [A] (verified)

Time = 0.30 (sec) , antiderivative size = 232, normalized size of antiderivative = 0.95

method	result
default	$-\frac{\left(\frac{7d}{3}-\frac{7f}{3}-\frac{4e}{3}-\frac{g}{3}\right)x^3+(-6d+4f+2g)x^2+\left(\frac{20d}{3}-\frac{13f}{3}+\frac{e}{3}-\frac{8g}{3}\right)x-4d+\frac{4f}{3}-2e+2g}{16(x^2-x+1)^2}-\frac{(27d-12f)\ln(x^2-x+1)}{96}-\frac{\left(-\frac{13d}{2}-16e-f\right.}{}$
risch	Expression too large to display

input
```
int((g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^3,x,method=_RETURNVERBOSE)
```

output
```
-1/16*((7/3*d-7/3*f-4/3*e-1/3*g)*x^3+(-6*d+4*f+2*g)*x^2+(20/3*d-13/3*f+1/3
*e-8/3*g)*x-4*d+4/3*f-2*e+2*g)/(x^2-x+1)^2-1/96*(27*d-12*f)*ln(x^2-x+1)-1/
72*(-13/2*d-16*e-f+8*g)*3^(1/2)*arctan(1/3*(2*x-1)*3^(1/2))+1/16*((-7/3*d+
7/3*f-4/3*e-1/3*g)*x^3+(-6*d+4*f-2*g)*x^2+(-20/3*d+13/3*f+1/3*e-8/3*g)*x-4
*d+4/3*f+2*e-2*g)/(x^2+x+1)^2+1/96*(27*d-12*f)*ln(x^2+x+1)+1/72*(13/2*d-16
*e+f+8*g)*arctan(1/3*(1+2*x)*3^(1/2))*3^(1/2)
```

3.49.5 Fricas [A] (verification not implemented)

Time = 0.49 (sec) , antiderivative size = 435, normalized size of antiderivative = 1.79

$$\int \frac{d+ex+fx^2+gx^3}{(1+x^2+x^4)^3}\,dx =$$
$$-\frac{84\,(d-f)x^7-48\,(2\,e-g)x^6+60\,(d-2\,f)x^5-72\,(2\,e-g)x^4+84\,(d-2\,f)x^3-96\,(2\,e-g)x^2-2\,\sqrt{}}{}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^3,x, algorithm="fricas")
```

output
```
-1/288*(84*(d - f)*x^7 - 48*(2*e - g)*x^6 + 60*(d - 2*f)*x^5 - 72*(2*e - g
)*x^4 + 84*(d - 2*f)*x^3 - 96*(2*e - g)*x^2 - 2*sqrt(3)*((13*d - 32*e + 2*
f + 16*g)*x^8 + 2*(13*d - 32*e + 2*f + 16*g)*x^6 + 3*(13*d - 32*e + 2*f +
16*g)*x^4 + 2*(13*d - 32*e + 2*f + 16*g)*x^2 + 13*d - 32*e + 2*f + 16*g)*a
rctan(1/3*sqrt(3)*(2*x + 1)) - 2*sqrt(3)*((13*d + 32*e + 2*f - 16*g)*x^8 +
 2*(13*d + 32*e + 2*f - 16*g)*x^6 + 3*(13*d + 32*e + 2*f - 16*g)*x^4 + 2*(
13*d + 32*e + 2*f - 16*g)*x^2 + 13*d + 32*e + 2*f - 16*g)*arctan(1/3*sqrt(
3)*(2*x - 1)) - 12*(4*d + 5*f)*x - 9*((9*d - 4*f)*x^8 + 2*(9*d - 4*f)*x^6
+ 3*(9*d - 4*f)*x^4 + 2*(9*d - 4*f)*x^2 + 9*d - 4*f)*log(x^2 + x + 1) + 9*
((9*d - 4*f)*x^8 + 2*(9*d - 4*f)*x^6 + 3*(9*d - 4*f)*x^4 + 2*(9*d - 4*f)*x
^2 + 9*d - 4*f)*log(x^2 - x + 1) - 72*e + 72*g)/(x^8 + 2*x^6 + 3*x^4 + 2*x
^2 + 1)
```

3.49.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3}{\left(1+x^2+x^4\right)^3}\,dx = \text{Timed out}$$

input
```
integrate((g*x**3+f*x**2+e*x+d)/(x**4+x**2+1)**3,x)
```

output
```
Timed out
```

3.49.7 Maxima [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 200, normalized size of antiderivative = 0.82

$$\int \frac{d+ex+fx^2+gx^3}{\left(1+x^2+x^4\right)^3}\,dx = \frac{1}{144}\,\sqrt{3}(13\,d-32\,e+2\,f+16\,g)\arctan\left(\frac{1}{3}\,\sqrt{3}(2\,x+1)\right)$$
$$+\frac{1}{144}\,\sqrt{3}(13\,d+32\,e+2\,f-16\,g)\arctan\left(\frac{1}{3}\,\sqrt{3}(2\,x-1)\right)$$
$$+\frac{1}{32}\,(9\,d-4\,f)\log\left(x^2+x+1\right)-\frac{1}{32}\,(9\,d-4\,f)\log\left(x^2-x+1\right)$$
$$-\frac{7\,(d-f)x^7-4\,(2\,e-g)x^6+5\,(d-2\,f)x^5-6\,(2\,e-g)x^4+7\,(d-2\,f)x^3-8\,(2\,e-g)x^2-(4\,d+5\,f)}{24\,(x^8+2\,x^6+3\,x^4+2\,x^2+1)}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^3,x, algorithm="maxima")
```

output
```
1/144*sqrt(3)*(13*d - 32*e + 2*f + 16*g)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1
/144*sqrt(3)*(13*d + 32*e + 2*f - 16*g)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/
32*(9*d - 4*f)*log(x^2 + x + 1) - 1/32*(9*d - 4*f)*log(x^2 - x + 1) - 1/24
*(7*(d - f)*x^7 - 4*(2*e - g)*x^6 + 5*(d - 2*f)*x^5 - 6*(2*e - g)*x^4 + 7*
(d - 2*f)*x^3 - 8*(2*e - g)*x^2 - (4*d + 5*f)*x - 6*e + 6*g)/(x^8 + 2*x^6
+ 3*x^4 + 2*x^2 + 1)
```

3.49.8 Giac [A] (verification not implemented)

Time = 0.31 (sec) , antiderivative size = 192, normalized size of antiderivative = 0.79

$$\int \frac{d+ex+fx^2+gx^3}{(1+x^2+x^4)^3}\,dx = \frac{1}{144}\,\sqrt{3}(13\,d-32\,e+2\,f+16\,g)\arctan\left(\frac{1}{3}\,\sqrt{3}(2\,x+1)\right)$$
$$+\frac{1}{144}\,\sqrt{3}(13\,d+32\,e+2\,f-16\,g)\arctan\left(\frac{1}{3}\,\sqrt{3}(2\,x-1)\right)$$
$$+\frac{1}{32}\,(9\,d-4\,f)\log\left(x^2+x+1\right)-\frac{1}{32}\,(9\,d-4\,f)\log\left(x^2-x+1\right)$$
$$-\frac{7\,dx^7-7\,fx^7-8\,ex^6+4\,gx^6+5\,dx^5-10\,fx^5-12\,ex^4+6\,gx^4+7\,dx^3-14\,fx^3-16\,ex^2+8\,gx^2-4}{24\left(x^4+x^2+1\right)^2}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^3,x, algorithm="giac")
```

output
```
1/144*sqrt(3)*(13*d - 32*e + 2*f + 16*g)*arctan(1/3*sqrt(3)*(2*x + 1)) + 1
/144*sqrt(3)*(13*d + 32*e + 2*f - 16*g)*arctan(1/3*sqrt(3)*(2*x - 1)) + 1/
32*(9*d - 4*f)*log(x^2 + x + 1) - 1/32*(9*d - 4*f)*log(x^2 - x + 1) - 1/24
*(7*d*x^7 - 7*f*x^7 - 8*e*x^6 + 4*g*x^6 + 5*d*x^5 - 10*f*x^5 - 12*e*x^4 +
6*g*x^4 + 7*d*x^3 - 14*f*x^3 - 16*e*x^2 + 8*g*x^2 - 4*d*x - 5*f*x - 6*e +
6*g)/(x^4 + x^2 + 1)^2
```

3.49.9 Mupad [B] (verification not implemented)

Time = 8.07 (sec) , antiderivative size = 295, normalized size of antiderivative = 1.21

$$\int \frac{d+ex+fx^2+gx^3}{(1+x^2+x^4)^3}\,dx$$
$$=\frac{\left(\frac{7\,f}{24}-\frac{7\,d}{24}\right)\,x^7+\left(\frac{e}{3}-\frac{g}{6}\right)\,x^6+\left(\frac{5\,f}{12}-\frac{5\,d}{24}\right)\,x^5+\left(\frac{e}{2}-\frac{g}{4}\right)\,x^4+\left(\frac{7\,f}{12}-\frac{7\,d}{24}\right)\,x^3+\left(\frac{2\,e}{3}-\frac{g}{3}\right)\,x^2+\left(\frac{d}{6}+\frac{5\,f}{24}\right)\,x+}{x^8+2\,x^6+3\,x^4+2\,x^2+1}$$
$$-\ln\left(x-\frac{1}{2}-\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\,\left(\frac{9\,d}{32}-\frac{f}{8}+\frac{\sqrt{3}\,d\,13\mathrm{i}}{288}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{144}-\frac{\sqrt{3}\,g\,1\mathrm{i}}{18}\right)$$
$$-\ln\left(x+\frac{1}{2}-\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\,\left(\frac{f}{8}-\frac{9\,d}{32}+\frac{\sqrt{3}\,d\,13\mathrm{i}}{288}-\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{144}+\frac{\sqrt{3}\,g\,1\mathrm{i}}{18}\right)$$
$$+\ln\left(x-\frac{1}{2}+\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\,\left(\frac{f}{8}-\frac{9\,d}{32}+\frac{\sqrt{3}\,d\,13\mathrm{i}}{288}+\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{144}-\frac{\sqrt{3}\,g\,1\mathrm{i}}{18}\right)$$
$$+\ln\left(x+\frac{1}{2}+\frac{\sqrt{3}\,1\mathrm{i}}{2}\right)\,\left(\frac{9\,d}{32}-\frac{f}{8}+\frac{\sqrt{3}\,d\,13\mathrm{i}}{288}-\frac{\sqrt{3}\,e\,1\mathrm{i}}{9}+\frac{\sqrt{3}\,f\,1\mathrm{i}}{144}+\frac{\sqrt{3}\,g\,1\mathrm{i}}{18}\right)$$

input
```
int((d + e*x + f*x^2 + g*x^3)/(x^2 + x^4 + 1)^3,x)
```

output
```
(e/4 - g/4 - x^5*((5*d)/24 - (5*f)/12) - x^3*((7*d)/24 - (7*f)/12) - x^7*(
(7*d)/24 - (7*f)/24) + x^2*((2*e)/3 - g/3) + x^4*(e/2 - g/4) + x^6*(e/3 -
g/6) + x*(d/6 + (5*f)/24))/(2*x^2 + 3*x^4 + 2*x^6 + x^8 + 1) - log(x - (3^
(1/2)*1i)/2 - 1/2)*((9*d)/32 - f/8 + (3^(1/2)*d*13i)/288 + (3^(1/2)*e*1i)/
9 + (3^(1/2)*f*1i)/144 - (3^(1/2)*g*1i)/18) - log(x - (3^(1/2)*1i)/2 + 1/2
)*(f/8 - (9*d)/32 + (3^(1/2)*d*13i)/288 - (3^(1/2)*e*1i)/9 + (3^(1/2)*f*1i
)/144 + (3^(1/2)*g*1i)/18) + log(x + (3^(1/2)*1i)/2 - 1/2)*(f/8 - (9*d)/32
 + (3^(1/2)*d*13i)/288 + (3^(1/2)*e*1i)/9 + (3^(1/2)*f*1i)/144 - (3^(1/2)*
g*1i)/18) + log(x + (3^(1/2)*1i)/2 + 1/2)*((9*d)/32 - f/8 + (3^(1/2)*d*13i
)/288 - (3^(1/2)*e*1i)/9 + (3^(1/2)*f*1i)/144 + (3^(1/2)*g*1i)/18)
```

3.49. $\int \frac{d+ex+fx^2+gx^3}{(1+x^2+x^4)^3}\,dx$

3.50 $\int \frac{d+ex+fx^2+gx^3+hx^4}{(1+x^2+x^4)^3}\,dx$

3.50.1 Optimal result

Integrand size = 31, antiderivative size = 263

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3+hx^4}{(1+x^2+x^4)^3}\,dx &= \frac{e-2g+(2e-g)x^2}{12(1+x^2+x^4)^2}+\frac{x(d+f-2h-(d-2f+h)x^2)}{12(1+x^2+x^4)^2}\\ &+\frac{(2e-g)(1+2x^2)}{12(1+x^2+x^4)}\\ &+\frac{x(2d+3f-h-(7d-7f+4h)x^2)}{24(1+x^2+x^4)}\\ &-\frac{(13d+2f+h)\arctan\left(\frac{1-2x}{\sqrt{3}}\right)}{48\sqrt{3}}\\ &+\frac{(13d+2f+h)\arctan\left(\frac{1+2x}{\sqrt{3}}\right)}{48\sqrt{3}}\\ &+\frac{(2e-g)\arctan\left(\frac{1+2x^2}{\sqrt{3}}\right)}{3\sqrt{3}}\\ &-\frac{1}{32}(9d-4f+3h)\log(1-x+x^2)\\ &+\frac{1}{32}(9d-4f+3h)\log(1+x+x^2)\end{aligned}$$

output
```
1/12*(e-2*g+(2*e-g)*x^2)/(x^4+x^2+1)^2+1/12*x*(d+f-2*h-(d-2*f+h)*x^2)/(x^4
+x^2+1)^2+1/12*(2*e-g)*(2*x^2+1)/(x^4+x^2+1)+1/24*x*(2*d+3*f-h-(7*d-7*f+4*
h)*x^2)/(x^4+x^2+1)-1/32*(9*d-4*f+3*h)*ln(x^2-x+1)+1/32*(9*d-4*f+3*h)*ln(x
^2+x+1)-1/144*(13*d+2*f+h)*arctan(1/3*(1-2*x)*3^(1/2))*3^(1/2)+1/144*(13*d
+2*f+h)*arctan(1/3*(1+2*x)*3^(1/2))*3^(1/2)+1/9*(2*e-g)*arctan(1/3*(2*x^2+
1)*3^(1/2))*3^(1/2)
```

3.50.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.57 (sec) , antiderivative size = 303, normalized size of antiderivative = 1.15

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(1+x^2+x^4)^3}\,dx$$

$$= \frac{1}{144}\left(-\frac{6(-4e(1+2x^2)+g(2+4x^2)+x(-2d-3f+h+7dx^2-7fx^2+4hx^2))}{1+x^2+x^4} + \frac{12(e+2ex^2-g(2+x^2)+x(d+f-dx^2+2fx^2-h(2+x^2)))}{(1+x^2+x^4)^2} - \frac{\left(\left(-47i+7\sqrt{3}\right)d+\left(17i-7\sqrt{3}\right)f+2\left(-7i+2\sqrt{3}\right)h\right)\arctan\left(\frac{1}{2}\left(-i+\sqrt{3}\right)x\right)}{\sqrt{\frac{1}{6}\left(1+i\sqrt{3}\right)}} - \frac{\left(\left(47i+7\sqrt{3}\right)d-\left(17i+7\sqrt{3}\right)f+2\left(7i+2\sqrt{3}\right)h\right)\arctan\left(\frac{1}{2}\left(i+\sqrt{3}\right)x\right)}{\sqrt{\frac{1}{6}\left(1-i\sqrt{3}\right)}} - 16\sqrt{3}(2e-g)\arctan\left(\frac{\sqrt{3}}{1+2x^2}\right)\right)$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4)/(1 + x^2 + x^4)^3,x]
```

output
```
((-6*(-4*e*(1 + 2*x^2) + g*(2 + 4*x^2) + x*(-2*d - 3*f + h + 7*d*x^2 - 7*f
*x^2 + 4*h*x^2)))/(1 + x^2 + x^4) + (12*(e + 2*e*x^2 - g*(2 + x^2) + x*(d
+ f - d*x^2 + 2*f*x^2 - h*(2 + x^2))))/(1 + x^2 + x^4)^2 - (((-47*I + 7*Sq
rt[3])*d + (17*I - 7*Sqrt[3])*f + 2*(-7*I + 2*Sqrt[3])*h)*ArcTan[((-I + Sq
rt[3])*x)/2])/Sqrt[(1 + I*Sqrt[3])/6] - (((47*I + 7*Sqrt[3])*d - (17*I + 7
*Sqrt[3])*f + 2*(7*I + 2*Sqrt[3])*h)*ArcTan[((I + Sqrt[3])*x)/2])/Sqrt[(1
- I*Sqrt[3])/6] - 16*Sqrt[3]*(2*e - g)*ArcTan[Sqrt[3]/(1 + 2*x^2)])/144
```

3.50.3 Rubi [A] (verified)

Time = 0.66 (sec) , antiderivative size = 280, normalized size of antiderivative = 1.06, number of steps used = 16, number of rules used = 15, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.484, Rules used = {2202, 1576, 1159, 1086, 1083, 217, 2206, 1492, 27, 1483, 1142, 25, 1083, 217, 1103}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(x^4+x^2+1)^3}dx$$

$$\Big\downarrow \text{ 2202}$$

$$\int \frac{hx^4+fx^2+d}{(x^4+x^2+1)^3}dx+\int \frac{x(gx^2+e)}{(x^4+x^2+1)^3}dx$$

$$\Big\downarrow \text{ 1576}$$

$$\int \frac{hx^4+fx^2+d}{(x^4+x^2+1)^3}dx+\frac{1}{2}\int \frac{gx^2+e}{(x^4+x^2+1)^3}dx^2$$

$$\Big\downarrow \text{ 1159}$$

$$\int \frac{hx^4+fx^2+d}{(x^4+x^2+1)^3}dx+\frac{1}{2}\left(\frac{1}{2}(2e-g)\int \frac{1}{(x^4+x^2+1)^2}dx^2+\frac{x^2(2e-g)+e-2g}{6(x^4+x^2+1)^2}\right)$$

$$\Big\downarrow \text{ 1086}$$

$$\int \frac{hx^4+fx^2+d}{(x^4+x^2+1)^3}dx+ \frac{1}{2}\left(\frac{1}{2}(2e-g)\left(\frac{2}{3}\int \frac{1}{x^4+x^2+1}dx^2+\frac{2x^2+1}{3(x^4+x^2+1)}\right)+\frac{x^2(2e-g)+e-2g}{6(x^4+x^2+1)^2}\right)$$

$$\Big\downarrow \text{ 1083}$$

$$\int \frac{hx^4+fx^2+d}{(x^4+x^2+1)^3}dx+ \frac{1}{2}\left(\frac{1}{2}(2e-g)\left(\frac{2x^2+1}{3(x^4+x^2+1)}-\frac{4}{3}\int \frac{1}{-x^4-3}d(2x^2+1)\right)+\frac{x^2(2e-g)+e-2g}{6(x^4+x^2+1)^2}\right)$$

$$\Big\downarrow \text{ 217}$$

$$\int \frac{hx^4+fx^2+d}{(x^4+x^2+1)^3}dx+ \frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3(x^4+x^2+1)}\right)(2e-g)+\frac{x^2(2e-g)+e-2g}{6(x^4+x^2+1)^2}\right)$$

$$\Big\downarrow \text{ 2206}$$

$$\frac{1}{12}\int \frac{-5(d-2f+h)x^2+11d-f+2h}{\left(x^4+x^2+1\right)^2}dx+\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g)+\frac{x^2(2e-g)+e-2g}{6\left(x^4+x^2+1\right)^2}\right)+\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \ 1492$$

$$\frac{1}{12}\left(\frac{1}{6}\int \frac{3\left(5(4d-f+h)-(7d-7f+4h)x^2\right)}{x^4+x^2+1}dx+\frac{x\left(-\left(x^2(7d-7f+4h)\right)+2d+3f-h\right)}{2\left(x^4+x^2+1\right)}\right)+\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g)+\frac{x^2(2e-g)+e-2g}{6\left(x^4+x^2+1\right)^2}\right)+\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \ 27$$

$$\frac{1}{12}\left(\frac{1}{2}\int \frac{5(4d-f+h)-(7d-7f+4h)x^2}{x^4+x^2+1}dx+\frac{x\left(-\left(x^2(7d-7f+4h)\right)+2d+3f-h\right)}{2\left(x^4+x^2+1\right)}\right)+\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g)+\frac{x^2(2e-g)+e-2g}{6\left(x^4+x^2+1\right)^2}\right)+\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \ 1483$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\int \frac{5(4d-f+h)-3(9d-4f+3h)x}{x^2-x+1}dx+\frac{1}{2}\int \frac{5(4d-f+h)+3(9d-4f+3h)x}{x^2+x+1}dx\right)+\frac{x\left(-\left(x^2(7d-\right.\right.}{2}\right.$$
$$\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g)+\frac{x^2(2e-g)+e-2g}{6\left(x^4+x^2+1\right)^2}\right)+\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \ 1142$$

3.50. $\int \frac{d+ex+fx^2+gx^3+hx^4}{\left(1+x^2+x^4\right)^3}\,dx$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}(13d+2f+h)\int\frac{1}{x^2-x+1}dx-\frac{3}{2}(9d-4f+3h)\int-\frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(13d+2f+h)\int\frac{}{x^2}\right.\right.\right.$$
$$\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g)+\frac{x^2(2e-g)+e-2g}{6\left(x^4+x^2+1\right)^2}\right)+$$
$$\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \; 25$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}(13d+2f+h)\int\frac{1}{x^2-x+1}dx+\frac{3}{2}(9d-4f+3h)\int\frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(13d+2f+h)\int\frac{}{x^2-}\right.\right.\right.$$
$$\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g)+\frac{x^2(2e-g)+e-2g}{6\left(x^4+x^2+1\right)^2}\right)+$$
$$\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \; 1083$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{3}{2}(9d-4f+3h)\int\frac{1-2x}{x^2-x+1}dx-(13d+2f+h)\int\frac{1}{-(2x-1)^2-3}d(2x-1)\right)+\frac{1}{2}\left(\frac{3}{2}(9d-4f+3\right.\right.\right.$$
$$\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g)+\frac{x^2(2e-g)+e-2g}{6\left(x^4+x^2+1\right)^2}\right)+$$
$$\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \; 217$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{3}{2}(9d-4f+3h)\int\frac{1-2x}{x^2-x+1}dx+\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(13d+2f+h)}{\sqrt{3}}\right)+\frac{1}{2}\left(\frac{3}{2}(9d-4f+3h)\int\frac{2x}{x^2+}\right.\right.\right.$$
$$\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g)+\frac{x^2(2e-g)+e-2g}{6\left(x^4+x^2+1\right)^2}\right)+$$
$$\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \; 1103$$

3.50. $\int \frac{d+ex+fx^2+gx^3+hx^4}{\left(1+x^2+x^4\right)^3}\,dx$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(13d+2f+h)}{\sqrt{3}}-\frac{3}{2}\log\left(x^2-x+1\right)(9d-4f+3h)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(13d+2f\right.\right.\right.$$

$$\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g)+\frac{x^2(2e-g)+e-2g}{6\left(x^4+x^2+1\right)^2}\right)+$$

$$\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{12\left(x^4+x^2+1\right)^2}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4)/(1 + x^2 + x^4)^3,x]
```

output
```
(x*(d + f - 2*h - (d - 2*f + h)*x^2))/(12*(1 + x^2 + x^4)^2) + ((e - 2*g +
 (2*e - g)*x^2)/(6*(1 + x^2 + x^4)^2) + ((2*e - g)*((1 + 2*x^2)/(3*(1 + x^
2 + x^4)) + (4*ArcTan[(1 + 2*x^2)/Sqrt[3]])/(3*Sqrt[3])))/2)/2 + ((x*(2*d
+ 3*f - h - (7*d - 7*f + 4*h)*x^2))/(2*(1 + x^2 + x^4)) + ((((13*d + 2*f +
 h)*ArcTan[(-1 + 2*x)/Sqrt[3]])/Sqrt[3] - (3*(9*d - 4*f + 3*h)*Log[1 - x +
 x^2])/2)/2 + (((13*d + 2*f + h)*ArcTan[(1 + 2*x)/Sqrt[3]])/Sqrt[3] + (3*(
9*d - 4*f + 3*h)*Log[1 + x + x^2])/2)/2)/2)/12
```

3.50.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 217
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1086
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] - Simp[2*c*((2*p +
 3)/((p + 1)*(b^2 - 4*a*c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; Fre
eQ[{a, b, c}, x] && ILtQ[p, -1]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1159
```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[((b*d - 2*a*e + (2*c*d - b*e)*x)/((p + 1)*(b^2 - 4*a*c)))*(a + b*
x + c*x^2)^(p + 1), x] - Simp[(2*p + 3)*((2*c*d - b*e)/((p + 1)*(b^2 - 4*a*
c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; FreeQ[{a, b, c, d, e}, x] &
& LtQ[p, -1] && NeQ[p, -3/2]
```

rule 1483
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[a/c, 2]}, With[{r = Rt[2*q - b/c, 2]}, Simp[1/(2*c*q*r)   In
t[(d*r - (d - e*q)*x)/(q - r*x + x^2), x], x] + Simp[1/(2*c*q*r)   Int[(d*r
 + (d - e*q)*x)/(q + r*x + x^2), x], x]]] /; FreeQ[{a, b, c, d, e}, x] && N
eQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NegQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

3.50. $\int \frac{d+ex+fx^2+gx^3+hx^4}{(1+x^2+x^4)^3}\,dx$

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2206
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> With[{d =
 Coeff[PolynomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 0], e = Coeff[Poly
nomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 2]}, Simp[x*(a + b*x^2 + c*x^
4)^(p + 1)*((a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)/(2*a*(p + 1)*(b
^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(a + b*x^2 + c
*x^4)^(p + 1)*ExpandToSum[2*a*(p + 1)*(b^2 - 4*a*c)*PolynomialQuotient[Px,
a + b*x^2 + c*x^4, x] + b^2*d*(2*p + 3) - 2*a*c*d*(4*p + 5) - a*b*e + c*(4*
p + 7)*(b*d - 2*a*e)*x^2, x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x
^2] && Expon[Px, x^2] > 1 && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1]
```

3.50.4 Maple [A] (verified)

Time = 0.43 (sec) , antiderivative size = 262, normalized size of antiderivative = 1.00

method	result
default	$-\frac{\left(\frac{7d}{3}-\frac{7f}{3}+\frac{4h}{3}-\frac{4e}{3}-\frac{g}{3}\right)x^3+(-6d+4f-2h+2g)x^2+\left(\frac{20d}{3}-\frac{13f}{3}+\frac{5h}{3}+\frac{e}{3}-\frac{8g}{3}\right)x-4d+\frac{4f}{3}-2e+2g}{16(x^2-x+1)^2}-\frac{(27d-12f+9h)\ln(x^2-x+1)}{96}$
risch	Expression too large to display

input
```
int((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^3,x,method=_RETURNVERBOSE)
```

output
```
-1/16*((7/3*d-7/3*f+4/3*h-4/3*e-1/3*g)*x^3+(-6*d+4*f-2*h+2*g)*x^2+(20/3*d-
13/3*f+5/3*h+1/3*e-8/3*g)*x-4*d+4/3*f-2*e+2*g)/(x^2-x+1)^2-1/96*(27*d-12*f
+9*h)*ln(x^2-x+1)-1/72*(-13/2*d-16*e-f+8*g-1/2*h)*3^(1/2)*arctan(1/3*(2*x-
1)*3^(1/2))+1/16*((-7/3*d+7/3*f-4/3*h-4/3*e-1/3*g)*x^3+(-6*d+4*f-2*h-2*g)*
x^2+(-20/3*d+13/3*f-5/3*h+1/3*e-8/3*g)*x-4*d+4/3*f+2*e-2*g)/(x^2+x+1)^2+1/
96*(27*d-12*f+9*h)*ln(x^2+x+1)+1/72*(13/2*d-16*e+f+8*g+1/2*h)*arctan(1/3*(
1+2*x)*3^(1/2))*3^(1/2)
```

3.50.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 485 vs. $2(236) = 472$.

Time = 1.22 (sec) , antiderivative size = 485, normalized size of antiderivative = 1.84

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4}{(1 + x^2 + x^4)^3}\, dx = -\frac{12\,(7\,d - 7\,f + 4\,h)x^7 - 48\,(2\,e - g)x^6 + 60\,(d - 2\,f + h)x^5 - 72\,(2\,e - g)x^4 + 84\,(d - 2\,f + h)x^3 - 96}{}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^3,x, algorithm="fricas")
```

output
```
-1/288*(12*(7*d - 7*f + 4*h)*x^7 - 48*(2*e - g)*x^6 + 60*(d - 2*f + h)*x^5
 - 72*(2*e - g)*x^4 + 84*(d - 2*f + h)*x^3 - 96*(2*e - g)*x^2 - 2*sqrt(3)*
((13*d - 32*e + 2*f + 16*g + h)*x^8 + 2*(13*d - 32*e + 2*f + 16*g + h)*x^6
 + 3*(13*d - 32*e + 2*f + 16*g + h)*x^4 + 2*(13*d - 32*e + 2*f + 16*g + h)
*x^2 + 13*d - 32*e + 2*f + 16*g + h)*arctan(1/3*sqrt(3)*(2*x + 1)) - 2*sqr
t(3)*((13*d + 32*e + 2*f - 16*g + h)*x^8 + 2*(13*d + 32*e + 2*f - 16*g + h
)*x^6 + 3*(13*d + 32*e + 2*f - 16*g + h)*x^4 + 2*(13*d + 32*e + 2*f - 16*g
 + h)*x^2 + 13*d + 32*e + 2*f - 16*g + h)*arctan(1/3*sqrt(3)*(2*x - 1)) -
12*(4*d + 5*f - 5*h)*x - 9*((9*d - 4*f + 3*h)*x^8 + 2*(9*d - 4*f + 3*h)*x^
6 + 3*(9*d - 4*f + 3*h)*x^4 + 2*(9*d - 4*f + 3*h)*x^2 + 9*d - 4*f + 3*h)*l
og(x^2 + x + 1) + 9*((9*d - 4*f + 3*h)*x^8 + 2*(9*d - 4*f + 3*h)*x^6 + 3*(
9*d - 4*f + 3*h)*x^4 + 2*(9*d - 4*f + 3*h)*x^2 + 9*d - 4*f + 3*h)*log(x^2
- x + 1) - 72*e + 72*g)/(x^8 + 2*x^6 + 3*x^4 + 2*x^2 + 1)
```

3.50.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4}{(1 + x^2 + x^4)^3}\, dx = \text{Timed out}$$

input
```
integrate((h*x**4+g*x**3+f*x**2+e*x+d)/(x**4+x**2+1)**3,x)
```

output
```
Timed out
```

3.50.7 Maxima [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 217, normalized size of antiderivative = 0.83

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(1+x^2+x^4)^3}\,dx$$
$$= \frac{1}{144}\sqrt{3}(13\,d-32\,e+2\,f+16\,g+h)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x+1)\right)$$
$$+ \frac{1}{144}\sqrt{3}(13\,d+32\,e+2\,f-16\,g+h)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x-1)\right)$$
$$+ \frac{1}{32}(9\,d-4\,f+3\,h)\log\left(x^2+x+1\right)-\frac{1}{32}(9\,d-4\,f+3\,h)\log\left(x^2-x+1\right)$$
$$- \frac{(7\,d-7\,f+4\,h)x^7-4\,(2\,e-g)x^6+5\,(d-2\,f+h)x^5-6\,(2\,e-g)x^4+7\,(d-2\,f+h)x^3-8\,(2\,e-g)}{24\,(x^8+2\,x^6+3\,x^4+2\,x^2+1)}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^3,x, algorithm="maxima")
```

output
```
1/144*sqrt(3)*(13*d - 32*e + 2*f + 16*g + h)*arctan(1/3*sqrt(3)*(2*x + 1))
 + 1/144*sqrt(3)*(13*d + 32*e + 2*f - 16*g + h)*arctan(1/3*sqrt(3)*(2*x -
1)) + 1/32*(9*d - 4*f + 3*h)*log(x^2 + x + 1) - 1/32*(9*d - 4*f + 3*h)*log
(x^2 - x + 1) - 1/24*((7*d - 7*f + 4*h)*x^7 - 4*(2*e - g)*x^6 + 5*(d - 2*f
 + h)*x^5 - 6*(2*e - g)*x^4 + 7*(d - 2*f + h)*x^3 - 8*(2*e - g)*x^2 - (4*d
 + 5*f - 5*h)*x - 6*e + 6*g)/(x^8 + 2*x^6 + 3*x^4 + 2*x^2 + 1)
```

3.50.8 Giac [A] (verification not implemented)

Time = 0.31 (sec) , antiderivative size = 222, normalized size of antiderivative = 0.84

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(1+x^2+x^4)^3}\,dx$$
$$= \frac{1}{144}\sqrt{3}(13\,d-32\,e+2\,f+16\,g+h)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x+1)\right)$$
$$+ \frac{1}{144}\sqrt{3}(13\,d+32\,e+2\,f-16\,g+h)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x-1)\right)$$
$$+ \frac{1}{32}(9\,d-4\,f+3\,h)\log\left(x^2+x+1\right)-\frac{1}{32}(9\,d-4\,f+3\,h)\log\left(x^2-x+1\right)$$
$$- \frac{7\,dx^7-7\,fx^7+4\,hx^7-8\,ex^6+4\,gx^6+5\,dx^5-10\,fx^5+5\,hx^5-12\,ex^4+6\,gx^4+7\,dx^3-14\,fx^3+7\,h}{24\,(x^4+x^2+1)^2}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^3,x, algorithm="giac")
```

output
```
1/144*sqrt(3)*(13*d - 32*e + 2*f + 16*g + h)*arctan(1/3*sqrt(3)*(2*x + 1))
 + 1/144*sqrt(3)*(13*d + 32*e + 2*f - 16*g + h)*arctan(1/3*sqrt(3)*(2*x -
1)) + 1/32*(9*d - 4*f + 3*h)*log(x^2 + x + 1) - 1/32*(9*d - 4*f + 3*h)*log
(x^2 - x + 1) - 1/24*(7*d*x^7 - 7*f*x^7 + 4*h*x^7 - 8*e*x^6 + 4*g*x^6 + 5*
d*x^5 - 10*f*x^5 + 5*h*x^5 - 12*e*x^4 + 6*g*x^4 + 7*d*x^3 - 14*f*x^3 + 7*h
*x^3 - 16*e*x^2 + 8*g*x^2 - 4*d*x - 5*f*x + 5*h*x - 6*e + 6*g)/(x^4 + x^2
+ 1)^2
```

3.50.9 Mupad [B] (verification not implemented)

Time = 3.44 (sec) , antiderivative size = 1611, normalized size of antiderivative = 6.13

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(1+x^2+x^4)^3}\,dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4)/(x^2 + x^4 + 1)^3,x)
```

output
```
(e/4 - g/4 + x^2*((2*e)/3 - g/3) + x^4*(e/2 - g/4) + x^6*(e/3 - g/6) + x*(
d/6 + (5*f)/24 - (5*h)/24) - x^7*((7*d)/24 - (7*f)/24 + h/6) - x^5*((5*d)/
24 - (5*f)/12 + (5*h)/24) - x^3*((7*d)/24 - (7*f)/12 + (7*h)/24))/(2*x^2 +
 3*x^4 + 2*x^6 + x^8 + 1) - log(960*d*g - 2763*d*f - 1920*d*e + 480*e*f +
1971*d*h - 480*e*h - 240*f*g - 981*f*h + 240*g*h + 3^(1/2)*d^2*1620i + 3^(
1/2)*f^2*180i + 3^(1/2)*h^2*135i - 3807*d^2*x - 612*f^2*x - 378*h^2*x + 27
54*d^2 + 684*f^2 + 351*h^2 + 3^(1/2)*d*e*1088i - 3^(1/2)*d*f*1125i - 3^(1/
2)*d*g*544i - 3^(1/2)*e*f*608i + 3^(1/2)*d*h*945i + 3^(1/2)*e*h*416i + 3^(
1/2)*f*g*304i - 3^(1/2)*f*h*315i - 3^(1/2)*g*h*208i - 672*d*e*x + 3069*d*f
*x + 336*d*g*x + 672*e*f*x - 2403*d*h*x - 384*e*h*x - 336*f*g*x + 963*f*h*
x + 192*g*h*x + 3^(1/2)*d^2*x*567i + 3^(1/2)*f^2*x*252i + 3^(1/2)*h^2*x*10
8i - 3^(1/2)*d*f*x*819i + 3^(1/2)*d*g*x*752i + 3^(1/2)*e*f*x*544i + 3^(1/2
)*d*h*x*513i - 3^(1/2)*e*h*x*448i - 3^(1/2)*f*g*x*272i - 3^(1/2)*f*h*x*333
i + 3^(1/2)*g*h*x*224i - 3^(1/2)*d*e*x*1504i)*((9*d)/32 - f/8 + (3*h)/32 +
 (3^(1/2)*d*13i)/288 + (3^(1/2)*e*1i)/9 + (3^(1/2)*f*1i)/144 - (3^(1/2)*g*
1i)/18 + (3^(1/2)*h*1i)/288) - log(1920*d*e - 2763*d*f - 960*d*g - 480*e*f
 + 1971*d*h + 480*e*h + 240*f*g - 981*f*h - 240*g*h - 3^(1/2)*d^2*1620i -
3^(1/2)*f^2*180i - 3^(1/2)*h^2*135i + 3807*d^2*x + 612*f^2*x + 378*h^2*x +
 2754*d^2 + 684*f^2 + 351*h^2 + 3^(1/2)*d*e*1088i + 3^(1/2)*d*f*1125i - 3^
(1/2)*d*g*544i - 3^(1/2)*e*f*608i - 3^(1/2)*d*h*945i + 3^(1/2)*e*h*416i...
```

3.51 $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^3}\,dx$

3.51.1 Optimal result

Integrand size = 36, antiderivative size = 269

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^3}\,dx &= \frac{x(d+f-2h-(d-2f+h)x^2)}{12\,(1+x^2+x^4)^2} \\ &+ \frac{e-2g+i+(2e-g-i)x^2}{12\,(1+x^2+x^4)^2} \\ &+ \frac{(2e-g+i)\,(1+2x^2)}{12\,(1+x^2+x^4)} \\ &+ \frac{x(2d+3f-h-(7d-7f+4h)x^2)}{24\,(1+x^2+x^4)} \\ &- \frac{(13d+2f+h)\arctan\left(\frac{1-2x}{\sqrt{3}}\right)}{48\sqrt{3}} \\ &+ \frac{(13d+2f+h)\arctan\left(\frac{1+2x}{\sqrt{3}}\right)}{48\sqrt{3}} \\ &+ \frac{(2e-g+i)\arctan\left(\frac{1+2x^2}{\sqrt{3}}\right)}{3\sqrt{3}} \\ &- \frac{1}{32}(9d-4f+3h)\log\left(1-x+x^2\right) \\ &+ \frac{1}{32}(9d-4f+3h)\log\left(1+x+x^2\right)\end{aligned}$$

output
```
1/12*x*(d+f-2*h-(d-2*f+h)*x^2)/(x^4+x^2+1)^2+1/12*(e-2*g+i+(2*e-g-i)*x^2)/
(x^4+x^2+1)^2+1/12*(2*e-g+i)*(2*x^2+1)/(x^4+x^2+1)+1/24*x*(2*d+3*f-h-(7*d-
7*f+4*h)*x^2)/(x^4+x^2+1)-1/32*(9*d-4*f+3*h)*ln(x^2-x+1)+1/32*(9*d-4*f+3*h
)*ln(x^2+x+1)-1/144*(13*d+2*f+h)*arctan(1/3*(1-2*x)*3^(1/2))*3^(1/2)+1/144
*(13*d+2*f+h)*arctan(1/3*(1+2*x)*3^(1/2))*3^(1/2)+1/9*(2*e-g+i)*arctan(1/3
*(2*x^2+1)*3^(1/2))*3^(1/2)
```

3.51.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 0.58 (sec) , antiderivative size = 325, normalized size of antiderivative = 1.21

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^3}\,dx$$
$$= \frac{1}{144}\left(\frac{12(e+i+dx+fx-2hx+2ex^2-ix^2-dx^3+2fx^3-hx^3-g(2+x^2))}{(1+x^2+x^4)^2} + \frac{6(2i+2dx+3fx-hx+4ix^2-7dx^3+7fx^3-4hx^3-2g(1+2x^2)+e(4+8x^2))}{1+x^2+x^4} - \frac{\left(\left(-47i+7\sqrt{3}\right)d+\left(17i-7\sqrt{3}\right)f+2\left(-7i+2\sqrt{3}\right)h\right)\arctan\left(\frac{1}{2}\left(-i+\sqrt{3}\right)x\right)}{\sqrt{\frac{1}{6}\left(1+i\sqrt{3}\right)}} - \frac{\left(\left(47i+7\sqrt{3}\right)d-\left(17i+7\sqrt{3}\right)f+2\left(7i+2\sqrt{3}\right)h\right)\arctan\left(\frac{1}{2}\left(i+\sqrt{3}\right)x\right)}{\sqrt{\frac{1}{6}\left(1-i\sqrt{3}\right)}} - 16\sqrt{3}(2e-g+i)\arctan\left(\frac{\sqrt{3}}{1+2x^2}\right)\right)$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(1 + x^2 + x^4)^3,x]
```

output
```
((12*(e + i + d*x + f*x - 2*h*x + 2*e*x^2 - i*x^2 - d*x^3 + 2*f*x^3 - h*x^
3 - g*(2 + x^2)))/(1 + x^2 + x^4)^2 + (6*(2*i + 2*d*x + 3*f*x - h*x + 4*i*
x^2 - 7*d*x^3 + 7*f*x^3 - 4*h*x^3 - 2*g*(1 + 2*x^2) + e*(4 + 8*x^2)))/(1 +
 x^2 + x^4) - (((-47*I + 7*Sqrt[3])*d + (17*I - 7*Sqrt[3])*f + 2*(-7*I + 2
*Sqrt[3])*h)*ArcTan[((-I + Sqrt[3])*x)/2])/Sqrt[(1 + I*Sqrt[3])/6] - (((47
*I + 7*Sqrt[3])*d - (17*I + 7*Sqrt[3])*f + 2*(7*I + 2*Sqrt[3])*h)*ArcTan[(
(I + Sqrt[3])*x)/2])/Sqrt[(1 - I*Sqrt[3])/6] - 16*Sqrt[3]*(2*e - g + i)*Ar
cTan[Sqrt[3]/(1 + 2*x^2)])/144
```

3.51.3 Rubi [A] (verified)

Time = 0.72 (sec) , antiderivative size = 285, normalized size of antiderivative = 1.06, number of steps used = 17, number of rules used = 16, $\frac{\text{number of rules}}{\text{integrand size}} = 0.444$, Rules used = {2202, 2194, 2191, 27, 1086, 1083, 217, 2206, 1492, 27, 1483, 1142, 25, 1083, 217, 1103}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{\left(x^4+x^2+1\right)^3}dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{hx^4+fx^2+d}{\left(x^4+x^2+1\right)^3}dx+\int \frac{x\left(ix^4+gx^2+e\right)}{\left(x^4+x^2+1\right)^3}dx$$

$$\Big\downarrow \text{2194}$$

$$\int \frac{hx^4+fx^2+d}{\left(x^4+x^2+1\right)^3}dx+\frac{1}{2}\int \frac{ix^4+gx^2+e}{\left(x^4+x^2+1\right)^3}dx^2$$

$$\Big\downarrow \text{2191}$$

$$\int \frac{hx^4+fx^2+d}{\left(x^4+x^2+1\right)^3}dx+\frac{1}{2}\left(\frac{1}{6}\int \frac{3(2e-g+i)}{\left(x^4+x^2+1\right)^2}dx^2+\frac{x^2(2e-g-i)+e-2g+i}{6\left(x^4+x^2+1\right)^2}\right)$$

$$\Big\downarrow \text{27}$$

$$\int \frac{hx^4+fx^2+d}{\left(x^4+x^2+1\right)^3}dx+\frac{1}{2}\left(\frac{1}{2}(2e-g+i)\int \frac{1}{\left(x^4+x^2+1\right)^2}dx^2+\frac{x^2(2e-g-i)+e-2g+i}{6\left(x^4+x^2+1\right)^2}\right)$$

$$\Big\downarrow \text{1086}$$

$$\int \frac{hx^4+fx^2+d}{\left(x^4+x^2+1\right)^3}dx+\frac{1}{2}\left(\frac{1}{2}(2e-g+i)\left(\frac{2}{3}\int \frac{1}{x^4+x^2+1}dx^2+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)+\frac{x^2(2e-g-i)+e-2g+i}{6\left(x^4+x^2+1\right)^2}\right)$$

$$\Big\downarrow \text{1083}$$

$$\int \frac{hx^4+fx^2+d}{\left(x^4+x^2+1\right)^3}dx+\frac{1}{2}\left(\frac{1}{2}(2e-g+i)\left(\frac{2x^2+1}{3\left(x^4+x^2+1\right)}-\frac{4}{3}\int \frac{1}{-x^4-3}d\left(2x^2+1\right)\right)+\frac{x^2(2e-g-i)+e-2g+i}{6\left(x^4+x^2+1\right)^2}\right)$$

$$\Big\downarrow \text{217}$$

$$\int \frac{hx^4+fx^2+d}{\left(x^4+x^2+1\right)^3}dx+\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g+i)+\frac{x^2(2e-g-i)+e-2g+i}{6\left(x^4+x^2+1\right)^2}\right)$$

$$\Big\downarrow \ 2206$$

$$\frac{1}{12}\int \frac{-5(d-2f+h)x^2+11d-f+2h}{\left(x^4+x^2+1\right)^2}dx+\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g+i)+\frac{x^2(2e-g-i)+e-2g+i}{6\left(x^4+x^2+1\right)^2}\right)+\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \ 1492$$

$$\frac{1}{12}\left(\frac{1}{6}\int \frac{3\left(5(4d-f+h)-(7d-7f+4h)x^2\right)}{x^4+x^2+1}dx+\frac{x\left(-\left(x^2(7d-7f+4h)\right)+2d+3f-h\right)}{2\left(x^4+x^2+1\right)}\right)+\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g+i)+\frac{x^2(2e-g-i)+e-2g+i}{6\left(x^4+x^2+1\right)^2}\right)+\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \ 27$$

$$\frac{1}{12}\left(\frac{1}{2}\int \frac{5(4d-f+h)-(7d-7f+4h)x^2}{x^4+x^2+1}dx+\frac{x\left(-\left(x^2(7d-7f+4h)\right)+2d+3f-h\right)}{2\left(x^4+x^2+1\right)}\right)+\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g+i)+\frac{x^2(2e-g-i)+e-2g+i}{6\left(x^4+x^2+1\right)^2}\right)+\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \ 1483$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\int \frac{5(4d-f+h)-3(9d-4f+3h)x}{x^2-x+1}dx+\frac{1}{2}\int \frac{5(4d-f+h)+3(9d-4f+3h)x}{x^2+x+1}dx\right)+\frac{x\left(-\left(x^2(7d-\right.\right.}{2}\right.$$

$$\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g+i)+\frac{x^2(2e-g-i)+e-2g+i}{6\left(x^4+x^2+1\right)^2}\right)+\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{12\left(x^4+x^2+1\right)^2}$$

3.51. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{\left(1+x^2+x^4\right)^3}\,dx$

$$\Big\downarrow \text{1142}$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}(13d+2f+h)\int\frac{1}{x^2-x+1}dx-\frac{3}{2}(9d-4f+3h)\int-\frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(13d+2f+h)\int\frac{}{x^2}\right.\right.\right.$$
$$\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g+i)+\frac{x^2(2e-g-i)+e-2g+i}{6\left(x^4+x^2+1\right)^2}\right)+$$
$$\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \text{25}$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{1}{2}(13d+2f+h)\int\frac{1}{x^2-x+1}dx+\frac{3}{2}(9d-4f+3h)\int\frac{1-2x}{x^2-x+1}dx\right)+\frac{1}{2}\left(\frac{1}{2}(13d+2f+h)\int\frac{}{x^2-}\right.\right.\right.$$
$$\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g+i)+\frac{x^2(2e-g-i)+e-2g+i}{6\left(x^4+x^2+1\right)^2}\right)+$$
$$\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \text{1083}$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{3}{2}(9d-4f+3h)\int\frac{1-2x}{x^2-x+1}dx-(13d+2f+h)\int\frac{1}{-(2x-1)^2-3}d(2x-1)\right)+\frac{1}{2}\left(\frac{3}{2}(9d-4f+3\right.\right.\right.$$
$$\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g+i)+\frac{x^2(2e-g-i)+e-2g+i}{6\left(x^4+x^2+1\right)^2}\right)+$$
$$\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \text{217}$$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{3}{2}(9d-4f+3h)\int\frac{1-2x}{x^2-x+1}dx+\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(13d+2f+h)}{\sqrt{3}}\right)+\frac{1}{2}\left(\frac{3}{2}(9d-4f+3h)\int\frac{2x}{x^2+}\right.\right.\right.$$
$$\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g+i)+\frac{x^2(2e-g-i)+e-2g+i}{6\left(x^4+x^2+1\right)^2}\right)+$$
$$\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{12\left(x^4+x^2+1\right)^2}$$

$$\Big\downarrow \text{1103}$$

3.51. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{\left(1+x^2+x^4\right)^3}\,dx$

$$\frac{1}{12}\left(\frac{1}{2}\left(\frac{1}{2}\left(\frac{\arctan\left(\frac{2x-1}{\sqrt{3}}\right)(13d+2f+h)}{\sqrt{3}}-\frac{3}{2}\log\left(x^2-x+1\right)(9d-4f+3h)\right)+\frac{1}{2}\left(\frac{\arctan\left(\frac{2x+1}{\sqrt{3}}\right)(13d+2f\right.\right.\right.$$

$$\frac{1}{2}\left(\frac{1}{2}\left(\frac{4\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)}{3\sqrt{3}}+\frac{2x^2+1}{3\left(x^4+x^2+1\right)}\right)(2e-g+i)+\frac{x^2(2e-g-i)+e-2g+i}{6\left(x^4+x^2+1\right)^2}\right)+$$

$$\frac{x\left(-\left(x^2(d-2f+h)\right)+d+f-2h\right)}{12\left(x^4+x^2+1\right)^2}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(1 + x^2 + x^4)^3,x]
```

output
```
(x*(d + f - 2*h - (d - 2*f + h)*x^2))/(12*(1 + x^2 + x^4)^2) + ((e - 2*g +
 i + (2*e - g - i)*x^2)/(6*(1 + x^2 + x^4)^2) + ((2*e - g + i)*((1 + 2*x^2
)/(3*(1 + x^2 + x^4)) + (4*ArcTan[(1 + 2*x^2)/Sqrt[3]])/(3*Sqrt[3])))/2)/2
 + ((x*(2*d + 3*f - h - (7*d - 7*f + 4*h)*x^2))/(2*(1 + x^2 + x^4)) + ((((
13*d + 2*f + h)*ArcTan[(-1 + 2*x)/Sqrt[3]])/Sqrt[3] - (3*(9*d - 4*f + 3*h)
*Log[1 - x + x^2])/2)/2 + (((13*d + 2*f + h)*ArcTan[(1 + 2*x)/Sqrt[3]])/Sq
rt[3] + (3*(9*d - 4*f + 3*h)*Log[1 + x + x^2])/2)/2)/2)/12
```

3.51.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 217
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1086
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] - Simp[2*c*((2*p +
 3)/((p + 1)*(b^2 - 4*a*c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; Fre
eQ[{a, b, c}, x] && ILtQ[p, -1]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1483
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[a/c, 2]}, With[{r = Rt[2*q - b/c, 2]}, Simp[1/(2*c*q*r)   In
t[(d*r - (d - e*q)*x)/(q - r*x + x^2), x], x] + Simp[1/(2*c*q*r)   Int[(d*r
 + (d - e*q)*x)/(q + r*x + x^2), x], x]]] /; FreeQ[{a, b, c, d, e}, x] && N
eQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NegQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 2191
```
Int[(Pq_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> With[{Q =
PolynomialQuotient[Pq, a + b*x + c*x^2, x], f = Coeff[PolynomialRemainder[P
q, a + b*x + c*x^2, x], x, 0], g = Coeff[PolynomialRemainder[Pq, a + b*x +
c*x^2, x], x, 1]}, Simp[(b*f - 2*a*g + (2*c*f - b*g)*x)*((a + b*x + c*x^2)^
(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] + Simp[1/((p + 1)*(b^2 - 4*a*c))   Int
[(a + b*x + c*x^2)^(p + 1)*ExpandToSum[(p + 1)*(b^2 - 4*a*c)*Q - (2*p + 3)*
(2*c*f - b*g), x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Pq, x] && NeQ[b^
2 - 4*a*c, 0] && LtQ[p, -1]
```

3.51. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^3}\,dx$

rule 2194
```
Int[(Pq_)*(x_)^(m_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :
> Simp[1/2   Subst[Int[x^((m - 1)/2)*SubstFor[x^2, Pq, x]*(a + b*x + c*x^2)
^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pq, x^2] && IntegerQ
[(m - 1)/2]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2206
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> With[{d =
 Coeff[PolynomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 0], e = Coeff[Poly
nomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 2]}, Simp[x*(a + b*x^2 + c*x^
4)^(p + 1)*((a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)/(2*a*(p + 1)*(b
^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(a + b*x^2 + c
*x^4)^(p + 1)*ExpandToSum[2*a*(p + 1)*(b^2 - 4*a*c)*PolynomialQuotient[Px,
a + b*x^2 + c*x^4, x] + b^2*d*(2*p + 3) - 2*a*c*d*(4*p + 5) - a*b*e + c*(4*
p + 7)*(b*d - 2*a*e)*x^2, x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x
^2] && Expon[Px, x^2] > 1 && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1]
```

3.51.4 Maple [A] (verified)

Time = 0.50 (sec) , antiderivative size = 292, normalized size of antiderivative = 1.09

method	result
default	$-\frac{\left(\frac{7d}{3}-\frac{7f}{3}+\frac{4h}{3}-\frac{4e}{3}-\frac{g}{3}+\frac{i}{3}\right)x^3+(-6d+4f-2h+2g-2i)x^2+\left(\frac{20d}{3}-\frac{13f}{3}+\frac{5h}{3}+\frac{e}{3}-\frac{8g}{3}+\frac{7i}{3}\right)x-4d+\frac{4f}{3}-2e+2g-\frac{4i}{3}}{16(x^2-x+1)^2}-\frac{(27d-12f+9h}{9}$
risch	Expression too large to display

input
```
int((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^3,x,method=_RETURNVERBOSE)
```

output
```
-1/16*((7/3*d-7/3*f+4/3*h-4/3*e-1/3*g+1/3*i)*x^3+(-6*d+4*f-2*h+2*g-2*i)*x^
2+(20/3*d-13/3*f+5/3*h+1/3*e-8/3*g+7/3*i)*x-4*d+4/3*f-2*e+2*g-4/3*i)/(x^2-
x+1)^2-1/96*(27*d-12*f+9*h)*ln(x^2-x+1)-1/72*(-13/2*d-16*e-f+8*g-1/2*h-8*i
)*3^(1/2)*arctan(1/3*(2*x-1)*3^(1/2))+1/16*((-7/3*d+7/3*f-4/3*h-4/3*e-1/3*
g+1/3*i)*x^3+(-6*d+4*f-2*h-2*g+2*i)*x^2+(-20/3*d+13/3*f-5/3*h+1/3*e-8/3*g+
7/3*i)*x-4*d+4/3*f+2*e-2*g+4/3*i)/(x^2+x+1)^2+1/96*(27*d-12*f+9*h)*ln(x^2+
x+1)+1/72*(13/2*d-16*e+f+8*g+1/2*h-8*i)*arctan(1/3*(1+2*x)*3^(1/2))*3^(1/2
)
```

3.51.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 521 vs. $2(242) = 484$.

Time = 5.49 (sec) , antiderivative size = 521, normalized size of antiderivative = 1.94

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^3}\,dx = -\frac{12\,(7\,d-7\,f+4\,h)x^7-48\,(2\,e-g+i)x^6+60\,(d-2\,f+h)x^5-72\,(2\,e-g+i)x^4+84\,(d-2\,f+h)}{}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^3,x, algorithm="fric
as")
```

output
```
-1/288*(12*(7*d - 7*f + 4*h)*x^7 - 48*(2*e - g + i)*x^6 + 60*(d - 2*f + h)
*x^5 - 72*(2*e - g + i)*x^4 + 84*(d - 2*f + h)*x^3 - 48*(4*e - 2*g + i)*x^
2 - 2*sqrt(3)*((13*d - 32*e + 2*f + 16*g + h - 16*i)*x^8 + 2*(13*d - 32*e
+ 2*f + 16*g + h - 16*i)*x^6 + 3*(13*d - 32*e + 2*f + 16*g + h - 16*i)*x^4
 + 2*(13*d - 32*e + 2*f + 16*g + h - 16*i)*x^2 + 13*d - 32*e + 2*f + 16*g
+ h - 16*i)*arctan(1/3*sqrt(3)*(2*x + 1)) - 2*sqrt(3)*((13*d + 32*e + 2*f
- 16*g + h + 16*i)*x^8 + 2*(13*d + 32*e + 2*f - 16*g + h + 16*i)*x^6 + 3*(
13*d + 32*e + 2*f - 16*g + h + 16*i)*x^4 + 2*(13*d + 32*e + 2*f - 16*g + h
 + 16*i)*x^2 + 13*d + 32*e + 2*f - 16*g + h + 16*i)*arctan(1/3*sqrt(3)*(2*
x - 1)) - 12*(4*d + 5*f - 5*h)*x - 9*((9*d - 4*f + 3*h)*x^8 + 2*(9*d - 4*f
 + 3*h)*x^6 + 3*(9*d - 4*f + 3*h)*x^4 + 2*(9*d - 4*f + 3*h)*x^2 + 9*d - 4*
f + 3*h)*log(x^2 + x + 1) + 9*((9*d - 4*f + 3*h)*x^8 + 2*(9*d - 4*f + 3*h)
*x^6 + 3*(9*d - 4*f + 3*h)*x^4 + 2*(9*d - 4*f + 3*h)*x^2 + 9*d - 4*f + 3*h
)*log(x^2 - x + 1) - 72*e + 72*g - 48*i)/(x^8 + 2*x^6 + 3*x^4 + 2*x^2 + 1)
```

3.51.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^3}\,dx = \text{Timed out}$$

input
```
integrate((i*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(x**4+x**2+1)**3,x)
```

output
```
Timed out
```

3.51.7 Maxima [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 229, normalized size of antiderivative = 0.85

$$\begin{aligned}
&\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^3}\,dx \\
&= \frac{1}{144}\sqrt{3}(13\,d-32\,e+2\,f+16\,g+h-16\,i)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x+1)\right) \\
&\quad+\frac{1}{144}\sqrt{3}(13\,d+32\,e+2\,f-16\,g+h+16\,i)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x-1)\right) \\
&\quad+\frac{1}{32}(9\,d-4\,f+3\,h)\log\left(x^2+x+1\right)-\frac{1}{32}(9\,d-4\,f+3\,h)\log\left(x^2-x+1\right) \\
&\quad-\frac{(7\,d-7\,f+4\,h)x^7-4\,(2\,e-g+i)x^6+5\,(d-2\,f+h)x^5-6\,(2\,e-g+i)x^4+7\,(d-2\,f+h)x^3-4\,(}{24\,(x^8+2\,x^6+3\,x^4+2\,x^2+1)}
\end{aligned}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^3,x, algorithm="maxi
ma")
```

output
```
1/144*sqrt(3)*(13*d - 32*e + 2*f + 16*g + h - 16*i)*arctan(1/3*sqrt(3)*(2*
x + 1)) + 1/144*sqrt(3)*(13*d + 32*e + 2*f - 16*g + h + 16*i)*arctan(1/3*s
qrt(3)*(2*x - 1)) + 1/32*(9*d - 4*f + 3*h)*log(x^2 + x + 1) - 1/32*(9*d -
4*f + 3*h)*log(x^2 - x + 1) - 1/24*((7*d - 7*f + 4*h)*x^7 - 4*(2*e - g + i
)*x^6 + 5*(d - 2*f + h)*x^5 - 6*(2*e - g + i)*x^4 + 7*(d - 2*f + h)*x^3 -
4*(4*e - 2*g + i)*x^2 - (4*d + 5*f - 5*h)*x - 6*e + 6*g - 4*i)/(x^8 + 2*x^
6 + 3*x^4 + 2*x^2 + 1)
```

3.51.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 249, normalized size of antiderivative = 0.93

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^3}\,dx$$
$$= \frac{1}{144}\sqrt{3}(13\,d-32\,e+2\,f+16\,g+h-16\,i)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x+1)\right)$$
$$+\frac{1}{144}\sqrt{3}(13\,d+32\,e+2\,f-16\,g+h+16\,i)\arctan\left(\frac{1}{3}\sqrt{3}(2\,x-1)\right)$$
$$+\frac{1}{32}\,(9\,d-4\,f+3\,h)\log\left(x^2+x+1\right)-\frac{1}{32}\,(9\,d-4\,f+3\,h)\log\left(x^2-x+1\right)$$
$$-\frac{7\,dx^7-7\,fx^7+4\,hx^7-8\,ex^6+4\,gx^6-4\,ix^6+5\,dx^5-10\,fx^5+5\,hx^5-12\,ex^4+6\,gx^4-6\,ix^4+7\,dx^3}{24\,(x^4+x^2+1)^2}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4+x^2+1)^3,x, algorithm="giac
")
```

output
```
1/144*sqrt(3)*(13*d - 32*e + 2*f + 16*g + h - 16*i)*arctan(1/3*sqrt(3)*(2*
x + 1)) + 1/144*sqrt(3)*(13*d + 32*e + 2*f - 16*g + h + 16*i)*arctan(1/3*s
qrt(3)*(2*x - 1)) + 1/32*(9*d - 4*f + 3*h)*log(x^2 + x + 1) - 1/32*(9*d -
4*f + 3*h)*log(x^2 - x + 1) - 1/24*(7*d*x^7 - 7*f*x^7 + 4*h*x^7 - 8*e*x^6
+ 4*g*x^6 - 4*i*x^6 + 5*d*x^5 - 10*f*x^5 + 5*h*x^5 - 12*e*x^4 + 6*g*x^4 -
6*i*x^4 + 7*d*x^3 - 14*f*x^3 + 7*h*x^3 - 16*e*x^2 + 8*g*x^2 - 4*i*x^2 - 4*
d*x - 5*f*x + 5*h*x - 6*e + 6*g - 4*i)/(x^4 + x^2 + 1)^2
```

3.51.9 Mupad [B] (verification not implemented)

Time = 12.84 (sec) , antiderivative size = 1963, normalized size of antiderivative = 7.30

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^3}\,dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(x^2 + x^4 + 1)^3,x)
```

output
```
(e/4 - g/4 + i/6 + x*(d/6 + (5*f)/24 - (5*h)/24) - x^7*((7*d)/24 - (7*f)/2
4 + h/6) - x^5*((5*d)/24 - (5*f)/12 + (5*h)/24) - x^3*((7*d)/24 - (7*f)/12
 + (7*h)/24) + x^4*(e/2 - g/4 + i/4) + x^2*((2*e)/3 - g/3 + i/6) + x^6*(e/
3 - g/6 + i/6))/(2*x^2 + 3*x^4 + 2*x^6 + x^8 + 1) - log(960*d*g - 2763*d*f
 - 1920*d*e + 480*e*f + 1971*d*h - 960*d*i - 480*e*h - 240*f*g - 981*f*h +
 240*f*i + 240*g*h - 240*h*i + 3^(1/2)*d^2*1620i + 3^(1/2)*f^2*180i + 3^(1
/2)*h^2*135i - 3807*d^2*x - 612*f^2*x - 378*h^2*x + 2754*d^2 + 684*f^2 + 3
51*h^2 + 3^(1/2)*d*e*1088i - 3^(1/2)*d*f*1125i - 3^(1/2)*d*g*544i - 3^(1/2
)*e*f*608i + 3^(1/2)*d*h*945i + 3^(1/2)*d*i*544i + 3^(1/2)*e*h*416i + 3^(1
/2)*f*g*304i - 3^(1/2)*f*h*315i - 3^(1/2)*f*i*304i - 3^(1/2)*g*h*208i + 3^
(1/2)*h*i*208i - 672*d*e*x + 3069*d*f*x + 336*d*g*x + 672*e*f*x - 2403*d*h
*x - 336*d*i*x - 384*e*h*x - 336*f*g*x + 963*f*h*x + 336*f*i*x + 192*g*h*x
 - 192*h*i*x + 3^(1/2)*d^2*x*567i + 3^(1/2)*f^2*x*252i + 3^(1/2)*h^2*x*108
i - 3^(1/2)*d*f*x*819i + 3^(1/2)*d*g*x*752i + 3^(1/2)*e*f*x*544i + 3^(1/2)
*d*h*x*513i - 3^(1/2)*d*i*x*752i - 3^(1/2)*e*h*x*448i - 3^(1/2)*f*g*x*272i
 - 3^(1/2)*f*h*x*333i + 3^(1/2)*f*i*x*272i + 3^(1/2)*g*h*x*224i - 3^(1/2)*
h*i*x*224i - 3^(1/2)*d*e*x*1504i)*((9*d)/32 - f/8 + (3*h)/32 + (3^(1/2)*d*
13i)/288 + (3^(1/2)*e*1i)/9 + (3^(1/2)*f*1i)/144 - (3^(1/2)*g*1i)/18 + (3^
(1/2)*h*1i)/288 + (3^(1/2)*i*1i)/18) - log(1920*d*e - 2763*d*f - 960*d*g -
 480*e*f + 1971*d*h + 960*d*i + 480*e*h + 240*f*g - 981*f*h - 240*f*i -...
```

3.51. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(1+x^2+x^4)^3}\,dx$

3.52 $\int \frac{d+ex}{(a+bx^2+cx^4)^3}\,dx$

3.52.1 Optimal result

Integrand size = 20, antiderivative size = 474

$$\begin{aligned}\int \frac{d+ex}{(a+bx^2+cx^4)^3}\,dx &= -\frac{e(b+2cx^2)}{4(b^2-4ac)(a+bx^2+cx^4)^2}+\frac{dx(b^2-2ac+bcx^2)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}\\ &\quad+\frac{3ce(b+2cx^2)}{2(b^2-4ac)^2(a+bx^2+cx^4)}+\frac{dx((b^2-7ac)(3b^2-4ac)+3bc(b^2-8ac)x^2)}{8a^2(b^2-4ac)^2(a+bx^2+cx^4)}\\ &\quad+\frac{3\sqrt{c}(b^4-10ab^2c+56a^2c^2+b(b^2-8ac)\sqrt{b^2-4ac})\,d\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{8\sqrt{2}a^2(b^2-4ac)^{5/2}\sqrt{b-\sqrt{b^2-4ac}}}\\ &\quad+\frac{3\sqrt{c}\left(b^3-8abc-\frac{b^4-10ab^2c+56a^2c^2}{\sqrt{b^2-4ac}}\right)d\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{8\sqrt{2}a^2(b^2-4ac)^2\sqrt{b+\sqrt{b^2-4ac}}}-\frac{6c^2e\,\mathrm{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{5/2}}\end{aligned}$$

output
```
-1/4*e*(2*c*x^2+b)/(-4*a*c+b^2)/(c*x^4+b*x^2+a)^2+1/4*d*x*(b*c*x^2-2*a*c+b
^2)/a/(-4*a*c+b^2)/(c*x^4+b*x^2+a)^2+3/2*c*e*(2*c*x^2+b)/(-4*a*c+b^2)^2/(c
*x^4+b*x^2+a)+1/8*d*x*((-7*a*c+b^2)*(-4*a*c+3*b^2)+3*b*c*(-8*a*c+b^2)*x^2)
/a^2/(-4*a*c+b^2)^2/(c*x^4+b*x^2+a)-6*c^2*e*arctanh((2*c*x^2+b)/(-4*a*c+b^
2)^(1/2))/(-4*a*c+b^2)^(5/2)+3/16*d*arctan(x*2^(1/2)*c^(1/2)/(b-(-4*a*c+b^
2)^(1/2))^(1/2))*c^(1/2)*(b^4-10*a*b^2*c+56*a^2*c^2+b*(-8*a*c+b^2)*(-4*a*c
+b^2)^(1/2))/a^2/(-4*a*c+b^2)^(5/2)*2^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2)+3
/16*d*arctan(x*2^(1/2)*c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2))*c^(1/2)*(b^3-
8*a*b*c+(-56*a^2*c^2+10*a*b^2*c-b^4)/(-4*a*c+b^2)^(1/2))/a^2/(-4*a*c+b^2)^
2*2^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2)
```

3.52.2 Mathematica [A] (verified)

Time = 1.14 (sec) , antiderivative size = 488, normalized size of antiderivative = 1.03

$$\int \frac{d+ex}{(a+bx^2+cx^4)^3}\,dx$$
$$= \frac{1}{16}\left(\frac{4abe+8acx(d+ex)-4bdx(b+cx^2)}{a\left(-b^2+4ac\right)\left(a+bx^2+cx^4\right)^2} + \frac{6b^3dx(b+cx^2)-2abcdx(25b+24cx^2)+8a^2c(3be+cx(7d+6ex))}{a^2\left(b^2-4ac\right)^2\left(a+bx^2+cx^4\right)} + \frac{3\sqrt{2}\sqrt{c}\left(b^4-10ab^2c+56a^2c^2+b^3\sqrt{b^2-4ac}-8abc\sqrt{b^2-4ac}\right)d\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{a^2\left(b^2-4ac\right)^{5/2}\sqrt{b-\sqrt{b^2-4ac}}} - \frac{3\sqrt{2}\sqrt{c}\left(b^4-10ab^2c+56a^2c^2-b^3\sqrt{b^2-4ac}+8abc\sqrt{b^2-4ac}\right)d\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{a^2\left(b^2-4ac\right)^{5/2}\sqrt{b+\sqrt{b^2-4ac}}} + \frac{48c^2e\log\left(-b+\sqrt{b^2-4ac}-2cx^2\right)}{\left(b^2-4ac\right)^{5/2}} - \frac{48c^2e\log\left(b+\sqrt{b^2-4ac}+2cx^2\right)}{\left(b^2-4ac\right)^{5/2}}\right)$$

input
```
Integrate[(d + e*x)/(a + b*x^2 + c*x^4)^3,x]
```

output
```
((4*a*b*e + 8*a*c*x*(d + e*x) - 4*b*d*x*(b + c*x^2))/(a*(-b^2 + 4*a*c)*(a
+ b*x^2 + c*x^4)^2) + (6*b^3*d*x*(b + c*x^2) - 2*a*b*c*d*x*(25*b + 24*c*x^
2) + 8*a^2*c*(3*b*e + c*x*(7*d + 6*e*x)))/(a^2*(b^2 - 4*a*c)^2*(a + b*x^2
+ c*x^4)) + (3*Sqrt[2]*Sqrt[c]*(b^4 - 10*a*b^2*c + 56*a^2*c^2 + b^3*Sqrt[b
^2 - 4*a*c] - 8*a*b*c*Sqrt[b^2 - 4*a*c])*d*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt
[b - Sqrt[b^2 - 4*a*c]]])/(a^2*(b^2 - 4*a*c)^(5/2)*Sqrt[b - Sqrt[b^2 - 4*a
*c]]) - (3*Sqrt[2]*Sqrt[c]*(b^4 - 10*a*b^2*c + 56*a^2*c^2 - b^3*Sqrt[b^2 -
 4*a*c] + 8*a*b*c*Sqrt[b^2 - 4*a*c])*d*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b +
 Sqrt[b^2 - 4*a*c]]])/(a^2*(b^2 - 4*a*c)^(5/2)*Sqrt[b + Sqrt[b^2 - 4*a*c]]
) + (48*c^2*e*Log[-b + Sqrt[b^2 - 4*a*c] - 2*c*x^2])/(b^2 - 4*a*c)^(5/2) -
 (48*c^2*e*Log[b + Sqrt[b^2 - 4*a*c] + 2*c*x^2])/(b^2 - 4*a*c)^(5/2))/16
```

3.52.3 Rubi [A] (verified)

Time = 0.85 (sec) , antiderivative size = 491, normalized size of antiderivative = 1.04, number of steps used = 14, number of rules used = 13, $\frac{\text{number of rules}}{\text{integrand size}} = 0.650$, Rules used = {2202, 27, 1405, 25, 1432, 1086, 1086, 1083, 219, 1492, 27, 1480, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex}{(a+bx^2+cx^4)^3}dx$$

$$\Big\downarrow \text{ 2202}$$

$$\int \frac{d}{(cx^4+bx^2+a)^3}dx+\int \frac{ex}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{ 27}$$

$$d\int \frac{1}{(cx^4+bx^2+a)^3}dx+e\int \frac{x}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{ 1405}$$

$$d\left(\frac{x(-2ac+b^2+bcx^2)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}-\frac{\int -\frac{3b^2+5cx^2b-14ac}{(cx^4+bx^2+a)^2}dx}{4a(b^2-4ac)}\right)+e\int \frac{x}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{ 25}$$

$$d\left(\frac{\int \frac{3b^2+5cx^2b-14ac}{(cx^4+bx^2+a)^2}dx}{4a(b^2-4ac)}+\frac{x(-2ac+b^2+bcx^2)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}\right)+e\int \frac{x}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{ 1432}$$

$$d\left(\frac{\int \frac{3b^2+5cx^2b-14ac}{(cx^4+bx^2+a)^2}dx}{4a(b^2-4ac)}+\frac{x(-2ac+b^2+bcx^2)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}\right)+\frac{1}{2}e\int \frac{1}{(cx^4+bx^2+a)^3}dx^2$$

$$\Big\downarrow \text{ 1086}$$

$$d\left(\frac{\int \frac{3b^2+5cx^2b-14ac}{(cx^4+bx^2+a)^2}dx}{4a(b^2-4ac)}+\frac{x(-2ac+b^2+bcx^2)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}\right)+\frac{1}{2}e\left(-\frac{3c\int \frac{1}{(cx^4+bx^2+a)^2}dx^2}{b^2-4ac}-\frac{b+2cx^2}{2(b^2-4ac)(a+bx^2+cx^4)^2}\right)$$

$$\Big\downarrow \text{ 1086}$$

$$d\left(\frac{\int\frac{3b^2+5cx^2b-14ac}{(cx^4+bx^2+a)^2}dx}{4a(b^2-4ac)}+\frac{x(-2ac+b^2+bcx^2)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}\right)+\frac{1}{2}e\left(-\frac{3c\left(-\frac{2c\int\frac{1}{cx^4+bx^2+a}dx^2}{b^2-4ac}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2(b^2-4ac)(a+bx^2+cx^4)^2}\right)$$

$$\Big\downarrow \text{1083}$$

$$d\left(\frac{\int\frac{3b^2+5cx^2b-14ac}{(cx^4+bx^2+a)^2}dx}{4a(b^2-4ac)}+\frac{x(-2ac+b^2+bcx^2)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}\right)+\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\int\frac{1}{-x^4+b^2-4ac}d(2cx^2+b)}{b^2-4ac}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2(b^2-4ac)(a+bx^2+cx^4)^2}\right)$$

$$\Big\downarrow \text{219}$$

$$d\left(\frac{\int\frac{3b^2+5cx^2b-14ac}{(cx^4+bx^2+a)^2}dx}{4a(b^2-4ac)}+\frac{x(-2ac+b^2+bcx^2)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}\right)+\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2(b^2-4ac)(a+bx^2+cx^4)^2}\right)$$

$$\Big\downarrow \text{1492}$$

$$d\left(\frac{\frac{x(3bcx^2(b^2-8ac)+(b^2-7ac)(3b^2-4ac))}{2a(b^2-4ac)(a+bx^2+cx^4)}-\frac{\int-\frac{3(b^4-9acb^2+c(b^2-8ac)x^2b+28a^2c^2)}{cx^4+bx^2+a}dx}{2a(b^2-4ac)}}{4a(b^2-4ac)}+\frac{x(-2ac+b^2+bcx^2)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}\right)+\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2(b^2-4ac)(a+bx^2+cx^4)^2}\right)$$

$$\Big\downarrow \text{27}$$

3.52. $\int\frac{d+ex}{(a+bx^2+cx^4)^3}\,dx$

$$d\left(\frac{\frac{3\int \frac{b^4-9acb^2+c\left(b^2-8ac\right)x^2b+28a^2c^2}{cx^4+bx^2+a}dx}{2a(b^2-4ac)}+\frac{x(3bcx^2(b^2-8ac)+(b^2-7ac)(3b^2-4ac))}{2a(b^2-4ac)(a+bx^2+cx^4)}}{4a\left(b^2-4ac\right)}+\frac{x\left(-2ac+b^2+bcx^2\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)+\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)$$

$\downarrow$ 1480

$$d\left(\frac{\frac{3\left(\frac{1}{2}c\left(\frac{56a^2c^2-10ab^2c+b^4}{\sqrt{b^2-4ac}}+b\left(b^2-8ac\right)\right)\int \frac{1}{cx^2+\frac{1}{2}\left(b-\sqrt{b^2-4ac}\right)}dx+\frac{1}{2}c\left(b\left(b^2-8ac\right)-\frac{56a^2c^2-10ab^2c+b^4}{\sqrt{b^2-4ac}}\right)\int \frac{1}{cx^2+\frac{1}{2}\left(b+\sqrt{b^2-4ac}\right)}dx\right)}{2a(b^2-4ac)}+\frac{x(3bc}{}}{4a\left(b^2-4ac\right)}\right.$$
$$\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)$$

$\downarrow$ 218

$$d\left(\frac{\frac{3\left(\frac{\sqrt{c}\left(\frac{56a^2c^2-10ab^2c+b^4}{\sqrt{b^2-4ac}}+b\left(b^2-8ac\right)\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\left(b\left(b^2-8ac\right)-\frac{56a^2c^2-10ab^2c+b^4}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)}{\sqrt{2}\sqrt{\sqrt{b^2-4ac}+b}}\right)}{2a(b^2-4ac)}+\frac{x(3bcx^2(b^2-}{2a(b}}{4a\left(b^2-4ac\right)}\right.$$
$$\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)$$

input
```
Int[(d + e*x)/(a + b*x^2 + c*x^4)^3,x]
```

3.52. $\int \frac{d+ex}{(a+bx^2+cx^4)^3}\,dx$

output
```
d*((x*(b^2 - 2*a*c + b*c*x^2))/(4*a*(b^2 - 4*a*c)*(a + b*x^2 + c*x^4)^2) +
 ((x*((b^2 - 7*a*c)*(3*b^2 - 4*a*c) + 3*b*c*(b^2 - 8*a*c)*x^2))/(2*a*(b^2
- 4*a*c)*(a + b*x^2 + c*x^4)) + (3*((Sqrt[c]*(b*(b^2 - 8*a*c) + (b^4 - 10*
a*b^2*c + 56*a^2*c^2)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b
 - Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + (Sqrt[c]*(
b*(b^2 - 8*a*c) - (b^4 - 10*a*b^2*c + 56*a^2*c^2)/Sqrt[b^2 - 4*a*c])*ArcTa
n[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[b + Sqrt
[b^2 - 4*a*c]])))/(2*a*(b^2 - 4*a*c)))/(4*a*(b^2 - 4*a*c))) + (e*(-1/2*(b
+ 2*c*x^2)/((b^2 - 4*a*c)*(a + b*x^2 + c*x^4)^2) - (3*c*(-((b + 2*c*x^2)/(
(b^2 - 4*a*c)*(a + b*x^2 + c*x^4))) + (4*c*ArcTanh[(b + 2*c*x^2)/Sqrt[b^2
- 4*a*c]])/(b^2 - 4*a*c)^(3/2)))/(b^2 - 4*a*c)))/2
```

3.52.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1086
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] - Simp[2*c*((2*p +
 3)/((p + 1)*(b^2 - 4*a*c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; Fre
eQ[{a, b, c}, x] && ILtQ[p, -1]
```

rule 1405
```
Int[((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[(-x)*(b^2
- 2*a*c + b*c*x^2)*((a + b*x^2 + c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))
), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(b^2 - 2*a*c + 2*(p + 1)*(
b^2 - 4*a*c) + b*c*(4*p + 7)*x^2)*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; Fr
eeQ[{a, b, c}, x] && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1] && IntegerQ[2*p]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.52.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.53 (sec) , antiderivative size = 499, normalized size of antiderivative = 1.05

method	result
risch	$\frac{-\frac{3b\,c^2 d\left(8ac-b^2\right)x^7}{8a^2\left(16a^2c^2-8a\,b^2c+b^4\right)}+\frac{3c^3e\,x^6}{16a^2c^2-8a\,b^2c+b^4}+\frac{cd\left(28a^2c^2-49a\,b^2c+6b^4\right)x^5}{8a^2\left(16a^2c^2-8a\,b^2c+b^4\right)}+\frac{9b\,c^2e\,x^4}{2\left(16a^2c^2-8a\,b^2c+b^4\right)}-\frac{bd\left(4a^2c^2+20a\,b^2c-3b^4\right)x^3}{8a^2\left(16a^2c^2-8a\,b^2c+b^4\right)}+\frac{(5ac-}{16a^2c^2}}{\left(c\,x^4+b\,x^2+a\right)^2}$
default	$64c^3\left(\frac{\frac{-\frac{3\left(24a^2c^2\sqrt{-4ac+b^2}-10a\,b^2c\sqrt{-4ac+b^2}+b^4\sqrt{-4ac+b^2}+32a^2b\,c^2-12a\,b^3c+b^5\right)d\,x^3}{64a^2c^3}+\frac{3e\left(4ac-b^2\right)x^2}{8c^2}-\frac{d\left(-20\sqrt{-4ac+b^2}\,abc+5\sqrt{-4\right.}{}}{\left(x^2+\frac{b}{2c}-\frac{\sqrt{-4ac+b^2}}{2c}\right)^2}}{}\right.$

input
```
int((e*x+d)/(c*x^4+b*x^2+a)^3,x,method=_RETURNVERBOSE)
```

output
```
(-3/8*b*c^2*d*(8*a*c-b^2)/a^2/(16*a^2*c^2-8*a*b^2*c+b^4)*x^7+3*c^3*e/(16*a
^2*c^2-8*a*b^2*c+b^4)*x^6+1/8/a^2*c*d*(28*a^2*c^2-49*a*b^2*c+6*b^4)/(16*a^
2*c^2-8*a*b^2*c+b^4)*x^5+9/2*b*c^2*e/(16*a^2*c^2-8*a*b^2*c+b^4)*x^4-1/8*b*
d*(4*a^2*c^2+20*a*b^2*c-3*b^4)/a^2/(16*a^2*c^2-8*a*b^2*c+b^4)*x^3+(5*a*c+b
^2)*c*e/(16*a^2*c^2-8*a*b^2*c+b^4)*x^2+1/8*d*(44*a^2*c^2-37*a*b^2*c+5*b^4)
/(16*a^2*c^2-8*a*b^2*c+b^4)/a*x+1/4*b*(10*a*c-b^2)*e/(16*a^2*c^2-8*a*b^2*c
+b^4))/(c*x^4+b*x^2+a)^2+3/16*sum((-b*c*d*(8*a*c-b^2)/a^2/(16*a^2*c^2-8*a*
b^2*c+b^4)*_R^2+16*c^2*e/(16*a^2*c^2-8*a*b^2*c+b^4)*_R+d*(28*a^2*c^2-9*a*b
^2*c+b^4)/a^2/(16*a^2*c^2-8*a*b^2*c+b^4))/(2*_R^3*c+_R*b)*ln(x-_R),_R=Root
Of(_Z^4*c+_Z^2*b+a))
```

3.52.5 Fricas [F(-1)]

Timed out.

$$\int \frac{d+ex}{\left(a+bx^2+cx^4\right)^3}\,dx = \text{Timed out}$$

input
```
integrate((e*x+d)/(c*x^4+b*x^2+a)^3,x, algorithm="fricas")
```

output
```
Timed out
```

3.52.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex}{(a+bx^2+cx^4)^3}\,dx = \text{Timed out}$$

input
```
integrate((e*x+d)/(c*x**4+b*x**2+a)**3,x)
```

output
```
Timed out
```

3.52.7 Maxima [F]

$$\int \frac{d+ex}{(a+bx^2+cx^4)^3}\,dx = \int \frac{ex+d}{(cx^4+bx^2+a)^3}\,dx$$

input
```
integrate((e*x+d)/(c*x^4+b*x^2+a)^3,x, algorithm="maxima")
```

output
```
1/8*(24*a^2*c^3*e*x^6 + 36*a^2*b*c^2*e*x^4 + 3*(b^3*c^2 - 8*a*b*c^3)*d*x^7
 + (6*b^4*c - 49*a*b^2*c^2 + 28*a^2*c^3)*d*x^5 + (3*b^5 - 20*a*b^3*c - 4*a
^2*b*c^2)*d*x^3 + 8*(a^2*b^2*c + 5*a^3*c^2)*e*x^2 + (5*a*b^4 - 37*a^2*b^2*
c + 44*a^3*c^2)*d*x - 2*(a^2*b^3 - 10*a^3*b*c)*e)/((a^2*b^4*c^2 - 8*a^3*b^
2*c^3 + 16*a^4*c^4)*x^8 + a^4*b^4 - 8*a^5*b^2*c + 16*a^6*c^2 + 2*(a^2*b^5*
c - 8*a^3*b^3*c^2 + 16*a^4*b*c^3)*x^6 + (a^2*b^6 - 6*a^3*b^4*c + 32*a^5*c^
3)*x^4 + 2*(a^3*b^5 - 8*a^4*b^3*c + 16*a^5*b*c^2)*x^2) - 3/8*integrate(-(1
6*a^2*c^2*e*x + (b^3*c - 8*a*b*c^2)*d*x^2 + (b^4 - 9*a*b^2*c + 28*a^2*c^2)
*d)/(c*x^4 + b*x^2 + a), x)/(a^2*b^4 - 8*a^3*b^2*c + 16*a^4*c^2)
```

3.52.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 3389 vs. 2(420) = 840.

Time = 3.11 (sec) , antiderivative size = 3389, normalized size of antiderivative = 7.15

$$\int \frac{d+ex}{(a+bx^2+cx^4)^3}\,dx = \text{Too large to display}$$

input
```
integrate((e*x+d)/(c*x^4+b*x^2+a)^3,x, algorithm="giac")
```

output
```
3/32*(sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^8 - 17*sqrt(2)*sqrt(b*c +
sqrt(b^2 - 4*a*c)*c)*a*b^6*c - 2*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b
^7*c - 2*b^8*c + 116*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^4*c^2 +
 26*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^5*c^2 + sqrt(2)*sqrt(b*c +
 sqrt(b^2 - 4*a*c)*c)*b^6*c^2 + 34*a*b^6*c^2 + 2*b^7*c^2 - 368*sqrt(2)*sqr
t(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*b^2*c^3 - 128*sqrt(2)*sqrt(b*c + sqrt(b^2
 - 4*a*c)*c)*a^2*b^3*c^3 - 13*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^
4*c^3 - 232*a^2*b^4*c^3 - 30*a*b^5*c^3 + 448*sqrt(2)*sqrt(b*c + sqrt(b^2 -
 4*a*c)*c)*a^4*c^4 + 224*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*b*c^4
 + 64*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^2*c^4 + 736*a^3*b^2*c^
4 + 176*a^2*b^3*c^4 - 112*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*c^5
- 896*a^4*c^5 - 352*a^3*b*c^5 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(
b^2 - 4*a*c)*c)*b^7 + 15*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4
*a*c)*c)*a*b^5*c + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c
)*c)*b^6*c - 88*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*
a^2*b^3*c^2 - 22*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)
*a*b^4*c^2 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^5
*c^2 + 176*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*b
*c^3 + 88*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^
2*c^3 + 11*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*...
```

3.52.9 Mupad [B] (verification not implemented)

Time = 8.77 (sec) , antiderivative size = 4225, normalized size of antiderivative = 8.91

$$\int \frac{d+ex}{(a+bx^2+cx^4)^3}\,dx = \text{Too large to display}$$

input
```
int((d + e*x)/(a + b*x^2 + c*x^4)^3,x)
```

output
```
symsum(log(root(56371445760*a^11*b^8*c^6*z^4 - 503316480*a^8*b^14*c^3*z^4
+ 47185920*a^7*b^16*c^2*z^4 - 171798691840*a^14*b^2*c^9*z^4 + 193273528320
*a^13*b^4*c^8*z^4 - 128849018880*a^12*b^6*c^7*z^4 - 16911433728*a^10*b^10*
c^5*z^4 + 3523215360*a^9*b^12*c^4*z^4 - 2621440*a^6*b^18*c*z^4 + 687194767
36*a^15*c^10*z^4 + 65536*a^5*b^20*z^4 + 6936330240*a^8*b^3*c^8*d^2*z^2 + 2
464874496*a^6*b^7*c^6*d^2*z^2 - 3963617280*a^9*b*c^9*d^2*z^2 - 1509949440*
a^9*b^2*c^8*e^2*z^2 - 5400428544*a^7*b^5*c^7*d^2*z^2 - 94464*a*b^17*c*d^2*
z^2 + 754974720*a^8*b^4*c^7*e^2*z^2 - 730054656*a^5*b^9*c^5*d^2*z^2 - 1887
43680*a^7*b^6*c^6*e^2*z^2 + 146165760*a^4*b^11*c^4*d^2*z^2 + 23592960*a^6*
b^8*c^5*e^2*z^2 - 19860480*a^3*b^13*c^3*d^2*z^2 - 1179648*a^5*b^10*c^4*e^2
*z^2 + 1771776*a^2*b^15*c^2*d^2*z^2 + 1207959552*a^10*c^9*e^2*z^2 + 2304*b
^19*d^2*z^2 - 428544*a*b^12*c^3*d^2*e*z + 1022754816*a^6*b^2*c^8*d^2*e*z -
 642318336*a^5*b^4*c^7*d^2*e*z + 223395840*a^4*b^6*c^6*d^2*e*z - 46725120*
a^3*b^8*c^5*d^2*e*z + 5930496*a^2*b^10*c^4*d^2*e*z - 693633024*a^7*c^9*d^2
*e*z + 13824*b^14*c^2*d^2*e*z + 34836480*a^4*b*c^8*d^2*e^2 - 435456*a*b^7*
c^5*d^2*e^2 - 17418240*a^3*b^3*c^7*d^2*e^2 + 3919104*a^2*b^5*c^6*d^2*e^2 +
 20736*b^9*c^4*d^2*e^2 - 27433728*a^3*b^2*c^8*d^4 + 6446304*a^2*b^4*c^7*d^
4 - 734832*a*b^6*c^6*d^4 + 49787136*a^4*c^9*d^4 + 5308416*a^5*c^8*e^4 + 35
721*b^8*c^5*d^4, z, k)*(root(56371445760*a^11*b^8*c^6*z^4 - 503316480*a^8*
b^14*c^3*z^4 + 47185920*a^7*b^16*c^2*z^4 - 171798691840*a^14*b^2*c^9*z^...
```

3.52. $\int \frac{d+ex}{(a+bx^2+cx^4)^3}\,dx$

3.53 $\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^3}\,dx$

3.53.1 Optimal result

Integrand size = 25, antiderivative size = 621

$$\begin{aligned}\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^3}\,dx =& -\frac{e(b+2cx^2)}{4(b^2-4ac)(a+bx^2+cx^4)^2} \\ &+ \frac{x(b^2d-2acd-abf+c(bd-2af)x^2)}{4a(b^2-4ac)(a+bx^2+cx^4)^2} + \frac{3ce(b+2cx^2)}{2(b^2-4ac)^2(a+bx^2+cx^4)} \\ &+ \frac{x(3b^4d-25ab^2cd+28a^2c^2d+ab^3f+8a^2bcf+c(3b^3d-24abcd+ab^2f+20a^2cf)x^2)}{8a^2(b^2-4ac)^2(a+bx^2+cx^4)} \\ &+ \frac{\sqrt{c}\left(3b^4d+b^3\left(3\sqrt{b^2-4ac}\,d+af\right)-4abc\left(6\sqrt{b^2-4ac}\,d+13af\right)-ab^2\left(30cd-\sqrt{b^2-4ac}\,f\right)+4a^2c(42}{8\sqrt{2}a^2(b^2-4ac)^{5/2}\sqrt{b-\sqrt{b^2-4ac}}} \\ &+ \frac{\sqrt{c}\left(3b^3d-24abcd+ab^2f+20a^2cf-\frac{3b^4d-30ab^2cd+168a^2c^2d+ab^3f-52a^2bcf}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{8\sqrt{2}a^2(b^2-4ac)^2\sqrt{b+\sqrt{b^2-4ac}}} \\ &- \frac{6c^2e\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{5/2}}\end{aligned}$$

output
```
-1/4*e*(2*c*x^2+b)/(-4*a*c+b^2)/(c*x^4+b*x^2+a)^2+1/4*x*(b^2*d-2*a*c*d-a*b
*f+c*(-2*a*f+b*d)*x^2)/a/(-4*a*c+b^2)/(c*x^4+b*x^2+a)^2+3/2*c*e*(2*c*x^2+b
)/(-4*a*c+b^2)^2/(c*x^4+b*x^2+a)+1/8*x*(3*b^4*d-25*a*b^2*c*d+28*a^2*c^2*d+
a*b^3*f+8*a^2*b*c*f+c*(20*a^2*c*f+a*b^2*f-24*a*b*c*d+3*b^3*d)*x^2)/a^2/(-4
*a*c+b^2)^2/(c*x^4+b*x^2+a)-6*c^2*e*arctanh((2*c*x^2+b)/(-4*a*c+b^2)^(1/2)
)/(-4*a*c+b^2)^(5/2)+1/16*arctan(x*2^(1/2)*c^(1/2)/(b-(-4*a*c+b^2)^(1/2))^
(1/2))*c^(1/2)*(3*b^4*d+b^3*(a*f+3*d*(-4*a*c+b^2)^(1/2))-4*a*b*c*(13*a*f+6
*d*(-4*a*c+b^2)^(1/2))-a*b^2*(30*c*d-f*(-4*a*c+b^2)^(1/2))+4*a^2*c*(42*c*d
+5*f*(-4*a*c+b^2)^(1/2)))/a^2/(-4*a*c+b^2)^(5/2)*2^(1/2)/(b-(-4*a*c+b^2)^(
1/2))^(1/2)+1/16*arctan(x*2^(1/2)*c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2))*c^
(1/2)*(3*b^3*d-24*a*b*c*d+a*b^2*f+20*a^2*c*f+(52*a^2*b*c*f-168*a^2*c^2*d-a
*b^3*f+30*a*b^2*c*d-3*b^4*d)/(-4*a*c+b^2)^(1/2))/a^2/(-4*a*c+b^2)^2*2^(1/2
)/(b+(-4*a*c+b^2)^(1/2))^(1/2)
```

3.53.2 Mathematica [A] (verified)

Time = 1.93 (sec) , antiderivative size = 625, normalized size of antiderivative = 1.01

$$\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^3}\,dx = \frac{1}{16}\left(\frac{4ab(e+fx)-4bdx(b+cx^2)+8acx(d+x(e+fx))}{a(-b^2+4ac)(a+bx^2+cx^4)^2} + \frac{6b^3dx(b+cx^2)+2abx(-25bcd+b^2f-24c^2dx^2+bcfx^2)+8a^2c(b(3e+2fx)+cx(7d+6ex+5fx^2))}{a^2(b^2-4ac)^2(a+bx^2+cx^4)} + \frac{\sqrt{2}\sqrt{c}\left(3b^4d+b^3\left(3\sqrt{b^2-4ac}d+af\right)-4abc\left(6\sqrt{b^2-4ac}d+13af\right)+ab^2\left(-30cd+\sqrt{b^2-4ac}f\right)+4a^2 \cdots}{a^2(b^2-4ac)^{5/2}\sqrt{b-\sqrt{b^2-4ac}}} + \frac{\sqrt{2}\sqrt{c}\left(-3b^4d+b^3\left(3\sqrt{b^2-4ac}d-af\right)+4abc\left(-6\sqrt{b^2-4ac}d+13af\right)+ab^2\left(30cd+\sqrt{b^2-4ac}f\right)+4 \cdots}{a^2(b^2-4ac)^{5/2}\sqrt{b+\sqrt{b^2-4ac}}} + \frac{48c^2e\log\left(-b+\sqrt{b^2-4ac}-2cx^2\right)}{(b^2-4ac)^{5/2}} - \frac{48c^2e\log\left(b+\sqrt{b^2-4ac}+2cx^2\right)}{(b^2-4ac)^{5/2}}\right)$$

input
```
Integrate[(d + e*x + f*x^2)/(a + b*x^2 + c*x^4)^3,x]
```

output
```
((4*a*b*(e + f*x) - 4*b*d*x*(b + c*x^2) + 8*a*c*x*(d + x*(e + f*x)))/(a*(-
b^2 + 4*a*c)*(a + b*x^2 + c*x^4)^2) + (6*b^3*d*x*(b + c*x^2) + 2*a*b*x*(-2
5*b*c*d + b^2*f - 24*c^2*d*x^2 + b*c*f*x^2) + 8*a^2*c*(b*(3*e + 2*f*x) + c
*x*(7*d + 6*e*x + 5*f*x^2)))/(a^2*(b^2 - 4*a*c)^2*(a + b*x^2 + c*x^4)) + (
Sqrt[2]*Sqrt[c]*(3*b^4*d + b^3*(3*Sqrt[b^2 - 4*a*c]*d + a*f) - 4*a*b*c*(6*
Sqrt[b^2 - 4*a*c]*d + 13*a*f) + a*b^2*(-30*c*d + Sqrt[b^2 - 4*a*c]*f) + 4*
a^2*c*(42*c*d + 5*Sqrt[b^2 - 4*a*c]*f))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b
- Sqrt[b^2 - 4*a*c]]])/(a^2*(b^2 - 4*a*c)^(5/2)*Sqrt[b - Sqrt[b^2 - 4*a*c]
]) + (Sqrt[2]*Sqrt[c]*(-3*b^4*d + b^3*(3*Sqrt[b^2 - 4*a*c]*d - a*f) + 4*a*
b*c*(-6*Sqrt[b^2 - 4*a*c]*d + 13*a*f) + a*b^2*(30*c*d + Sqrt[b^2 - 4*a*c]*
f) + 4*a^2*c*(-42*c*d + 5*Sqrt[b^2 - 4*a*c]*f))*ArcTan[(Sqrt[2]*Sqrt[c]*x)
/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(a^2*(b^2 - 4*a*c)^(5/2)*Sqrt[b + Sqrt[b^2
- 4*a*c]]) + (48*c^2*e*Log[-b + Sqrt[b^2 - 4*a*c] - 2*c*x^2])/(b^2 - 4*a*c
)^(5/2) - (48*c^2*e*Log[b + Sqrt[b^2 - 4*a*c] + 2*c*x^2])/(b^2 - 4*a*c)^(5
/2))/16
```

3.53.3 Rubi [A] (verified)

Time = 1.07 (sec) , antiderivative size = 605, normalized size of antiderivative = 0.97, number of steps used = 14, number of rules used = 13, $\frac{\text{number of rules}}{\text{integrand size}} = 0.520$, Rules used = {2202, 27, 1432, 1086, 1086, 1083, 219, 1492, 25, 1492, 25, 1480, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^3} \, dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^3}dx+\int \frac{ex}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{27}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^3}dx+e\int \frac{x}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{1432}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^3}dx+\frac{1}{2}e\int \frac{1}{(cx^4+bx^2+a)^3}dx^2$$

$$\Big\downarrow \text{1086}$$

$$\frac{1}{2}e\left(-\frac{3c\int \frac{1}{\left(cx^4+bx^2+a\right)^2}dx^2}{b^2-4ac}-\frac{b+2cx^2}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)+\int \frac{fx^2+d}{\left(cx^4+bx^2+a\right)^3}dx$$

$\Big\downarrow$ 1086

$$\frac{1}{2}e\left(-\frac{3c\left(-\frac{2c\int \frac{1}{cx^4+bx^2+a}dx^2}{b^2-4ac}-\frac{b+2cx^2}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)+\int \frac{fx^2+d}{\left(cx^4+bx^2+a\right)^3}dx$$

$\Big\downarrow$ 1083

$$\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\int \frac{1}{-x^4+b^2-4ac}d\left(2cx^2+b\right)}{b^2-4ac}-\frac{b+2cx^2}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)+\int \frac{fx^2+d}{\left(cx^4+bx^2+a\right)^3}dx$$

$\Big\downarrow$ 219

$$\int \frac{fx^2+d}{\left(cx^4+bx^2+a\right)^3}dx+\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{b+2cx^2}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)$$

$\Big\downarrow$ 1492

$$-\frac{\int -\frac{3db^2+afb+5c(bd-2af)x^2-14acd}{\left(cx^4+bx^2+a\right)^2}dx}{4a\left(b^2-4ac\right)}+\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{b+2cx^2}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$\Big\downarrow$ 25

3.53. $\int \frac{d+ex+fx^2}{\left(a+bx^2+cx^4\right)^3}\,dx$

$$\frac{\int \frac{3db^2+afb+5c(bd-2af)x^2-14acd}{(cx^4+bx^2+a)^2}dx}{4a(b^2-4ac)}+\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2(b^2-4ac)(a+bx^2+cx^4)^2}\right)+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}$$

$\Big\downarrow$ 1492

$$\frac{\frac{x(cx^2(20a^2cf+ab^2f-24abcd+3b^3d)+8a^2bcf+28a^2c^2d+ab^3f-25ab^2cd+3b^4d)}{2a(b^2-4ac)(a+bx^2+cx^4)}-\frac{\int -\frac{3db^4+afb^3-27acdb^2-16a^2cfb+c\left(3db^3+afb^2-24acdb+20a^2cf\right)\cdots}{cx^4+bx^2+a}\cdots}{2a(b^2-4ac)}}{4a(b^2-4ac)}+\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2(b^2-4ac)(a+bx^2+cx^4)^2}\right)+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}$$

$\Big\downarrow$ 25

$$\frac{\frac{\int \frac{3db^4+afb^3-27acdb^2-16a^2cfb+c\left(3db^3+afb^2-24acdb+20a^2cf\right)x^2+84a^2c^2d}{cx^4+bx^2+a}dx}{2a(b^2-4ac)}+\frac{x(cx^2(20a^2cf+ab^2f-24abcd+3b^3d)+8a^2bcf+28a^2c^2d+ab^3f-25\cdots}{2a(b^2-4ac)(a+bx^2+cx^4)}}{4a(b^2-4ac)}+\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2(b^2-4ac)(a+bx^2+cx^4)^2}\right)+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}$$

$\Big\downarrow$ 1480

3.53. $\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^3}\,dx$

$$\frac{\frac{\frac{1}{2}c\left(\frac{-52a^2bcf+168a^2c^2d+ab^3f-30ab^2cd+3b^4d}{\sqrt{b^2-4ac}}+20a^2cf+ab^2f-24abcd+3b^3d\right)\int\frac{1}{cx^2+\frac{1}{2}\left(b-\sqrt{b^2-4ac}\right)}dx+\frac{1}{2}c\left(-\frac{-52a^2bcf+168a^2c^2d+ab^3f-30ab^2cd+}{\sqrt{b^2-4ac}}\right.}{2a(b^2-4ac)}}{4a\,(b^2}$$

$$\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2\,(b^2-4ac)\,(a+bx^2+cx^4)^2}\right)+$$

$$\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\,(b^2-4ac)\,(a+bx^2+cx^4)^2}$$

$$\Big\downarrow \ 218$$

$$\frac{\frac{\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{-52a^2bcf+168a^2c^2d+ab^3f-30ab^2cd+3b^4d}{\sqrt{b^2-4ac}}+20a^2cf+ab^2f-24abcd+3b^3d\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{-52a^2bcf+168a^2c^2}{\sqrt{b}}\right.}{\sqrt{2}\sqrt{\sqrt{}}}}{2a(b^2-4ac)}}{4a\,(b^2-4a}$$

$$\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2\,(b^2-4ac)\,(a+bx^2+cx^4)^2}\right)+$$

$$\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\,(b^2-4ac)\,(a+bx^2+cx^4)^2}$$

input
```
Int[(d + e*x + f*x^2)/(a + b*x^2 + c*x^4)^3,x]
```

3.53. $\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^3}\,dx$

output
```
(x*(b^2*d - 2*a*c*d - a*b*f + c*(b*d - 2*a*f)*x^2))/(4*a*(b^2 - 4*a*c)*(a
+ b*x^2 + c*x^4)^2) + ((x*(3*b^4*d - 25*a*b^2*c*d + 28*a^2*c^2*d + a*b^3*f
 + 8*a^2*b*c*f + c*(3*b^3*d - 24*a*b*c*d + a*b^2*f + 20*a^2*c*f)*x^2))/(2*
a*(b^2 - 4*a*c)*(a + b*x^2 + c*x^4)) + ((Sqrt[c]*(3*b^3*d - 24*a*b*c*d + a
*b^2*f + 20*a^2*c*f + (3*b^4*d - 30*a*b^2*c*d + 168*a^2*c^2*d + a*b^3*f -
52*a^2*b*c*f)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[
b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + (Sqrt[c]*(3*b^3*d
- 24*a*b*c*d + a*b^2*f + 20*a^2*c*f - (3*b^4*d - 30*a*b^2*c*d + 168*a^2*c^
2*d + a*b^3*f - 52*a^2*b*c*f)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x
)/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[b + Sqrt[b^2 - 4*a*c]]))/(2*
a*(b^2 - 4*a*c)))/(4*a*(b^2 - 4*a*c)) + (e*(-1/2*(b + 2*c*x^2)/((b^2 - 4*a
*c)*(a + b*x^2 + c*x^4)^2) - (3*c*(-((b + 2*c*x^2)/((b^2 - 4*a*c)*(a + b*x
^2 + c*x^4))) + (4*c*ArcTanh[(b + 2*c*x^2)/Sqrt[b^2 - 4*a*c]])/(b^2 - 4*a*
c)^(3/2)))/(b^2 - 4*a*c)))/2
```

3.53.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1086
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] - Simp[2*c*((2*p +
 3)/((p + 1)*(b^2 - 4*a*c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; Fre
eQ[{a, b, c}, x] && ILtQ[p, -1]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.53. $\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^3}\,dx$

3.53.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.65 (sec) , antiderivative size = 607, normalized size of antiderivative = 0.98

method	result
risch	$\frac{\frac{c^{2}\left(20a^{2}cf+a\,b^{2}f-24abcd+3b^{3}d\right)x^{7}}{8a^{2}\left(16a^{2}c^{2}-8a\,b^{2}c+b^{4}\right)}+\frac{3c^{3}e\,x^{6}}{16a^{2}c^{2}-8a\,b^{2}c+b^{4}}+\frac{c\left(28a^{2}bcf+28a^{2}c^{2}d+2a\,b^{3}f-49a\,b^{2}cd+6d\,b^{4}\right)x^{5}}{8a^{2}\left(16a^{2}c^{2}-8a\,b^{2}c+b^{4}\right)}+\frac{9b\,c^{2}e\,x^{4}}{2\left(16a^{2}c^{2}-8a\,b^{2}c+b^{4}\right)}+\frac{(36a^{3}c^{2}}{}}{(c\,x^{4}+b\,x}$
default	Expression too large to display

input
```
int((f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x,method=_RETURNVERBOSE)
```

output
```
(1/8*c^2*(20*a^2*c*f+a*b^2*f-24*a*b*c*d+3*b^3*d)/a^2/(16*a^2*c^2-8*a*b^2*c
+b^4)*x^7+3*c^3*e/(16*a^2*c^2-8*a*b^2*c+b^4)*x^6+1/8/a^2*c*(28*a^2*b*c*f+2
8*a^2*c^2*d+2*a*b^3*f-49*a*b^2*c*d+6*b^4*d)/(16*a^2*c^2-8*a*b^2*c+b^4)*x^5
+9/2*b*c^2*e/(16*a^2*c^2-8*a*b^2*c+b^4)*x^4+1/8*(36*a^3*c^2*f+5*a^2*b^2*c*
f-4*a^2*b*c^2*d+a*b^4*f-20*a*b^3*c*d+3*b^5*d)/a^2/(16*a^2*c^2-8*a*b^2*c+b^
4)*x^3+(5*a*c+b^2)*c*e/(16*a^2*c^2-8*a*b^2*c+b^4)*x^2+1/8*(16*a^2*b*c*f+44
*a^2*c^2*d-a*b^3*f-37*a*b^2*c*d+5*b^4*d)/(16*a^2*c^2-8*a*b^2*c+b^4)/a*x+1/
4*b*(10*a*c-b^2)*e/(16*a^2*c^2-8*a*b^2*c+b^4))/(c*x^4+b*x^2+a)^2+1/16*sum(
(c*(20*a^2*c*f+a*b^2*f-24*a*b*c*d+3*b^3*d)/a^2/(16*a^2*c^2-8*a*b^2*c+b^4)*
_R^2+48*c^2*e/(16*a^2*c^2-8*a*b^2*c+b^4)*_R-(16*a^2*b*c*f-84*a^2*c^2*d-a*b
^3*f+27*a*b^2*c*d-3*b^4*d)/a^2/(16*a^2*c^2-8*a*b^2*c+b^4))/(2*_R^3*c+_R*b)
*ln(x-_R),_R=RootOf(_Z^4*c+_Z^2*b+a))
```

3.53.5 Fricas [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2}{\left(a+bx^2+cx^4\right)^3}\,dx=\text{Timed out}$$

input
```
integrate((f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x, algorithm="fricas")
```

output
```
Timed out
```

3.53.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^3}\,dx = \text{Timed out}$$

input
```
integrate((f*x**2+e*x+d)/(c*x**4+b*x**2+a)**3,x)
```

output
```
Timed out
```

3.53.7 Maxima [F]

$$\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^3}\,dx = \int \frac{fx^2+ex+d}{(cx^4+bx^2+a)^3}\,dx$$

input
```
integrate((f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x, algorithm="maxima")
```

output
```
1/8*(24*a^2*c^3*e*x^6 + 36*a^2*b*c^2*e*x^4 + (3*(b^3*c^2 - 8*a*b*c^3)*d +
(a*b^2*c^2 + 20*a^2*c^3)*f)*x^7 + ((6*b^4*c - 49*a*b^2*c^2 + 28*a^2*c^3)*d
 + 2*(a*b^3*c + 14*a^2*b*c^2)*f)*x^5 + 8*(a^2*b^2*c + 5*a^3*c^2)*e*x^2 + (
(3*b^5 - 20*a*b^3*c - 4*a^2*b*c^2)*d + (a*b^4 + 5*a^2*b^2*c + 36*a^3*c^2)*
f)*x^3 - 2*(a^2*b^3 - 10*a^3*b*c)*e + ((5*a*b^4 - 37*a^2*b^2*c + 44*a^3*c^
2)*d - (a^2*b^3 - 16*a^3*b*c)*f)*x)/((a^2*b^4*c^2 - 8*a^3*b^2*c^3 + 16*a^4
*c^4)*x^8 + a^4*b^4 - 8*a^5*b^2*c + 16*a^6*c^2 + 2*(a^2*b^5*c - 8*a^3*b^3*
c^2 + 16*a^4*b*c^3)*x^6 + (a^2*b^6 - 6*a^3*b^4*c + 32*a^5*c^3)*x^4 + 2*(a^
3*b^5 - 8*a^4*b^3*c + 16*a^5*b*c^2)*x^2) + 1/8*integrate((48*a^2*c^2*e*x +
 (3*(b^3*c - 8*a*b*c^2)*d + (a*b^2*c + 20*a^2*c^2)*f)*x^2 + 3*(b^4 - 9*a*b
^2*c + 28*a^2*c^2)*d + (a*b^3 - 16*a^2*b*c)*f)/(c*x^4 + b*x^2 + a), x)/(a^
2*b^4 - 8*a^3*b^2*c + 16*a^4*c^2)
```

3.53.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 5284 vs. 2(561) = 1122.

Time = 2.44 (sec) , antiderivative size = 5284, normalized size of antiderivative = 8.51

$$\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^3}\,dx = \text{Too large to display}$$

input
```
integrate((f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x, algorithm="giac")
```

output
```
-3*(b^2*c^4 - 4*a*c^5 - 2*b*c^5 + c^6)*sqrt(b^2 - 4*a*c)*e*log(x^2 + 1/2*(
a^2*b^5 - 8*a^3*b^3*c + 16*a^4*b*c^2 + sqrt((a^2*b^5 - 8*a^3*b^3*c + 16*a^
4*b*c^2)^2 - 4*(a^3*b^4 - 8*a^4*b^2*c + 16*a^5*c^2)*(a^2*b^4*c - 8*a^3*b^2
*c^2 + 16*a^4*c^3)))/(a^2*b^4*c - 8*a^3*b^2*c^2 + 16*a^4*c^3))/((b^8 - 16*
a*b^6*c - 2*b^7*c + 96*a^2*b^4*c^2 + 24*a*b^5*c^2 + b^6*c^2 - 256*a^3*b^2*
c^3 - 96*a^2*b^3*c^3 - 12*a*b^4*c^3 + 256*a^4*c^4 + 128*a^3*b*c^4 + 48*a^2
*b^2*c^4 - 64*a^3*c^5)*c^2) + 3*(b^2*c^4 - 4*a*c^5 - 2*b*c^5 + c^6)*sqrt(b
^2 - 4*a*c)*e*log(x^2 + 1/2*(a^2*b^5 - 8*a^3*b^3*c + 16*a^4*b*c^2 - sqrt((
a^2*b^5 - 8*a^3*b^3*c + 16*a^4*b*c^2)^2 - 4*(a^3*b^4 - 8*a^4*b^2*c + 16*a^
5*c^2)*(a^2*b^4*c - 8*a^3*b^2*c^2 + 16*a^4*c^3)))/(a^2*b^4*c - 8*a^3*b^2*c
^2 + 16*a^4*c^3))/((b^8 - 16*a*b^6*c - 2*b^7*c + 96*a^2*b^4*c^2 + 24*a*b^5
*c^2 + b^6*c^2 - 256*a^3*b^2*c^3 - 96*a^2*b^3*c^3 - 12*a*b^4*c^3 + 256*a^4
*c^4 + 128*a^3*b*c^4 + 48*a^2*b^2*c^4 - 64*a^3*c^5)*c^2) + 1/32*(3*(sqrt(2
)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^8 - 17*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4
*a*c)*c)*a*b^6*c - 2*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^7*c - 2*b^8
*c + 116*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^4*c^2 + 26*sqrt(2)*
sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^5*c^2 + sqrt(2)*sqrt(b*c + sqrt(b^2 -
4*a*c)*c)*b^6*c^2 + 34*a*b^6*c^2 + 2*b^7*c^2 - 368*sqrt(2)*sqrt(b*c + sqrt
(b^2 - 4*a*c)*c)*a^3*b^2*c^3 - 128*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)
*a^2*b^3*c^3 - 13*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^4*c^3 - 2...
```

3.53.9 Mupad [B] (verification not implemented)

Time = 9.38 (sec) , antiderivative size = 8689, normalized size of antiderivative = 13.99

$$\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^3}\,dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2)/(a + b*x^2 + c*x^4)^3,x)
```

output
```
((x^2*(5*a*c^2*e + b^2*c*e))/(b^4 + 16*a^2*c^2 - 8*a*b^2*c) - (b^3*e - 10*
a*b*c*e)/(4*(b^4 + 16*a^2*c^2 - 8*a*b^2*c)) + (x^5*(28*a^2*c^3*d + 6*b^4*c
*d + 2*a*b^3*c*f - 49*a*b^2*c^2*d + 28*a^2*b*c^2*f))/(8*a^2*(b^4 + 16*a^2*
c^2 - 8*a*b^2*c)) + (x*(5*b^4*d + 44*a^2*c^2*d - a*b^3*f - 37*a*b^2*c*d +
16*a^2*b*c*f))/(8*a*(b^4 + 16*a^2*c^2 - 8*a*b^2*c)) + (3*c^3*e*x^6)/(b^4 +
 16*a^2*c^2 - 8*a*b^2*c) + (x^3*(3*b^5*d + 36*a^3*c^2*f + a*b^4*f - 20*a*b
^3*c*d - 4*a^2*b*c^2*d + 5*a^2*b^2*c*f))/(8*a^2*(b^4 + 16*a^2*c^2 - 8*a*b^
2*c)) + (9*b*c^2*e*x^4)/(2*(b^4 + 16*a^2*c^2 - 8*a*b^2*c)) + (c*x^7*(20*a^
2*c^2*f + 3*b^3*c*d - 24*a*b*c^2*d + a*b^2*c*f))/(8*a^2*(b^4 + 16*a^2*c^2
- 8*a*b^2*c)))/(x^4*(2*a*c + b^2) + a^2 + c^2*x^8 + 2*a*b*x^2 + 2*b*c*x^6)
 + symsum(log(root(56371445760*a^11*b^8*c^6*z^4 - 503316480*a^8*b^14*c^3*z
^4 + 47185920*a^7*b^16*c^2*z^4 - 171798691840*a^14*b^2*c^9*z^4 + 193273528
320*a^13*b^4*c^8*z^4 - 128849018880*a^12*b^6*c^7*z^4 - 16911433728*a^10*b^
10*c^5*z^4 + 3523215360*a^9*b^12*c^4*z^4 - 2621440*a^6*b^18*c*z^4 + 687194
76736*a^15*c^10*z^4 + 65536*a^5*b^20*z^4 - 73728*a^2*b^16*c*d*f*z^2 - 1321
205760*a^9*b^2*c^8*d*f*z^2 + 732168192*a^7*b^6*c^6*d*f*z^2 - 366280704*a^6
*b^8*c^5*d*f*z^2 - 330301440*a^8*b^4*c^7*d*f*z^2 + 96583680*a^5*b^10*c^4*d
*f*z^2 - 15175680*a^4*b^12*c^3*d*f*z^2 + 1428480*a^3*b^14*c^2*d*f*z^2 - 44
0401920*a^10*b*c^8*f^2*z^2 + 1761607680*a^10*c^9*d*f*z^2 - 14080*a^3*b^15*
c*f^2*z^2 + 6936330240*a^8*b^3*c^8*d^2*z^2 + 2464874496*a^6*b^7*c^6*d^2...
```

3.53. $\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^3}\,dx$

3.54 $\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^3}\,dx$

3.54.1 Optimal result

Integrand size = 30, antiderivative size = 646

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^3}\,dx = \frac{x(b^2d-2acd-abf+c(bd-2af)x^2)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}$$
$$-\frac{be-2ag+(2ce-bg)x^2}{4(b^2-4ac)(a+bx^2+cx^4)^2}+\frac{3(2ce-bg)(b+2cx^2)}{4(b^2-4ac)^2(a+bx^2+cx^4)}$$
$$+\frac{x(3b^4d-25ab^2cd+28a^2c^2d+ab^3f+8a^2bcf+c(3b^3d-24abcd+ab^2f+20a^2cf)x^2)}{8a^2(b^2-4ac)^2(a+bx^2+cx^4)}$$
$$+\frac{\sqrt{c}(3b^4d+b^3(3\sqrt{b^2-4ac}d+af)-4abc(6\sqrt{b^2-4ac}d+13af)-ab^2(30cd-\sqrt{b^2-4ac}f)+4a^2c(42}{8\sqrt{2}a^2(b^2-4ac)^{5/2}\sqrt{b-\sqrt{b^2-4ac}}}$$
$$+\frac{\sqrt{c}\left(3b^3d-24abcd+ab^2f+20a^2cf-\frac{3b^4d-30ab^2cd+168a^2c^2d+ab^3f-52a^2bcf}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{8\sqrt{2}a^2(b^2-4ac)^2\sqrt{b+\sqrt{b^2-4ac}}}$$
$$-\frac{3c(2ce-bg)\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{5/2}}$$

output
```
1/4*x*(b^2*d-2*a*c*d-a*b*f+c*(-2*a*f+b*d)*x^2)/a/(-4*a*c+b^2)/(c*x^4+b*x^2
+a)^2+1/4*(-b*e+2*a*g-(-b*g+2*c*e)*x^2)/(-4*a*c+b^2)/(c*x^4+b*x^2+a)^2+3/4
*(-b*g+2*c*e)*(2*c*x^2+b)/(-4*a*c+b^2)^2/(c*x^4+b*x^2+a)+1/8*x*(3*b^4*d-25
*a*b^2*c*d+28*a^2*c^2*d+a*b^3*f+8*a^2*b*c*f+c*(20*a^2*c*f+a*b^2*f-24*a*b*c
*d+3*b^3*d)*x^2)/a^2/(-4*a*c+b^2)^2/(c*x^4+b*x^2+a)-3*c*(-b*g+2*c*e)*arcta
nh((2*c*x^2+b)/(-4*a*c+b^2)^(1/2))/(-4*a*c+b^2)^(5/2)+1/16*arctan(x*2^(1/2
)*c^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2))*c^(1/2)*(3*b^4*d+b^3*(a*f+3*d*(-4*
a*c+b^2)^(1/2))-4*a*b*c*(13*a*f+6*d*(-4*a*c+b^2)^(1/2))-a*b^2*(30*c*d-f*(-
4*a*c+b^2)^(1/2))+4*a^2*c*(42*c*d+5*f*(-4*a*c+b^2)^(1/2)))/a^2/(-4*a*c+b^2
)^(5/2)*2^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2)+1/16*arctan(x*2^(1/2)*c^(1/2)
/(b+(-4*a*c+b^2)^(1/2))^(1/2))*c^(1/2)*(3*b^3*d-24*a*b*c*d+a*b^2*f+20*a^2*
c*f+(52*a^2*b*c*f-168*a^2*c^2*d-a*b^3*f+30*a*b^2*c*d-3*b^4*d)/(-4*a*c+b^2)
^(1/2))/a^2/(-4*a*c+b^2)^2*2^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2)
```

3.54.2 Mathematica [A] (verified)

Time = 2.36 (sec) , antiderivative size = 661, normalized size of antiderivative = 1.02

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^3}\,dx$$

$$= \frac{1}{16}\left(\frac{-8a^2g-4bdx(b+cx^2)+8acx(d+x(e+fx))+4ab(e+x(f-gx))}{a(-b^2+4ac)(a+bx^2+cx^4)^2}\right.$$

$$+\frac{2(3b^3dx(b+cx^2)+abx(-25bcd+b^2f-24c^2dx^2+bcfx^2)+a^2(-6b^2g+4c^2x(7d+6ex+5fx^2)+4bc}{a^2(b^2-4ac)^2(a+bx^2+cx^4)}$$

$$+\frac{\sqrt{2}\sqrt{c}\left(3b^4d+b^3\left(3\sqrt{b^2-4ac}d+af\right)-4abc\left(6\sqrt{b^2-4ac}d+13af\right)+ab^2\left(-30cd+\sqrt{b^2-4ac}f\right)+4a}{a^2(b^2-4ac)^{5/2}\sqrt{b-\sqrt{b^2-4ac}}}$$

$$+\frac{\sqrt{2}\sqrt{c}\left(-3b^4d+b^3\left(3\sqrt{b^2-4ac}d-af\right)+4abc\left(-6\sqrt{b^2-4ac}d+13af\right)+ab^2\left(30cd+\sqrt{b^2-4ac}f\right)+4}{a^2(b^2-4ac)^{5/2}\sqrt{b+\sqrt{b^2-4ac}}}$$

$$-\frac{24c(-2ce+bg)\log\left(-b+\sqrt{b^2-4ac}-2cx^2\right)}{(b^2-4ac)^{5/2}}$$

$$\left.+\frac{24c(-2ce+bg)\log\left(b+\sqrt{b^2-4ac}+2cx^2\right)}{(b^2-4ac)^{5/2}}\right)$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4)^3,x]
```

output
```
((-8*a^2*g - 4*b*d*x*(b + c*x^2) + 8*a*c*x*(d + x*(e + f*x)) + 4*a*b*(e +
x*(f - g*x)))/(a*(-b^2 + 4*a*c)*(a + b*x^2 + c*x^4)^2) + (2*(3*b^3*d*x*(b
+ c*x^2) + a*b*x*(-25*b*c*d + b^2*f - 24*c^2*d*x^2 + b*c*f*x^2) + a^2*(-6*
b^2*g + 4*c^2*x*(7*d + 6*e*x + 5*f*x^2) + 4*b*c*(3*e + 2*f*x - 3*g*x^2))))
/(a^2*(b^2 - 4*a*c)^2*(a + b*x^2 + c*x^4)) + (Sqrt[2]*Sqrt[c]*(3*b^4*d + b
^3*(3*Sqrt[b^2 - 4*a*c]*d + a*f) - 4*a*b*c*(6*Sqrt[b^2 - 4*a*c]*d + 13*a*f
) + a*b^2*(-30*c*d + Sqrt[b^2 - 4*a*c]*f) + 4*a^2*c*(42*c*d + 5*Sqrt[b^2 -
 4*a*c]*f))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]])/(a^2*
(b^2 - 4*a*c)^(5/2)*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + (Sqrt[2]*Sqrt[c]*(-3*b^
4*d + b^3*(3*Sqrt[b^2 - 4*a*c]*d - a*f) + 4*a*b*c*(-6*Sqrt[b^2 - 4*a*c]*d
+ 13*a*f) + a*b^2*(30*c*d + Sqrt[b^2 - 4*a*c]*f) + 4*a^2*c*(-42*c*d + 5*Sq
rt[b^2 - 4*a*c]*f))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqrt[b^2 - 4*a*c]]
])/(a^2*(b^2 - 4*a*c)^(5/2)*Sqrt[b + Sqrt[b^2 - 4*a*c]]) - (24*c*(-2*c*e +
 b*g)*Log[-b + Sqrt[b^2 - 4*a*c] - 2*c*x^2])/(b^2 - 4*a*c)^(5/2) + (24*c*(
-2*c*e + b*g)*Log[b + Sqrt[b^2 - 4*a*c] + 2*c*x^2])/(b^2 - 4*a*c)^(5/2))/1
6
```

3.54.3 Rubi [A] (verified)

Time = 1.02 (sec) , antiderivative size = 627, normalized size of antiderivative = 0.97, number of steps used = 13, number of rules used = 12, $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$, Rules used = {2202, 1492, 25, 1492, 25, 1480, 218, 1576, 1159, 1086, 1083, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^3}dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^3}dx+\int \frac{x(gx^2+e)}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{1492}$$

$$-\frac{\int -\frac{3db^2+afb+5c(bd-2af)x^2-14acd}{(cx^4+bx^2+a)^2}dx}{4a(b^2-4ac)}+\int \frac{x(gx^2+e)}{(cx^4+bx^2+a)^3}dx+\frac{x(cx^2(bd-2af)-abf-2acd+b^2d)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}$$

$$\Big\downarrow \text{25}$$

$$\frac{\int \frac{3db^2+afb+5c(bd-2af)x^2-14acd}{(cx^4+bx^2+a)^2}dx}{4a\left(b^2-4ac\right)}+\int \frac{x\left(gx^2+e\right)}{\left(cx^4+bx^2+a\right)^3}dx+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$\big\downarrow$ 1492

$$\frac{\frac{x\left(cx^2\left(20a^2cf+ab^2f-24abcd+3b^3d\right)+8a^2bcf+28a^2c^2d+ab^3f-25ab^2cd+3b^4d\right)}{2a(b^2-4ac)(a+bx^2+cx^4)}-\frac{\int -\frac{3db^4+afb^3-27acdb^2-16a^2cfb+c\left(3db^3+afb^2-24acdb+20a^2cf\right)\ldots}{cx^4+bx^2+a}\ldots}{2a(b^2-4ac)}}{4a\left(b^2-4ac\right)}$$

$$\int \frac{x\left(gx^2+e\right)}{\left(cx^4+bx^2+a\right)^3}dx+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$\big\downarrow$ 25

$$\frac{\frac{\int \frac{3db^4+afb^3-27acdb^2-16a^2cfb+c\left(3db^3+afb^2-24acdb+20a^2cf\right)x^2+84a^2c^2d}{cx^4+bx^2+a}dx}{2a(b^2-4ac)}+\frac{x\left(cx^2\left(20a^2cf+ab^2f-24abcd+3b^3d\right)+8a^2bcf+28a^2c^2d+ab^3f-25\ldots\right)}{2a(b^2-4ac)(a+bx^2+cx^4)}}{4a\left(b^2-4ac\right)}$$

$$\int \frac{x\left(gx^2+e\right)}{\left(cx^4+bx^2+a\right)^3}dx+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$\big\downarrow$ 1480

$$\frac{\frac{\frac{1}{2}c\left(\frac{-52a^2bcf+168a^2c^2d+ab^3f-30ab^2cd+3b^4d}{\sqrt{b^2-4ac}}+20a^2cf+ab^2f-24abcd+3b^3d\right)\int \frac{1}{cx^2+\frac{1}{2}\left(b-\sqrt{b^2-4ac}\right)}dx+\frac{1}{2}c\left(-\frac{-52a^2bcf+168a^2c^2d+ab^3f-30ab^2cd+\ldots}{\sqrt{b^2-4ac}}\ldots\right.}{2a(b^2-4ac)}\ldots}{4a\left(b^2\ldots\right.}$$

$$\int \frac{x\left(gx^2+e\right)}{\left(cx^4+bx^2+a\right)^3}dx+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$\big\downarrow$ 218

$$\int \frac{x\left(gx^2+e\right)}{\left(cx^4+bx^2+a\right)^3}dx+$$

$$\frac{\frac{\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{-52a^2bcf+168a^2c^2d+ab^3f-30ab^2cd+3b^4d}{\sqrt{b^2-4ac}}+20a^2cf+ab^2f-24abcd+3b^3d\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{-52a^2bcf+168a^2c^2\ldots}{\sqrt{b\ldots}}\ldots\right.}{\sqrt{2}\sqrt{\ldots}}}{2a(b^2-4ac)}\ldots}{4a\left(b^2-4a\ldots\right.}$$

$$\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$\big\downarrow$ 1576

3.54. $\int \frac{d+ex+fx^2+gx^3}{\left(a+bx^2+cx^4\right)^3}\,dx$

$$\frac{1}{2}\int \frac{gx^2+e}{(cx^4+bx^2+a)^3}dx^2+$$

$$\frac{\frac{\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{-52a^2bcf+168a^2c^2d+ab^3f-30ab^2cd+3b^4d}{\sqrt{b^2-4ac}}+20a^2cf+ab^2f-24abcd+3b^3d\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{-52a^2bcf+168a^2c^2\cdots}{\sqrt{b\cdots}}\right.}{\sqrt{2}\sqrt{\sqrt{\cdots}}}}{2a(b^2-4ac)}}{4a\left(b^2-4a\cdots\right.}$$

$$\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$$\Big\downarrow \; 1159$$

$$\frac{1}{2}\left(-\frac{3(2ce-bg)\int \frac{1}{(cx^4+bx^2+a)^2}dx^2}{2\left(b^2-4ac\right)}-\frac{-2ag+x^2(2ce-bg)+be}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)+$$

$$\frac{\frac{\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{-52a^2bcf+168a^2c^2d+ab^3f-30ab^2cd+3b^4d}{\sqrt{b^2-4ac}}+20a^2cf+ab^2f-24abcd+3b^3d\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{-52a^2bcf+168a^2c^2\cdots}{\sqrt{b\cdots}}\right.}{\sqrt{2}\sqrt{\sqrt{\cdots}}}}{2a(b^2-4ac)}}{4a\left(b^2-4a\cdots\right.}$$

$$\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$$\Big\downarrow \; 1086$$

$$\frac{1}{2}\left(-\frac{3(2ce-bg)\left(-\frac{2c\int \frac{1}{cx^4+bx^2+a}dx^2}{b^2-4ac}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{2\left(b^2-4ac\right)}-\frac{-2ag+x^2(2ce-bg)+be}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)+$$

$$\frac{\frac{\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{-52a^2bcf+168a^2c^2d+ab^3f-30ab^2cd+3b^4d}{\sqrt{b^2-4ac}}+20a^2cf+ab^2f-24abcd+3b^3d\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{-52a^2bcf+168a^2c^2\cdots}{\sqrt{b\cdots}}\right.}{\sqrt{2}\sqrt{\sqrt{\cdots}}}}{2a(b^2-4ac)}}{4a\left(b^2-4a\cdots\right.}$$

$$\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$$\Big\downarrow \; 1083$$

$$\frac{1}{2}\left(-\frac{3(2ce-bg)\left(\frac{4c\int \frac{1}{-x^4+b^2-4ac}d(2cx^2+b)}{b^2-4ac}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{2\left(b^2-4ac\right)}-\frac{-2ag+x^2(2ce-bg)+be}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)+$$

$$\frac{\frac{\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{-52a^2bcf+168a^2c^2d+ab^3f-30ab^2cd+3b^4d}{\sqrt{b^2-4ac}}+20a^2cf+ab^2f-24abcd+3b^3d\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{-52a^2bcf+168a^2c^2\cdots}{\sqrt{b\cdots}}\right.}{\sqrt{2}\sqrt{\sqrt{\cdots}}}}{2a(b^2-4ac)}}{4a\left(b^2-4a\cdots\right.}$$

$$\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

3.54. $\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^3}\,dx$

$\downarrow$ 219

$$\frac{\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{-52a^2bcf+168a^2c^2d+ab^3f-30ab^2cd+3b^4d}{\sqrt{b^2-4ac}}+20a^2cf+ab^2f-24abcd+3b^3d\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{-52a^2bcf+168a^2c^2\cdots}{\sqrt{b\cdots}}\right.}{\sqrt{2}\sqrt{\cdots}}}{2a(b^2-4ac)}$$

$$4a\left(b^2-4a\cdots\right.$$

$$\frac{1}{2}\left(-\frac{3(2ce-bg)\left(\frac{4c\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{2\left(b^2-4ac\right)}-\frac{-2ag+x^2(2ce-bg)+be}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)+$$

$$\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4)^3,x]
```

output
```
(x*(b^2*d - 2*a*c*d - a*b*f + c*(b*d - 2*a*f)*x^2))/(4*a*(b^2 - 4*a*c)*(a
+ b*x^2 + c*x^4)^2) + ((x*(3*b^4*d - 25*a*b^2*c*d + 28*a^2*c^2*d + a*b^3*f
 + 8*a^2*b*c*f + c*(3*b^3*d - 24*a*b*c*d + a*b^2*f + 20*a^2*c*f)*x^2))/(2*
a*(b^2 - 4*a*c)*(a + b*x^2 + c*x^4)) + ((Sqrt[c]*(3*b^3*d - 24*a*b*c*d + a
*b^2*f + 20*a^2*c*f + (3*b^4*d - 30*a*b^2*c*d + 168*a^2*c^2*d + a*b^3*f -
52*a^2*b*c*f)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[
b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + (Sqrt[c]*(3*b^3*d
- 24*a*b*c*d + a*b^2*f + 20*a^2*c*f - (3*b^4*d - 30*a*b^2*c*d + 168*a^2*c^
2*d + a*b^3*f - 52*a^2*b*c*f)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x
)/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[b + Sqrt[b^2 - 4*a*c]]))/(2*
a*(b^2 - 4*a*c)))/(4*a*(b^2 - 4*a*c)) + (-1/2*(b*e - 2*a*g + (2*c*e - b*g)
*x^2)/((b^2 - 4*a*c)*(a + b*x^2 + c*x^4)^2) - (3*(2*c*e - b*g)*(-((b + 2*c
*x^2)/((b^2 - 4*a*c)*(a + b*x^2 + c*x^4))) + (4*c*ArcTanh[(b + 2*c*x^2)/Sq
rt[b^2 - 4*a*c]])/(b^2 - 4*a*c)^(3/2)))/(2*(b^2 - 4*a*c)))/2
```

3.54. $\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^3}\,dx$

3.54.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1086
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] - Simp[2*c*((2*p +
 3)/((p + 1)*(b^2 - 4*a*c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; Fre
eQ[{a, b, c}, x] && ILtQ[p, -1]
```

rule 1159
```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[((b*d - 2*a*e + (2*c*d - b*e)*x)/((p + 1)*(b^2 - 4*a*c)))*(a + b*
x + c*x^2)^(p + 1), x] - Simp[(2*p + 3)*((2*c*d - b*e)/((p + 1)*(b^2 - 4*a*
c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; FreeQ[{a, b, c, d, e}, x] &
& LtQ[p, -1] && NeQ[p, -3/2]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.54.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.58 (sec) , antiderivative size = 644, normalized size of antiderivative = 1.00

method	result
risch	$\frac{\frac{c^{2}\left(20a^{2}cf+a\,b^{2}f-24abcd+3b^{3}d\right)x^{7}}{8a^{2}\left(16a^{2}c^{2}-8a\,b^{2}c+b^{4}\right)}-\frac{3(bg-2ec)c^{2}x^{6}}{2\left(16a^{2}c^{2}-8a\,b^{2}c+b^{4}\right)}+\frac{c\left(28a^{2}bcf+28a^{2}c^{2}d+2a\,b^{3}f-49a\,b^{2}cd+6d\,b^{4}\right)x^{5}}{8a^{2}\left(16a^{2}c^{2}-8a\,b^{2}c+b^{4}\right)}-\frac{9b(bg-2ec)c\,x^{4}}{4\left(16a^{2}c^{2}-8a\,b^{2}c+b^{4}\right)}+\frac{(36\cdots}{(c\,x\cdots}$
default	Expression too large to display

input
```
int((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x,method=_RETURNVERBOSE)
```

output
```
(1/8*c^2*(20*a^2*c*f+a*b^2*f-24*a*b*c*d+3*b^3*d)/a^2/(16*a^2*c^2-8*a*b^2*c
+b^4)*x^7-3/2*(b*g-2*c*e)*c^2/(16*a^2*c^2-8*a*b^2*c+b^4)*x^6+1/8/a^2*c*(28
*a^2*b*c*f+28*a^2*c^2*d+2*a*b^3*f-49*a*b^2*c*d+6*b^4*d)/(16*a^2*c^2-8*a*b^
2*c+b^4)*x^5-9/4*b*(b*g-2*c*e)*c/(16*a^2*c^2-8*a*b^2*c+b^4)*x^4+1/8*(36*a^
3*c^2*f+5*a^2*b^2*c*f-4*a^2*b*c^2*d+a*b^4*f-20*a*b^3*c*d+3*b^5*d)/a^2/(16*
a^2*c^2-8*a*b^2*c+b^4)*x^3-1/2*(5*a*c+b^2)*(b*g-2*c*e)/(16*a^2*c^2-8*a*b^2
*c+b^4)*x^2+1/8*(16*a^2*b*c*f+44*a^2*c^2*d-a*b^3*f-37*a*b^2*c*d+5*b^4*d)/(
16*a^2*c^2-8*a*b^2*c+b^4)/a*x-1/4*(8*a^2*c*g+a*b^2*g-10*a*b*c*e+b^3*e)/(16
*a^2*c^2-8*a*b^2*c+b^4))/(c*x^4+b*x^2+a)^2+1/16*sum((c*(20*a^2*c*f+a*b^2*f
-24*a*b*c*d+3*b^3*d)/a^2/(16*a^2*c^2-8*a*b^2*c+b^4)*_R^2-24*(b*g-2*c*e)*c/
(16*a^2*c^2-8*a*b^2*c+b^4)*_R-(16*a^2*b*c*f-84*a^2*c^2*d-a*b^3*f+27*a*b^2*
c*d-3*b^4*d)/a^2/(16*a^2*c^2-8*a*b^2*c+b^4))/(2*_R^3*c+_R*b)*ln(x-_R),_R=R
ootOf(_Z^4*c+_Z^2*b+a))
```

3.54.5 Fricas [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^3}\,dx = \text{Timed out}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x, algorithm="fricas")
```

output
```
Timed out
```

3.54.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^3}\,dx = \text{Timed out}$$

input
```
integrate((g*x**3+f*x**2+e*x+d)/(c*x**4+b*x**2+a)**3,x)
```

output
```
Timed out
```

3.54.7 Maxima [F]

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^3}\,dx=\int \frac{gx^3+fx^2+ex+d}{(cx^4+bx^2+a)^3}\,dx$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x, algorithm="maxima")
```

output
```
1/8*((3*(b^3*c^2 - 8*a*b*c^3)*d + (a*b^2*c^2 + 20*a^2*c^3)*f)*x^7 + 12*(2*
a^2*c^3*e - a^2*b*c^2*g)*x^6 + ((6*b^4*c - 49*a*b^2*c^2 + 28*a^2*c^3)*d +
2*(a*b^3*c + 14*a^2*b*c^2)*f)*x^5 + 18*(2*a^2*b*c^2*e - a^2*b^2*c*g)*x^4 +
 ((3*b^5 - 20*a*b^3*c - 4*a^2*b*c^2)*d + (a*b^4 + 5*a^2*b^2*c + 36*a^3*c^2
)*f)*x^3 + 4*(2*(a^2*b^2*c + 5*a^3*c^2)*e - (a^2*b^3 + 5*a^3*b*c)*g)*x^2 -
 2*(a^2*b^3 - 10*a^3*b*c)*e - 2*(a^3*b^2 + 8*a^4*c)*g + ((5*a*b^4 - 37*a^2
*b^2*c + 44*a^3*c^2)*d - (a^2*b^3 - 16*a^3*b*c)*f)*x)/((a^2*b^4*c^2 - 8*a^
3*b^2*c^3 + 16*a^4*c^4)*x^8 + a^4*b^4 - 8*a^5*b^2*c + 16*a^6*c^2 + 2*(a^2*
b^5*c - 8*a^3*b^3*c^2 + 16*a^4*b*c^3)*x^6 + (a^2*b^6 - 6*a^3*b^4*c + 32*a^
5*c^3)*x^4 + 2*(a^3*b^5 - 8*a^4*b^3*c + 16*a^5*b*c^2)*x^2) + 1/8*integrate
(((3*(b^3*c - 8*a*b*c^2)*d + (a*b^2*c + 20*a^2*c^2)*f)*x^2 + 3*(b^4 - 9*a*
b^2*c + 28*a^2*c^2)*d + (a*b^3 - 16*a^2*b*c)*f + 24*(2*a^2*c^2*e - a^2*b*c
*g)*x)/(c*x^4 + b*x^2 + a), x)/(a^2*b^4 - 8*a^3*b^2*c + 16*a^4*c^2)
```

3.54.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 5619 vs. $2(586) = 1172$.

Time = 2.55 (sec) , antiderivative size = 5619, normalized size of antiderivative = 8.70

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^3}\,dx=\text{Too large to display}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x, algorithm="giac")
```

output
```
1/32*(3*(sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^8 - 17*sqrt(2)*sqrt(b*c
 + sqrt(b^2 - 4*a*c)*c)*a*b^6*c - 2*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c
)*b^7*c - 2*b^8*c + 116*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^4*c^
2 + 26*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^5*c^2 + sqrt(2)*sqrt(b*
c + sqrt(b^2 - 4*a*c)*c)*b^6*c^2 + 34*a*b^6*c^2 - 2*b^7*c^2 - 368*sqrt(2)*
sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*b^2*c^3 - 128*sqrt(2)*sqrt(b*c + sqrt(
b^2 - 4*a*c)*c)*a^2*b^3*c^3 - 13*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a
*b^4*c^3 - 232*a^2*b^4*c^3 + 30*a*b^5*c^3 + 448*sqrt(2)*sqrt(b*c + sqrt(b^
2 - 4*a*c)*c)*a^4*c^4 + 224*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*b*
c^4 + 64*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^2*c^4 + 736*a^3*b^2
*c^4 - 176*a^2*b^3*c^4 - 112*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*c
^5 - 896*a^4*c^5 + 352*a^3*b*c^5 + sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sq
rt(b^2 - 4*a*c)*c)*b^7 - 15*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2
- 4*a*c)*c)*a*b^5*c - 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*
a*c)*c)*b^6*c + 88*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*
c)*a^2*b^3*c^2 + 22*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)
*c)*a*b^4*c^2 + sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*
b^5*c^2 - 176*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^
3*b*c^3 - 88*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2
*b^2*c^3 - 11*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)...
```

3.54.9 Mupad [B] (verification not implemented)

Time = 10.42 (sec) , antiderivative size = 13431, normalized size of antiderivative = 20.79

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^3}\,dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4)^3,x)
```

output

```
symsum(log((x*(13824*a^4*c^8*e^3 - 54*b^7*c^5*d^2*e + 27*b^8*c^4*d^2*g - 1
728*a^4*b^3*c^5*g^3 - 20160*a^4*c^8*d*e*f + 972*a*b^5*c^6*d^2*e + 24192*a^
3*b*c^8*d^2*e - 486*a*b^6*c^5*d^2*g + 6240*a^4*b*c^7*e*f^2 - 20736*a^4*b*c
^7*e^2*g - 7344*a^2*b^3*c^7*d^2*e + 3672*a^2*b^4*c^6*d^2*g - 6*a^2*b^5*c^5
*e*f^2 - 12096*a^3*b^2*c^7*d^2*g + 192*a^3*b^3*c^6*e*f^2 + 10368*a^4*b^2*c
^6*e*g^2 + 3*a^2*b^6*c^4*f^2*g - 96*a^3*b^4*c^5*f^2*g - 3120*a^4*b^2*c^6*f
^2*g - 36*a*b^6*c^5*d*e*f + 18*a*b^7*c^4*d*f*g + 10080*a^4*b*c^7*d*f*g + 9
00*a^2*b^4*c^6*d*e*f - 4896*a^3*b^2*c^7*d*e*f - 450*a^2*b^5*c^5*d*f*g + 24
48*a^3*b^3*c^6*d*f*g))/(64*(a^4*b^12 + 4096*a^10*c^6 - 24*a^5*b^10*c + 240
*a^6*b^8*c^2 - 1280*a^7*b^6*c^3 + 3840*a^8*b^4*c^4 - 6144*a^9*b^2*c^5)) -
root(56371445760*a^11*b^8*c^6*z^4 - 503316480*a^8*b^14*c^3*z^4 + 47185920*
a^7*b^16*c^2*z^4 - 171798691840*a^14*b^2*c^9*z^4 + 193273528320*a^13*b^4*c
^8*z^4 - 128849018880*a^12*b^6*c^7*z^4 - 16911433728*a^10*b^10*c^5*z^4 + 3
523215360*a^9*b^12*c^4*z^4 - 2621440*a^6*b^18*c*z^4 + 68719476736*a^15*c^1
0*z^4 + 65536*a^5*b^20*z^4 - 73728*a^2*b^16*c*d*f*z^2 + 1509949440*a^9*b^3
*c^7*e*g*z^2 - 1321205760*a^9*b^2*c^8*d*f*z^2 - 754974720*a^8*b^5*c^6*e*g*
z^2 + 732168192*a^7*b^6*c^6*d*f*z^2 - 366280704*a^6*b^8*c^5*d*f*z^2 - 3303
01440*a^8*b^4*c^7*d*f*z^2 + 188743680*a^7*b^7*c^5*e*g*z^2 + 96583680*a^5*b
^10*c^4*d*f*z^2 - 23592960*a^6*b^9*c^4*e*g*z^2 + 1179648*a^5*b^11*c^3*e*g*
z^2 - 15175680*a^4*b^12*c^3*d*f*z^2 + 1428480*a^3*b^14*c^2*d*f*z^2 - 12...
```

3.54. $\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^3}\,dx$

3.55 $\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^3}\,dx$

3.55.1 Optimal result

Integrand size = 35, antiderivative size = 679

$$
\begin{aligned}
\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^3}\,dx &= -\frac{be-2ag+(2ce-bg)x^2}{4(b^2-4ac)(a+bx^2+cx^4)^2} \\
&+ \frac{x(b^2d-abf-2a(cd-ah)+(bcd-2acf+abh)x^2)}{4a(b^2-4ac)(a+bx^2+cx^4)^2} + \frac{3(2ce-bg)(b+2cx^2)}{4(b^2-4ac)^2(a+bx^2+cx^4)} \\
&+ \frac{x(3b^4d+ab^3f+8a^2bcf+4a^2c(7cd+ah)-ab^2(25cd+7ah)+c(3b^3d+ab^2f+20a^2cf-12ab(2cd+}{8a^2(b^2-4ac)^2(a+bx^2+cx^4)} \\
&+ \frac{\sqrt{c}\left(3b^3d+ab^2f+20a^2cf-12ab(2cd+ah)+\frac{3b^4d+ab^3f-52a^2bcf-6ab^2(5cd-3ah)+24a^2c(7cd+ah)}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{}}{\sqrt{b-}}\right.}{8\sqrt{2}a^2(b^2-4ac)^2\sqrt{b-\sqrt{b^2-4ac}}} \\
&+ \frac{\sqrt{c}\left(3b^3d+ab^2f+20a^2cf-12ab(2cd+ah)-\frac{3b^4d+ab^3f-52a^2bcf-6ab^2(5cd-3ah)+24a^2c(7cd+ah)}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{}}{\sqrt{b+}}\right.}{8\sqrt{2}a^2(b^2-4ac)^2\sqrt{b+\sqrt{b^2-4ac}}} \\
&- \frac{3c(2ce-bg)\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{5/2}}
\end{aligned}
$$

output
```
1/4*(-b*e+2*a*g-(-b*g+2*c*e)*x^2)/(-4*a*c+b^2)/(c*x^4+b*x^2+a)^2+1/4*x*(b^
2*d-a*b*f-2*a*(-a*h+c*d)+(a*b*h-2*a*c*f+b*c*d)*x^2)/a/(-4*a*c+b^2)/(c*x^4+
b*x^2+a)^2+3/4*(-b*g+2*c*e)*(2*c*x^2+b)/(-4*a*c+b^2)^2/(c*x^4+b*x^2+a)+1/8
*x*(3*b^4*d+a*b^3*f+8*a^2*b*c*f+4*a^2*c*(a*h+7*c*d)-a*b^2*(7*a*h+25*c*d)+c
*(3*b^3*d+a*b^2*f+20*a^2*c*f-12*a*b*(a*h+2*c*d))*x^2)/a^2/(-4*a*c+b^2)^2/(
c*x^4+b*x^2+a)-3*c*(-b*g+2*c*e)*arctanh((2*c*x^2+b)/(-4*a*c+b^2)^(1/2))/(-
4*a*c+b^2)^(5/2)+1/16*arctan(x*2^(1/2)*c^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2
))*c^(1/2)*(3*b^3*d+a*b^2*f+20*a^2*c*f-12*a*b*(a*h+2*c*d)+(3*b^4*d+a*b^3*f
-52*a^2*b*c*f-6*a*b^2*(-3*a*h+5*c*d)+24*a^2*c*(a*h+7*c*d))/(-4*a*c+b^2)^(1
/2))/a^2/(-4*a*c+b^2)^2*2^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2)+1/16*arctan(x
*2^(1/2)*c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2))*c^(1/2)*(3*b^3*d+a*b^2*f+20
*a^2*c*f-12*a*b*(a*h+2*c*d)+(-3*b^4*d-a*b^3*f+52*a^2*b*c*f+6*a*b^2*(-3*a*h
+5*c*d)-24*a^2*c*(a*h+7*c*d))/(-4*a*c+b^2)^(1/2))/a^2/(-4*a*c+b^2)^2*2^(1/
2)/(b+(-4*a*c+b^2)^(1/2))^(1/2)
```

3.55.2 Mathematica [A] (verified)

Time = 3.90 (sec) , antiderivative size = 739, normalized size of antiderivative = 1.09

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{\left(a+bx^2+cx^4\right)^3}\,dx$$

$$= \frac{1}{16}\Bigg(\frac{-8a^2(g+hx)-4bdx(b+cx^2)+8acx(d+x(e+fx))+4ab(e+x(f-x(g+hx)))}{a\left(-b^2+4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$$+\frac{2(4a^3chx+3b^3dx(b+cx^2)+abx(b^2f-24c^2dx^2+bc(-25d+fx^2))+a^2(-b^2(6g+7hx)+4c^2x(7d+}{a^2\left(b^2-4ac\right)^2\left(a+bx^2+cx^4\right)}$$

$$+\frac{\sqrt{2}\sqrt{c}\big(3b^4d+b^3\big(3\sqrt{b^2-4ac}d+af\big)+4a^2c\big(42cd+5\sqrt{b^2-4ac}f+6ah\big)+ab^2\big(-30cd+\sqrt{b^2-4ac}f}{a^2\left(b^2-4ac\right)^{5/2}\sqrt{b-\sqrt{b^2-}}}$$

$$-\frac{\sqrt{2}\sqrt{c}\big(3b^4d+b^3\big(-3\sqrt{b^2-4ac}d+af\big)+4a^2c\big(42cd-5\sqrt{b^2-4ac}f+6ah\big)+ab^2\big(-30cd-\sqrt{b^2-4ac}}{a^2\left(b^2-4ac\right)^{5/2}\sqrt{b+\sqrt{b^2}}}$$

$$-\frac{24c(-2ce+bg)\log\left(-b+\sqrt{b^2-4ac}-2cx^2\right)}{\left(b^2-4ac\right)^{5/2}}$$

$$+\frac{24c(-2ce+bg)\log\left(b+\sqrt{b^2-4ac}+2cx^2\right)}{\left(b^2-4ac\right)^{5/2}}\Bigg)$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4)/(a + b*x^2 + c*x^4)^3,x]
```

output
```
((-8*a^2*(g + h*x) - 4*b*d*x*(b + c*x^2) + 8*a*c*x*(d + x*(e + f*x)) + 4*a
*b*(e + x*(f - x*(g + h*x))))/(a*(-b^2 + 4*a*c)*(a + b*x^2 + c*x^4)^2) + (
2*(4*a^3*c*h*x + 3*b^3*d*x*(b + c*x^2) + a*b*x*(b^2*f - 24*c^2*d*x^2 + b*c
*(-25*d + f*x^2)) + a^2*(-(b^2*(6*g + 7*h*x)) + 4*c^2*x*(7*d + x*(6*e + 5*
f*x)) + 4*b*c*(3*e + x*(2*f - 3*x*(g + h*x))))))/(a^2*(b^2 - 4*a*c)^2*(a +
 b*x^2 + c*x^4)) + (Sqrt[2]*Sqrt[c]*(3*b^4*d + b^3*(3*Sqrt[b^2 - 4*a*c]*d
+ a*f) + 4*a^2*c*(42*c*d + 5*Sqrt[b^2 - 4*a*c]*f + 6*a*h) + a*b^2*(-30*c*d
 + Sqrt[b^2 - 4*a*c]*f + 18*a*h) - 4*a*b*(6*c*Sqrt[b^2 - 4*a*c]*d + 13*a*c
*f + 3*a*Sqrt[b^2 - 4*a*c]*h))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[b^
2 - 4*a*c]]])/(a^2*(b^2 - 4*a*c)^(5/2)*Sqrt[b - Sqrt[b^2 - 4*a*c]]) - (Sqr
t[2]*Sqrt[c]*(3*b^4*d + b^3*(-3*Sqrt[b^2 - 4*a*c]*d + a*f) + 4*a^2*c*(42*c
*d - 5*Sqrt[b^2 - 4*a*c]*f + 6*a*h) + a*b^2*(-30*c*d - Sqrt[b^2 - 4*a*c]*f
 + 18*a*h) + 4*a*b*(6*c*Sqrt[b^2 - 4*a*c]*d - 13*a*c*f + 3*a*Sqrt[b^2 - 4*
a*c]*h))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(a^2*(b^
2 - 4*a*c)^(5/2)*Sqrt[b + Sqrt[b^2 - 4*a*c]]) - (24*c*(-2*c*e + b*g)*Log[-
b + Sqrt[b^2 - 4*a*c] - 2*c*x^2])/(b^2 - 4*a*c)^(5/2) + (24*c*(-2*c*e + b*
g)*Log[b + Sqrt[b^2 - 4*a*c] + 2*c*x^2])/(b^2 - 4*a*c)^(5/2))/16
```

3.55.3 Rubi [A] (verified)

Time = 1.40 (sec) , antiderivative size = 692, normalized size of antiderivative = 1.02, number of steps used = 13, number of rules used = 12, $\frac{\text{number of rules}}{\text{integrand size}} = 0.343$, Rules used = {2202, 1576, 1159, 1086, 1083, 219, 2206, 25, 1492, 25, 1480, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^3}dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{hx^4+fx^2+d}{(cx^4+bx^2+a)^3}dx+\int \frac{x(gx^2+e)}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{1576}$$

$$\int \frac{hx^4+fx^2+d}{(cx^4+bx^2+a)^3}dx+\frac{1}{2}\int \frac{gx^2+e}{(cx^4+bx^2+a)^3}dx^2$$

$$\Big\downarrow \text{1159}$$

$$\frac{1}{2}\left(-\frac{3(2ce-bg)\int \frac{1}{(cx^4+bx^2+a)^2}dx^2}{2(b^2-4ac)}-\frac{-2ag+x^2(2ce-bg)+be}{2(b^2-4ac)(a+bx^2+cx^4)^2}\right)+\int \frac{hx^4+fx^2+d}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{1086}$$

$$\frac{1}{2}\left(-\frac{3(2ce-bg)\left(-\frac{2c\int \frac{1}{cx^4+bx^2+a}dx^2}{b^2-4ac}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{2\left(b^2-4ac\right)}-\frac{-2ag+x^2(2ce-bg)+be}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)+\int \frac{hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^3}dx$$

$$\Big\downarrow \text{1083}$$

$$\frac{1}{2}\left(-\frac{3(2ce-bg)\left(\frac{4c\int \frac{1}{-x^4+b^2-4ac}d(2cx^2+b)}{b^2-4ac}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{2\left(b^2-4ac\right)}-\frac{-2ag+x^2(2ce-bg)+be}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)+\int \frac{hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^3}dx$$

$$\Big\downarrow \text{219}$$

$$\int \frac{hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^3}dx+\frac{1}{2}\left(-\frac{3(2ce-bg)\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{2\left(b^2-4ac\right)}-\frac{-2ag+x^2(2ce-bg)+be}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)$$

$$\Big\downarrow \text{2206}$$

$$-\frac{\int -\frac{3db^2+afb+5(bcd-2acf+abh)x^2-2a(7cd+ah)}{\left(cx^4+bx^2+a\right)^2}dx}{4a\left(b^2-4ac\right)}+\frac{1}{2}\left(-\frac{3(2ce-bg)\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{2\left(b^2-4ac\right)}-\frac{-2ag+x^2(2ce-bg)+be}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)+\frac{x\left(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$$\Big\downarrow \text{25}$$

3.55. $\int \frac{d+ex+fx^2+gx^3+hx^4}{\left(a+bx^2+cx^4\right)^3}\,dx$

$$\frac{\int \frac{3db^2+afb+5(bcd-2acf+abh)x^2-2a(7cd+ah)}{(cx^4+bx^2+a)^2}dx}{4a(b^2-4ac)}+$$

$$\frac{1}{2}\left(-\frac{3(2ce-bg)\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{2(b^2-4ac)}-\frac{-2ag+x^2(2ce-bg)+be}{2(b^2-4ac)(a+bx^2+cx^4)^2}\right)+$$

$$\frac{x(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}$$

$\big\downarrow$ 1492

$$\frac{\frac{x(cx^2(20a^2cf+ab^2f-12ab(ah+2cd)+3b^3d)+8a^2bcf+4a^2c(ah+7cd)+ab^3f-ab^2(7ah+25cd)+3b^4d)}{2a(b^2-4ac)(a+bx^2+cx^4)}-\int -\frac{3db^4+afb^3-3a(9cd-ah)b^2-16a^2cfb+c(\cdots}{\cdots}}{4a(b^2-4ac)}$$

$$\frac{1}{2}\left(-\frac{3(2ce-bg)\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{2(b^2-4ac)}-\frac{-2ag+x^2(2ce-bg)+be}{2(b^2-4ac)(a+bx^2+cx^4)^2}\right)+$$

$$\frac{x(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}$$

$\big\downarrow$ 25

$$\frac{\frac{\int \frac{3db^4+afb^3-3a(9cd-ah)b^2-16a^2cfb+c\left(3db^3+afb^2-12a(2cd+ah)b+20a^2cf\right)x^2+12a^2c(7cd+ah)}{cx^4+bx^2+a}dx}{2a(b^2-4ac)}+\frac{x(cx^2(20a^2cf+ab^2f-12ab(ah+2cd)+3b^3d)+8a\cdots}{2a(b^2-4ac\cdots}}{4a(b^2-4ac)}$$

$$\frac{1}{2}\left(-\frac{3(2ce-bg)\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{2(b^2-4ac)}-\frac{-2ag+x^2(2ce-bg)+be}{2(b^2-4ac)(a+bx^2+cx^4)^2}\right)+$$

$$\frac{x(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}$$

$\big\downarrow$ 1480

3.55. $\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^3}\,dx$

$$\frac{\frac{1}{2}c\left(\frac{-52a^2bcf+24a^2c(ah+7cd)+ab^3f-6ab^2(5cd-3ah)+3b^4d}{\sqrt{b^2-4ac}}+20a^2cf+ab^2f-12ab(ah+2cd)+3b^3d\right)\int\frac{1}{cx^2+\frac{1}{2}\left(b-\sqrt{b^2-4ac}\right)}dx+\frac{1}{2}c\left(-\frac{-52a^2bcf+24a^2c}{}\right.}{2a(b^2-4ac)}$$

$$\frac{1}{2}\left(-\frac{3(2ce-bg)\left(\frac{4c\,\mathrm{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{2(b^2-4ac)}-\frac{-2ag+x^2(2ce-bg)+be}{2(b^2-4ac)(a+bx^2+cx^4)^2}\right)+$$

$$\frac{x\left(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d\right)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}$$

$$\Big\downarrow \; 218$$

$$\frac{\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{-52a^2bcf+24a^2c(ah+7cd)+ab^3f-6ab^2(5cd-3ah)+3b^4d}{\sqrt{b^2-4ac}}+20a^2cf+ab^2f-12ab(ah+2cd)+3b^3d\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\right.}{}}{2a(b^2-4ac)}$$

$$\frac{1}{2}\left(-\frac{3(2ce-bg)\left(\frac{4c\,\mathrm{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{2(b^2-4ac)}-\frac{-2ag+x^2(2ce-bg)+be}{2(b^2-4ac)(a+bx^2+cx^4)^2}\right)+$$

$$\frac{x\left(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d\right)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4)/(a + b*x^2 + c*x^4)^3,x]
```

3.55. $\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^3}\,dx$

output
```
(x*(b^2*d - a*b*f - 2*a*(c*d - a*h) + (b*c*d - 2*a*c*f + a*b*h)*x^2))/(4*a
*(b^2 - 4*a*c)*(a + b*x^2 + c*x^4)^2) + ((x*(3*b^4*d + a*b^3*f + 8*a^2*b*c
*f + 4*a^2*c*(7*c*d + a*h) - a*b^2*(25*c*d + 7*a*h) + c*(3*b^3*d + a*b^2*f
 + 20*a^2*c*f - 12*a*b*(2*c*d + a*h))*x^2))/(2*a*(b^2 - 4*a*c)*(a + b*x^2
+ c*x^4)) + ((Sqrt[c]*(3*b^3*d + a*b^2*f + 20*a^2*c*f - 12*a*b*(2*c*d + a*
h) + (3*b^4*d + a*b^3*f - 52*a^2*b*c*f - 6*a*b^2*(5*c*d - 3*a*h) + 24*a^2*
c*(7*c*d + a*h))/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sq
rt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + (Sqrt[c]*(3*b^3
*d + a*b^2*f + 20*a^2*c*f - 12*a*b*(2*c*d + a*h) - (3*b^4*d + a*b^3*f - 52
*a^2*b*c*f - 6*a*b^2*(5*c*d - 3*a*h) + 24*a^2*c*(7*c*d + a*h))/Sqrt[b^2 -
4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*
Sqrt[b + Sqrt[b^2 - 4*a*c]]))/(2*a*(b^2 - 4*a*c)))/(4*a*(b^2 - 4*a*c)) + (
-1/2*(b*e - 2*a*g + (2*c*e - b*g)*x^2)/((b^2 - 4*a*c)*(a + b*x^2 + c*x^4)^
2) - (3*(2*c*e - b*g)*(-((b + 2*c*x^2)/((b^2 - 4*a*c)*(a + b*x^2 + c*x^4))
) + (4*c*ArcTanh[(b + 2*c*x^2)/Sqrt[b^2 - 4*a*c]])/(b^2 - 4*a*c)^(3/2)))/(
2*(b^2 - 4*a*c)))/2
```

3.55.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1086
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] - Simp[2*c*((2*p +
 3)/((p + 1)*(b^2 - 4*a*c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; Fre
eQ[{a, b, c}, x] && ILtQ[p, -1]
```

rule 1159
```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[((b*d - 2*a*e + (2*c*d - b*e)*x)/((p + 1)*(b^2 - 4*a*c)))*(a + b*
x + c*x^2)^(p + 1), x] - Simp[(2*p + 3)*((2*c*d - b*e)/((p + 1)*(b^2 - 4*a*
c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; FreeQ[{a, b, c, d, e}, x] &
& LtQ[p, -1] && NeQ[p, -3/2]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.55. $\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^3} \, dx$

rule 2206
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> With[{d =
 Coeff[PolynomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 0], e = Coeff[Poly
nomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 2]}, Simp[x*(a + b*x^2 + c*x^
4)^(p + 1)*((a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)/(2*a*(p + 1)*(b
^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(a + b*x^2 + c
*x^4)^(p + 1)*ExpandToSum[2*a*(p + 1)*(b^2 - 4*a*c)*PolynomialQuotient[Px,
a + b*x^2 + c*x^4, x] + b^2*d*(2*p + 3) - 2*a*c*d*(4*p + 5) - a*b*e + c*(4*
p + 7)*(b*d - 2*a*e)*x^2, x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x
^2] && Expon[Px, x^2] > 1 && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1]
```

3.55.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.24 (sec) , antiderivative size = 724, normalized size of antiderivative = 1.07

method	result
risch	$\frac{-\frac{c^2\left(12a^2bh-20a^2cf-a\,b^2f+24abcd-3b^3d\right)x^7}{8a^2\left(16a^2c^2-8a\,b^2c+b^4\right)}-\frac{3(bg-2ec)c^2x^6}{2\left(16a^2c^2-8a\,b^2c+b^4\right)}+\frac{c\left(4a^3ch-19a^2b^2h+28a^2bcf+28a^2c^2d+2a\,b^3f-49a\,b^2cd+6d\,b^4\right)x^5}{8a^2\left(16a^2c^2-8a\,b^2c+b^4\right)}-\cdots}{\cdots}$
default	Expression too large to display

input
```
int((h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x,method=_RETURNVERBOSE)
```

output
```
(-1/8*c^2*(12*a^2*b*h-20*a^2*c*f-a*b^2*f+24*a*b*c*d-3*b^3*d)/a^2/(16*a^2*c
^2-8*a*b^2*c+b^4)*x^7-3/2*(b*g-2*c*e)*c^2/(16*a^2*c^2-8*a*b^2*c+b^4)*x^6+1
/8/a^2*c*(4*a^3*c*h-19*a^2*b^2*h+28*a^2*b*c*f+28*a^2*c^2*d+2*a*b^3*f-49*a*
b^2*c*d+6*b^4*d)/(16*a^2*c^2-8*a*b^2*c+b^4)*x^5-9/4*b*(b*g-2*c*e)*c/(16*a^
2*c^2-8*a*b^2*c+b^4)*x^4-1/8*(16*a^3*b*c*h-36*a^3*c^2*f+5*a^2*b^3*h-5*a^2*
b^2*c*f+4*a^2*b*c^2*d-a*b^4*f+20*a*b^3*c*d-3*b^5*d)/a^2/(16*a^2*c^2-8*a*b^
2*c+b^4)*x^3-1/2*(5*a*c+b^2)*(b*g-2*c*e)/(16*a^2*c^2-8*a*b^2*c+b^4)*x^2-1/
8*(12*a^3*c*h+3*a^2*b^2*h-16*a^2*b*c*f-44*a^2*c^2*d+a*b^3*f+37*a*b^2*c*d-5
*b^4*d)/(16*a^2*c^2-8*a*b^2*c+b^4)/a*x-1/4*(8*a^2*c*g+a*b^2*g-10*a*b*c*e+b
^3*e)/(16*a^2*c^2-8*a*b^2*c+b^4))/(c*x^4+b*x^2+a)^2+1/16*sum((-c*(12*a^2*b
*h-20*a^2*c*f-a*b^2*f+24*a*b*c*d-3*b^3*d)/a^2/(16*a^2*c^2-8*a*b^2*c+b^4)*_
R^2-24*(b*g-2*c*e)*c/(16*a^2*c^2-8*a*b^2*c+b^4)*_R+(12*a^3*c*h+3*a^2*b^2*h
-16*a^2*b*c*f+84*a^2*c^2*d+a*b^3*f-27*a*b^2*c*d+3*b^4*d)/a^2/(16*a^2*c^2-8
*a*b^2*c+b^4))/(2*_R^3*c+_R*b)*ln(x-_R),_R=RootOf(_Z^4*c+_Z^2*b+a))
```

3.55.5 Fricas [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^3}\,dx = \text{Timed out}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x, algorithm="fricas
")
```

output
```
Timed out
```

3.55.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^3}\,dx = \text{Timed out}$$

input
```
integrate((h*x**4+g*x**3+f*x**2+e*x+d)/(c*x**4+b*x**2+a)**3,x)
```

output
```
Timed out
```

3.55.7 Maxima [F]

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^3}\,dx = \int \frac{hx^4+gx^3+fx^2+ex+d}{(cx^4+bx^2+a)^3}\,dx$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x, algorithm="maxima
")
```

output
```
-1/8*((12*a^2*b*c^2*h - 3*(b^3*c^2 - 8*a*b*c^3)*d - (a*b^2*c^2 + 20*a^2*c^
3)*f)*x^7 - 12*(2*a^2*c^3*e - a^2*b*c^2*g)*x^6 - ((6*b^4*c - 49*a*b^2*c^2
+ 28*a^2*c^3)*d + 2*(a*b^3*c + 14*a^2*b*c^2)*f - (19*a^2*b^2*c - 4*a^3*c^2
)*h)*x^5 - 18*(2*a^2*b*c^2*e - a^2*b^2*c*g)*x^4 - ((3*b^5 - 20*a*b^3*c - 4
*a^2*b*c^2)*d + (a*b^4 + 5*a^2*b^2*c + 36*a^3*c^2)*f - (5*a^2*b^3 + 16*a^3
*b*c)*h)*x^3 - 4*(2*(a^2*b^2*c + 5*a^3*c^2)*e - (a^2*b^3 + 5*a^3*b*c)*g)*x
^2 + 2*(a^2*b^3 - 10*a^3*b*c)*e + 2*(a^3*b^2 + 8*a^4*c)*g - ((5*a*b^4 - 37
*a^2*b^2*c + 44*a^3*c^2)*d - (a^2*b^3 - 16*a^3*b*c)*f - 3*(a^3*b^2 + 4*a^4
*c)*h)*x)/((a^2*b^4*c^2 - 8*a^3*b^2*c^3 + 16*a^4*c^4)*x^8 + a^4*b^4 - 8*a^
5*b^2*c + 16*a^6*c^2 + 2*(a^2*b^5*c - 8*a^3*b^3*c^2 + 16*a^4*b*c^3)*x^6 +
(a^2*b^6 - 6*a^3*b^4*c + 32*a^5*c^3)*x^4 + 2*(a^3*b^5 - 8*a^4*b^3*c + 16*a
^5*b*c^2)*x^2) - 1/8*integrate(((12*a^2*b*c*h - 3*(b^3*c - 8*a*b*c^2)*d -
(a*b^2*c + 20*a^2*c^2)*f)*x^2 - 3*(b^4 - 9*a*b^2*c + 28*a^2*c^2)*d - (a*b^
3 - 16*a^2*b*c)*f - 3*(a^2*b^2 + 4*a^3*c)*h - 24*(2*a^2*c^2*e - a^2*b*c*g)
*x)/(c*x^4 + b*x^2 + a), x)/(a^2*b^4 - 8*a^3*b^2*c + 16*a^4*c^2)
```

3.55.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 6854 vs. $2(627) = 1254$.

Time = 3.09 (sec) , antiderivative size = 6854, normalized size of antiderivative = 10.09

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^3}\,dx = \text{Too large to display}$$

input
```
integrate((h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x, algorithm="giac")
```

output
```
1/32*(3*(sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^8 - 17*sqrt(2)*sqrt(b*c
 + sqrt(b^2 - 4*a*c)*c)*a*b^6*c - 2*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c
)*b^7*c - 2*b^8*c + 116*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^4*c^
2 + 26*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^5*c^2 + sqrt(2)*sqrt(b*
c + sqrt(b^2 - 4*a*c)*c)*b^6*c^2 + 34*a*b^6*c^2 + 2*b^7*c^2 - 368*sqrt(2)*
sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*b^2*c^3 - 128*sqrt(2)*sqrt(b*c + sqrt(
b^2 - 4*a*c)*c)*a^2*b^3*c^3 - 13*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a
*b^4*c^3 - 232*a^2*b^4*c^3 - 30*a*b^5*c^3 + 448*sqrt(2)*sqrt(b*c + sqrt(b^
2 - 4*a*c)*c)*a^4*c^4 + 224*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*b*
c^4 + 64*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^2*c^4 + 736*a^3*b^2
*c^4 + 176*a^2*b^3*c^4 - 112*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*c
^5 - 896*a^4*c^5 - 352*a^3*b*c^5 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sq
rt(b^2 - 4*a*c)*c)*b^7 + 15*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2
- 4*a*c)*c)*a*b^5*c + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*
a*c)*c)*b^6*c - 88*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*
c)*a^2*b^3*c^2 - 22*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)
*c)*a*b^4*c^2 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*
b^5*c^2 + 176*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^
3*b*c^3 + 88*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2
*b^2*c^3 + 11*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)...
```

3.55.9 Mupad [B] (verification not implemented)

Time = 10.96 (sec) , antiderivative size = 23811, normalized size of antiderivative = 35.07

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^3}\,dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4)/(a + b*x^2 + c*x^4)^3,x)
```

output
```
((9*x^4*(2*b*c^2*e - b^2*c*g))/(4*(b^4 + 16*a^2*c^2 - 8*a*b^2*c)) - (x^2*(
b^3*g - 10*a*c^2*e - 2*b^2*c*e + 5*a*b*c*g))/(2*(b^4 + 16*a^2*c^2 - 8*a*b^
2*c)) - (b^3*e + a*b^2*g + 8*a^2*c*g - 10*a*b*c*e)/(4*(b^4 + 16*a^2*c^2 -
8*a*b^2*c)) + (x^5*(28*a^2*c^3*d + 4*a^3*c^2*h + 6*b^4*c*d + 2*a*b^3*c*f -
 49*a*b^2*c^2*d + 28*a^2*b*c^2*f - 19*a^2*b^2*c*h))/(8*a^2*(b^4 + 16*a^2*c
^2 - 8*a*b^2*c)) + (x^3*(3*b^5*d + 36*a^3*c^2*f - 5*a^2*b^3*h + a*b^4*f -
20*a*b^3*c*d - 16*a^3*b*c*h - 4*a^2*b*c^2*d + 5*a^2*b^2*c*f))/(8*a^2*(b^4
+ 16*a^2*c^2 - 8*a*b^2*c)) - (x*(3*a^2*b^2*h - 44*a^2*c^2*d - 5*b^4*d + a*
b^3*f + 12*a^3*c*h + 37*a*b^2*c*d - 16*a^2*b*c*f))/(8*a*(b^4 + 16*a^2*c^2
- 8*a*b^2*c)) + (3*c^2*x^6*(2*c*e - b*g))/(2*(b^4 + 16*a^2*c^2 - 8*a*b^2*c
)) + (c*x^7*(20*a^2*c^2*f + 3*b^3*c*d - 24*a*b*c^2*d + a*b^2*c*f - 12*a^2*
b*c*h))/(8*a^2*(b^4 + 16*a^2*c^2 - 8*a*b^2*c)))/(x^4*(2*a*c + b^2) + a^2 +
 c^2*x^8 + 2*a*b*x^2 + 2*b*c*x^6) + symsum(log((10368*a*b^5*c^6*d^3 - 8000
*a^5*c^7*f^3 - 567*b^7*c^5*d^3 + 169344*a^3*b*c^8*d^3 + 193536*a^4*c^8*d*e
^2 - 141120*a^4*c^8*d^2*f + 1728*a^6*b*c^5*h^3 + 315*b^8*c^4*d^2*f + 27648
*a^5*c^7*e^2*h - 135*b^9*c^3*d^2*h - 2880*a^6*c^6*f*h^2 - 67824*a^2*b^3*c^
7*d^3 + 35*a^2*b^6*c^4*f^3 + 84*a^3*b^4*c^5*f^3 - 12720*a^4*b^2*c^6*f^3 +
540*a^4*b^5*c^3*h^3 + 4320*a^5*b^3*c^4*h^3 - 40320*a^5*c^7*d*f*h - 6237*a*
b^6*c^5*d^2*f + 210*a*b^7*c^4*d*f^2 + 116160*a^4*b*c^7*d*f^2 - 36864*a^4*b
*c^7*e^2*f + 2430*a*b^7*c^4*d^2*h + 133056*a^4*b*c^7*d^2*h + 27648*a^5*...
```

3.55. $\int \frac{d+ex+fx^2+gx^3+hx^4}{(a+bx^2+cx^4)^3}\,dx$

3.56 $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^3}\,dx$

3.56.1 Optimal result

Integrand size = 40, antiderivative size = 728

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^3}\,dx$$
$$= \frac{x(b^2d-abf-2a(cd-ah)+(bcd-2acf+abh)x^2)}{4a(b^2-4ac)(a+bx^2+cx^4)^2}$$
$$+\frac{2acg-b(ce+ai)-(2c^2e-bcg+b^2i-2aci)x^2}{4c(b^2-4ac)(a+bx^2+cx^4)^2}+\frac{\left(6ce-3bg+2ai+\frac{b^2i}{c}\right)(b+2cx^2)}{4(b^2-4ac)^2(a+bx^2+cx^4)}$$
$$+\frac{x(3b^4d+ab^3f+8a^2bcf+4a^2c(7cd+ah)-ab^2(25cd+7ah)+c(3b^3d+ab^2f+20a^2cf-12ab(2cd+}{8a^2(b^2-4ac)^2(a+bx^2+cx^4)}$$
$$+\frac{\sqrt{c}\left(3b^3d+ab^2f+20a^2cf-12ab(2cd+ah)+\frac{3b^4d+ab^3f-52a^2bcf-6ab^2(5cd-3ah)+24a^2c(7cd+ah)}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{}}{\sqrt{b-}}\right.}{8\sqrt{2}a^2(b^2-4ac)^2\sqrt{b-\sqrt{b^2-4ac}}}$$
$$+\frac{\sqrt{c}\left(3b^3d+ab^2f+20a^2cf-12ab(2cd+ah)-\frac{3b^4d+ab^3f-52a^2bcf-6ab^2(5cd-3ah)+24a^2c(7cd+ah)}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{}}{\sqrt{b+}}\right.}{8\sqrt{2}a^2(b^2-4ac)^2\sqrt{b+\sqrt{b^2-4ac}}}$$
$$-\frac{(6c^2e-3bcg+b^2i+2aci)\,\mathrm{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{5/2}}$$

output
```
1/4*x*(b^2*d-a*b*f-2*a*(-a*h+c*d)+(a*b*h-2*a*c*f+b*c*d)*x^2)/a/(-4*a*c+b^2
)/(c*x^4+b*x^2+a)^2+1/4*(2*a*c*g-b*(a*i+c*e)-(-2*a*c*i+b^2*i-b*c*g+2*c^2*e
)*x^2)/c/(-4*a*c+b^2)/(c*x^4+b*x^2+a)^2+1/4*(6*c*e-3*b*g+2*a*i+b^2*i/c)*(2
*c*x^2+b)/(-4*a*c+b^2)^2/(c*x^4+b*x^2+a)+1/8*x*(3*b^4*d+a*b^3*f+8*a^2*b*c*
f+4*a^2*c*(a*h+7*c*d)-a*b^2*(7*a*h+25*c*d)+c*(3*b^3*d+a*b^2*f+20*a^2*c*f-1
2*a*b*(a*h+2*c*d))*x^2)/a^2/(-4*a*c+b^2)^2/(c*x^4+b*x^2+a)-(2*a*c*i+b^2*i-
3*b*c*g+6*c^2*e)*arctanh((2*c*x^2+b)/(-4*a*c+b^2)^(1/2))/(-4*a*c+b^2)^(5/2
)+1/16*arctan(x*2^(1/2)*c^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2))*c^(1/2)*(3*b
^3*d+a*b^2*f+20*a^2*c*f-12*a*b*(a*h+2*c*d)+(3*b^4*d+a*b^3*f-52*a^2*b*c*f-6
*a*b^2*(-3*a*h+5*c*d)+24*a^2*c*(a*h+7*c*d))/(-4*a*c+b^2)^(1/2))/a^2/(-4*a*
c+b^2)^2*2^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2)+1/16*arctan(x*2^(1/2)*c^(1/2
)/(b+(-4*a*c+b^2)^(1/2))^(1/2))*c^(1/2)*(3*b^3*d+a*b^2*f+20*a^2*c*f-12*a*b
*(a*h+2*c*d)+(-3*b^4*d-a*b^3*f+52*a^2*b*c*f+6*a*b^2*(-3*a*h+5*c*d)-24*a^2*
c*(a*h+7*c*d))/(-4*a*c+b^2)^(1/2))/a^2/(-4*a*c+b^2)^2*2^(1/2)/(b+(-4*a*c+b
^2)^(1/2))^(1/2)
```

3.56.2 Mathematica [A] (verified)

Time = 5.35 (sec) , antiderivative size = 821, normalized size of antiderivative = 1.13

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^3}\,dx$$

$$= \frac{1}{16}\left(\frac{2(3b^3cdx(b+cx^2)+4a^3c(bi+cx(h+2ix))+abcx(b^2f-24c^2dx^2+bc(-25d+fx^2))+a^2(2b^3i+4\cdots}{a^2c\,(b^2-4ac)^2\,(a+b\cdots}\right.$$

$$+\frac{4(-bcdx(b+cx^2)+a^2(bi-2c(g+x(h+ix)))+a(b^2ix^2+2c^2x(d+x(e+fx))+bc(e+x(f-x(g+\cdots}{ac\,(-b^2+4ac)\,(a+bx^2+cx^4)^2}$$

$$+\frac{\sqrt{2}\sqrt{c}\big(3b^4d+b^3\big(3\sqrt{b^2-4ac}\,d+af\big)+4a^2c\big(42cd+5\sqrt{b^2-4ac}\,f+6ah\big)+ab^2\big(-30cd+\sqrt{b^2-4ac}\,f\cdots}{a^2\,(b^2-4ac)^{5/2}\,\sqrt{b-\sqrt{b^2-\cdots}}}$$

$$-\frac{\sqrt{2}\sqrt{c}\big(3b^4d+b^3\big(-3\sqrt{b^2-4ac}\,d+af\big)+4a^2c\big(42cd-5\sqrt{b^2-4ac}\,f+6ah\big)+ab^2\big(-30cd-\sqrt{b^2-4ac}\cdots}{a^2\,(b^2-4ac)^{5/2}\,\sqrt{b+\sqrt{b^2\cdots}}}$$

$$+\frac{8(6c^2e-3bcg+b^2i+2aci)\log\left(-b+\sqrt{b^2-4ac}-2cx^2\right)}{(b^2-4ac)^{5/2}}$$

$$\left.-\frac{8(6c^2e-3bcg+b^2i+2aci)\log\left(b+\sqrt{b^2-4ac}+2cx^2\right)}{(b^2-4ac)^{5/2}}\right)$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(a + b*x^2 + c*x^4)^3,
x]
```

output
```
((2*(3*b^3*c*d*x*(b + c*x^2) + 4*a^3*c*(b*i + c*x*(h + 2*i*x)) + a*b*c*x*(
b^2*f - 24*c^2*d*x^2 + b*c*(-25*d + f*x^2)) + a^2*(2*b^3*i + 4*c^3*x*(7*d
+ x*(6*e + 5*f*x)) + b^2*c*(-6*g + x*(-7*h + 4*i*x)) + 4*b*c^2*(3*e + x*(2
*f - 3*x*(g + h*x))))))/(a^2*c*(b^2 - 4*a*c)^2*(a + b*x^2 + c*x^4)) + (4*(
-(b*c*d*x*(b + c*x^2)) + a^2*(b*i - 2*c*(g + x*(h + i*x))) + a*(b^2*i*x^2
+ 2*c^2*x*(d + x*(e + f*x)) + b*c*(e + x*(f - x*(g + h*x))))))/(a*c*(-b^2
+ 4*a*c)*(a + b*x^2 + c*x^4)^2) + (Sqrt[2]*Sqrt[c]*(3*b^4*d + b^3*(3*Sqrt[
b^2 - 4*a*c]*d + a*f) + 4*a^2*c*(42*c*d + 5*Sqrt[b^2 - 4*a*c]*f + 6*a*h) +
 a*b^2*(-30*c*d + Sqrt[b^2 - 4*a*c]*f + 18*a*h) - 4*a*b*(6*c*Sqrt[b^2 - 4*
a*c]*d + 13*a*c*f + 3*a*Sqrt[b^2 - 4*a*c]*h))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/S
qrt[b - Sqrt[b^2 - 4*a*c]]])/(a^2*(b^2 - 4*a*c)^(5/2)*Sqrt[b - Sqrt[b^2 -
4*a*c]]) - (Sqrt[2]*Sqrt[c]*(3*b^4*d + b^3*(-3*Sqrt[b^2 - 4*a*c]*d + a*f)
+ 4*a^2*c*(42*c*d - 5*Sqrt[b^2 - 4*a*c]*f + 6*a*h) + a*b^2*(-30*c*d - Sqrt
[b^2 - 4*a*c]*f + 18*a*h) + 4*a*b*(6*c*Sqrt[b^2 - 4*a*c]*d - 13*a*c*f + 3*
a*Sqrt[b^2 - 4*a*c]*h))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqrt[b^2 - 4*a
*c]]])/(a^2*(b^2 - 4*a*c)^(5/2)*Sqrt[b + Sqrt[b^2 - 4*a*c]]) + (8*(6*c^2*e
 - 3*b*c*g + b^2*i + 2*a*c*i)*Log[-b + Sqrt[b^2 - 4*a*c] - 2*c*x^2])/(b^2
- 4*a*c)^(5/2) - (8*(6*c^2*e - 3*b*c*g + b^2*i + 2*a*c*i)*Log[b + Sqrt[b^2
 - 4*a*c] + 2*c*x^2])/(b^2 - 4*a*c)^(5/2))/16
```

3.56.3 Rubi [A] (verified)

Time = 1.37 (sec) , antiderivative size = 732, normalized size of antiderivative = 1.01, number of steps used = 14, number of rules used = 13, $\frac{\text{number of rules}}{\text{integrand size}} = 0.325$, Rules used = {2202, 2194, 2191, 27, 1086, 1083, 219, 2206, 25, 1492, 25, 1480, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^3}dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{hx^4+fx^2+d}{(cx^4+bx^2+a)^3}dx+\int \frac{x(ix^4+gx^2+e)}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{2194}$$

$$\int \frac{hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^3}dx+\frac{1}{2}\int \frac{ix^4+gx^2+e}{\left(cx^4+bx^2+a\right)^3}dx^2$$

$$\Big\downarrow \; 2191$$

$$\frac{1}{2}\left(\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{2c\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}-\frac{\int \frac{\frac{ib^2}{c}-3gb+6ce+2ai}{\left(cx^4+bx^2+a\right)^2}dx^2}{2\left(b^2-4ac\right)}\right)+\int \frac{hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^3}dx$$

$$\Big\downarrow \; 27$$

$$\frac{1}{2}\left(\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{2c\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}-\frac{\left(2ai+\frac{b^2i}{c}-3bg+6ce\right)\int \frac{1}{\left(cx^4+bx^2+a\right)^2}dx^2}{2\left(b^2-4ac\right)}\right)+\int \frac{hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^3}dx$$

$$\Big\downarrow \; 1086$$

$$\frac{1}{2}\left(\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{2c\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}-\frac{\left(2ai+\frac{b^2i}{c}-3bg+6ce\right)\left(-\frac{2c\int \frac{1}{cx^4+bx^2+a}dx^2}{b^2-4ac}-\frac{b+}{(b^2-4ac)(}\right.}{2\left(b^2-4ac\right)}\right.$$

$$\int \frac{hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^3}dx$$

$$\Big\downarrow \; 1083$$

$$\frac{1}{2}\left(\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{2c\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}-\frac{\left(2ai+\frac{b^2i}{c}-3bg+6ce\right)\left(\frac{4c\int \frac{1}{-x^4+b^2-4ac}d\left(2cx^2+b\right)}{b^2-4ac}-\frac{}{(b^2-}\right.}{2\left(b^2-4ac\right)}\right.$$

$$\int \frac{hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^3}dx$$

$$\Big\downarrow \; 219$$

3.56. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{\left(a+bx^2+cx^4\right)^3}\,dx$

$$\int \frac{hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^3}dx+$$

$$\frac{1}{2}\left(\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{2c\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}-\frac{\left(\frac{4c\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{b+2cx^2}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)\left(2ai+\frac{b^2}{c}\right.}{2\left(b^2-4ac\right)}\right.$$

$\downarrow$ 2206

$$-\frac{\int -\frac{3db^2+afb+5(bcd-2acf+abh)x^2-2a(7cd+ah)}{\left(cx^4+bx^2+a\right)^2}dx}{4a\left(b^2-4ac\right)}+$$

$$\frac{1}{2}\left(\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{2c\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}-\frac{\left(\frac{4c\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{b+2cx^2}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)\left(2ai+\frac{b^2}{c}\right.}{2\left(b^2-4ac\right)}\right.$$

$$\frac{x\left(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$\downarrow$ 25

$$\frac{\int \frac{3db^2+afb+5(bcd-2acf+abh)x^2-2a(7cd+ah)}{\left(cx^4+bx^2+a\right)^2}dx}{4a\left(b^2-4ac\right)}+$$

$$\frac{1}{2}\left(\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{2c\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}-\frac{\left(\frac{4c\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{b+2cx^2}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)\left(2ai+\frac{b^2}{c}\right.}{2\left(b^2-4ac\right)}\right.$$

$$\frac{x\left(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$\downarrow$ 1492

3.56. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{\left(a+bx^2+cx^4\right)^3}\,dx$

$$\frac{\frac{x\left(cx^2\left(20a^2cf+ab^2f-12ab(ah+2cd)+3b^3d\right)+8a^2bcf+4a^2c(ah+7cd)+ab^3f-ab^2(7ah+25cd)+3b^4d\right)}{2a(b^2-4ac)(a+bx^2+cx^4)}-\int -\frac{3db^4+afb^3-3a(9cd-ah)b^2-16a^2cfb+c(}{}}{4a\left(b^2-4ac\right)}$$

$$\frac{1}{2}\left(\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{2c\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}-\frac{\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)\left(2ai+\frac{b^2}{c}\right.}{2\left(b^2-4ac\right)}\right.$$

$$\frac{x\left(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$$\Big\downarrow \; 25$$

$$\frac{\frac{\int \frac{3db^4+afb^3-3a(9cd-ah)b^2-16a^2cfb+c\left(3db^3+afb^2-12a(2cd+ah)b+20a^2cf\right)x^2+12a^2c(7cd+ah)}{cx^4+bx^2+a}dx}{2a(b^2-4ac)}+\frac{x\left(cx^2\left(20a^2cf+ab^2f-12ab(ah+2cd)+3b^3d\right)+8a\right.}{2a(b^2-4ac}}{4a\left(b^2-4ac\right)}$$

$$\frac{1}{2}\left(\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{2c\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}-\frac{\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)\left(2ai+\frac{b^2}{c}\right.}{2\left(b^2-4ac\right)}\right.$$

$$\frac{x\left(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$$\Big\downarrow \; 1480$$

$$\frac{\frac{1}{2}c\left(\frac{-52a^2bcf+24a^2c(ah+7cd)+ab^3f-6ab^2(5cd-3ah)+3b^4d}{\sqrt{b^2-4ac}}+20a^2cf+ab^2f-12ab(ah+2cd)+3b^3d\right)\int\frac{1}{cx^2+\frac{1}{2}\left(b-\sqrt{b^2-4ac}\right)}dx+\frac{1}{2}c\left(-\frac{-52a^2bcf+24a^2c}{}\right.}{2a(b^2-4ac)}$$

$$\frac{1}{2}\left(\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{2c\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}-\frac{\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)\left(2ai+\frac{b^2}{c}\right.}{2\left(b^2-4ac\right)}\right.$$

$$\frac{x\left(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$$\Big\downarrow \; 218$$

3.56. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^3}\,dx$

$$\frac{\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{-52a^2bcf+24a^2c(ah+7cd)+ab^3f-6ab^2(5cd-3ah)+3b^4d}{\sqrt{b^2-4ac}}+20a^2cf+ab^2f-12ab(ah+2cd)+3b^3d\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\right.}{}}{2a(b^2-4ac)}$$

$$\frac{1}{2}\left(\frac{c\left(2ag-b\left(\frac{ai}{c}+e\right)\right)-x^2\left(-2aci+b^2i-bcg+2c^2e\right)}{2c\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}-\frac{\left(\frac{4c\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)\left(2ai+\frac{b^2}{c}\right.}{2\left(b^2-4ac\right)}\right.$$

$$\frac{x\left(x^2(abh-2acf+bcd)-abf-2a(cd-ah)+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(a + b*x^2 + c*x^4)^3,x]
```

output
```
(x*(b^2*d - a*b*f - 2*a*(c*d - a*h) + (b*c*d - 2*a*c*f + a*b*h)*x^2))/(4*a
*(b^2 - 4*a*c)*(a + b*x^2 + c*x^4)^2) + ((x*(3*b^4*d + a*b^3*f + 8*a^2*b*c
*f + 4*a^2*c*(7*c*d + a*h) - a*b^2*(25*c*d + 7*a*h) + c*(3*b^3*d + a*b^2*f
 + 20*a^2*c*f - 12*a*b*(2*c*d + a*h))*x^2))/(2*a*(b^2 - 4*a*c)*(a + b*x^2
+ c*x^4)) + ((Sqrt[c]*(3*b^3*d + a*b^2*f + 20*a^2*c*f - 12*a*b*(2*c*d + a*
h) + (3*b^4*d + a*b^3*f - 52*a^2*b*c*f - 6*a*b^2*(5*c*d - 3*a*h) + 24*a^2*
c*(7*c*d + a*h))/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sq
rt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + (Sqrt[c]*(3*b^3
*d + a*b^2*f + 20*a^2*c*f - 12*a*b*(2*c*d + a*h) - (3*b^4*d + a*b^3*f - 52
*a^2*b*c*f - 6*a*b^2*(5*c*d - 3*a*h) + 24*a^2*c*(7*c*d + a*h))/Sqrt[b^2 -
4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*
Sqrt[b + Sqrt[b^2 - 4*a*c]]))/(2*a*(b^2 - 4*a*c)))/(4*a*(b^2 - 4*a*c)) + (
(c*(2*a*g - b*(e + (a*i)/c)) - (2*c^2*e - b*c*g + b^2*i - 2*a*c*i)*x^2)/(2
*c*(b^2 - 4*a*c)*(a + b*x^2 + c*x^4)^2) - ((6*c*e - 3*b*g + 2*a*i + (b^2*i
)/c)*(-((b + 2*c*x^2)/((b^2 - 4*a*c)*(a + b*x^2 + c*x^4))) + (4*c*ArcTanh[
(b + 2*c*x^2)/Sqrt[b^2 - 4*a*c]])/(b^2 - 4*a*c)^(3/2)))/(2*(b^2 - 4*a*c)))
/2
```

3.56. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^3}\,dx$

3.56.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1086
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] - Simp[2*c*((2*p +
 3)/((p + 1)*(b^2 - 4*a*c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; Fre
eQ[{a, b, c}, x] && ILtQ[p, -1]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

3.56. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^3} dx$

rule 2191
```
Int[(Pq_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> With[{Q =
PolynomialQuotient[Pq, a + b*x + c*x^2, x], f = Coeff[PolynomialRemainder[P
q, a + b*x + c*x^2, x], x, 0], g = Coeff[PolynomialRemainder[Pq, a + b*x +
c*x^2, x], x, 1]}, Simp[(b*f - 2*a*g + (2*c*f - b*g)*x)*((a + b*x + c*x^2)^
(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] + Simp[1/((p + 1)*(b^2 - 4*a*c))   Int
[(a + b*x + c*x^2)^(p + 1)*ExpandToSum[(p + 1)*(b^2 - 4*a*c)*Q - (2*p + 3)*
(2*c*f - b*g), x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Pq, x] && NeQ[b^
2 - 4*a*c, 0] && LtQ[p, -1]
```

rule 2194
```
Int[(Pq_)*(x_)^(m_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :
> Simp[1/2   Subst[Int[x^((m - 1)/2)*SubstFor[x^2, Pq, x]*(a + b*x + c*x^2)
^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pq, x^2] && IntegerQ
[(m - 1)/2]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2206
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> With[{d =
 Coeff[PolynomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 0], e = Coeff[Poly
nomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 2]}, Simp[x*(a + b*x^2 + c*x^
4)^(p + 1)*((a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)/(2*a*(p + 1)*(b
^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(a + b*x^2 + c
*x^4)^(p + 1)*ExpandToSum[2*a*(p + 1)*(b^2 - 4*a*c)*PolynomialQuotient[Px,
a + b*x^2 + c*x^4, x] + b^2*d*(2*p + 3) - 2*a*c*d*(4*p + 5) - a*b*e + c*(4*
p + 7)*(b*d - 2*a*e)*x^2, x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x
^2] && Expon[Px, x^2] > 1 && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1]
```

3.56. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^3}\,dx$

3.56.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.30 (sec) , antiderivative size = 795, normalized size of antiderivative = 1.09

method	result
risch	$-\frac{c^{2}\left(12a^{2}bh-20a^{2}cf-a\,b^{2}f+24abcd-3b^{3}d\right)x^{7}}{8a^{2}\left(16a^{2}c^{2}-8a\,b^{2}c+b^{4}\right)}+\frac{c\left(2aci+b^{2}i-3gbc+6e\,c^{2}\right)x^{6}}{32a^{2}c^{2}-16a\,b^{2}c+2b^{4}}+\frac{c\left(4a^{3}ch-19a^{2}b^{2}h+28a^{2}bcf+28a^{2}c^{2}d+2a\,b^{3}f-49a\,b^{2}cd+6d\,b^{4}\right)}{8a^{2}\left(16a^{2}c^{2}-8a\,b^{2}c+b^{4}\right)}$
default	Expression too large to display

input
```
int((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x,method=_RETURNVERB
OSE)
```

output
```
(-1/8*c^2*(12*a^2*b*h-20*a^2*c*f-a*b^2*f+24*a*b*c*d-3*b^3*d)/a^2/(16*a^2*c
^2-8*a*b^2*c+b^4)*x^7+1/2*c*(2*a*c*i+b^2*i-3*b*c*g+6*c^2*e)/(16*a^2*c^2-8*
a*b^2*c+b^4)*x^6+1/8/a^2*c*(4*a^3*c*h-19*a^2*b^2*h+28*a^2*b*c*f+28*a^2*c^2
*d+2*a*b^3*f-49*a*b^2*c*d+6*b^4*d)/(16*a^2*c^2-8*a*b^2*c+b^4)*x^5+3/4*b*(2
*a*c*i+b^2*i-3*b*c*g+6*c^2*e)/(16*a^2*c^2-8*a*b^2*c+b^4)*x^4-1/8*(16*a^3*b
*c*h-36*a^3*c^2*f+5*a^2*b^3*h-5*a^2*b^2*c*f+4*a^2*b*c^2*d-a*b^4*f+20*a*b^3
*c*d-3*b^5*d)/a^2/(16*a^2*c^2-8*a*b^2*c+b^4)*x^3-1/2*(2*a^2*c*i-5*a*b^2*i+
5*a*b*c*g-10*a*c^2*e+b^3*g-2*b^2*c*e)/(16*a^2*c^2-8*a*b^2*c+b^4)*x^2-1/8*(
12*a^3*c*h+3*a^2*b^2*h-16*a^2*b*c*f-44*a^2*c^2*d+a*b^3*f+37*a*b^2*c*d-5*b^
4*d)/(16*a^2*c^2-8*a*b^2*c+b^4)/a*x+1/4*(6*a^2*b*i-8*a^2*c*g-a*b^2*g+10*a*
b*c*e-b^3*e)/(16*a^2*c^2-8*a*b^2*c+b^4))/(c*x^4+b*x^2+a)^2+1/16*sum((-c*(1
2*a^2*b*h-20*a^2*c*f-a*b^2*f+24*a*b*c*d-3*b^3*d)/a^2/(16*a^2*c^2-8*a*b^2*c
+b^4)*_R^2+8*(2*a*c*i+b^2*i-3*b*c*g+6*c^2*e)/(16*a^2*c^2-8*a*b^2*c+b^4)*_R
+(12*a^3*c*h+3*a^2*b^2*h-16*a^2*b*c*f+84*a^2*c^2*d+a*b^3*f-27*a*b^2*c*d+3*
b^4*d)/a^2/(16*a^2*c^2-8*a*b^2*c+b^4))/(2*_R^3*c+_R*b)*ln(x-_R),_R=RootOf(
_Z^4*c+_Z^2*b+a))
```

3.56.5 Fricas [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^3}\,dx = \text{Timed out}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x, algorithm="
fricas")
```

output
```
Timed out
```

3.56.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^3}\,dx = \text{Timed out}$$

input
```
integrate((i*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(c*x**4+b*x**2+a)**3,x)
```

output
```
Timed out
```

3.56.7 Maxima [F]

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^3}\,dx = \int \frac{ix^5+hx^4+gx^3+fx^2+ex+d}{(cx^4+bx^2+a)^3}\,dx$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x, algorithm="
maxima")
```

output
```
-1/8*((12*a^2*b*c^2*h - 3*(b^3*c^2 - 8*a*b*c^3)*d - (a*b^2*c^2 + 20*a^2*c^
3)*f)*x^7 - 4*(6*a^2*c^3*e - 3*a^2*b*c^2*g + (a^2*b^2*c + 2*a^3*c^2)*i)*x^
6 - 12*a^4*b*i - ((6*b^4*c - 49*a*b^2*c^2 + 28*a^2*c^3)*d + 2*(a*b^3*c + 1
4*a^2*b*c^2)*f - (19*a^2*b^2*c - 4*a^3*c^2)*h)*x^5 - 6*(6*a^2*b*c^2*e - 3*
a^2*b^2*c*g + (a^2*b^3 + 2*a^3*b*c)*i)*x^4 - ((3*b^5 - 20*a*b^3*c - 4*a^2*
b*c^2)*d + (a*b^4 + 5*a^2*b^2*c + 36*a^3*c^2)*f - (5*a^2*b^3 + 16*a^3*b*c)
*h)*x^3 - 4*(2*(a^2*b^2*c + 5*a^3*c^2)*e - (a^2*b^3 + 5*a^3*b*c)*g + (5*a^
3*b^2 - 2*a^4*c)*i)*x^2 + 2*(a^2*b^3 - 10*a^3*b*c)*e + 2*(a^3*b^2 + 8*a^4*
c)*g - ((5*a*b^4 - 37*a^2*b^2*c + 44*a^3*c^2)*d - (a^2*b^3 - 16*a^3*b*c)*f
 - 3*(a^3*b^2 + 4*a^4*c)*h)*x)/((a^2*b^4*c^2 - 8*a^3*b^2*c^3 + 16*a^4*c^4)
*x^8 + a^4*b^4 - 8*a^5*b^2*c + 16*a^6*c^2 + 2*(a^2*b^5*c - 8*a^3*b^3*c^2 +
 16*a^4*b*c^3)*x^6 + (a^2*b^6 - 6*a^3*b^4*c + 32*a^5*c^3)*x^4 + 2*(a^3*b^5
 - 8*a^4*b^3*c + 16*a^5*b*c^2)*x^2) - 1/8*integrate(((12*a^2*b*c*h - 3*(b^
3*c - 8*a*b*c^2)*d - (a*b^2*c + 20*a^2*c^2)*f)*x^2 - 3*(b^4 - 9*a*b^2*c +
28*a^2*c^2)*d - (a*b^3 - 16*a^2*b*c)*f - 3*(a^2*b^2 + 4*a^3*c)*h - 8*(6*a^
2*c^2*e - 3*a^2*b*c*g + (a^2*b^2 + 2*a^3*c)*i)*x)/(c*x^4 + b*x^2 + a), x)/
(a^2*b^4 - 8*a^3*b^2*c + 16*a^4*c^2)
```

3.56.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 7340 vs. $2(676) = 1352$.

Time = 3.38 (sec) , antiderivative size = 7340, normalized size of antiderivative = 10.08

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^3}\,dx = \text{Too large to display}$$

input
```
integrate((i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x, algorithm="
giac")
```

output
```
1/32*(3*(sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^8 - 17*sqrt(2)*sqrt(b*c
 + sqrt(b^2 - 4*a*c)*c)*a*b^6*c - 2*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c
)*b^7*c - 2*b^8*c + 116*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^4*c^
2 + 26*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^5*c^2 + sqrt(2)*sqrt(b*
c + sqrt(b^2 - 4*a*c)*c)*b^6*c^2 + 34*a*b^6*c^2 - 2*b^7*c^2 - 368*sqrt(2)*
sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*b^2*c^3 - 128*sqrt(2)*sqrt(b*c + sqrt(
b^2 - 4*a*c)*c)*a^2*b^3*c^3 - 13*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a
*b^4*c^3 - 232*a^2*b^4*c^3 + 30*a*b^5*c^3 + 448*sqrt(2)*sqrt(b*c + sqrt(b^
2 - 4*a*c)*c)*a^4*c^4 + 224*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*b*
c^4 + 64*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^2*c^4 + 736*a^3*b^2
*c^4 - 176*a^2*b^3*c^4 - 112*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*c
^5 - 896*a^4*c^5 + 352*a^3*b*c^5 + sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sq
rt(b^2 - 4*a*c)*c)*b^7 - 15*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2
- 4*a*c)*c)*a*b^5*c - 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*
a*c)*c)*b^6*c + 88*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*
c)*a^2*b^3*c^2 + 22*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)
*c)*a*b^4*c^2 + sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*
b^5*c^2 - 176*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^
3*b*c^3 - 88*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2
*b^2*c^3 - 11*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)...
```

3.56.9 Mupad [B] (verification not implemented)

Time = 12.37 (sec) , antiderivative size = 36653, normalized size of antiderivative = 50.35

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^3}\,dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)/(a + b*x^2 + c*x^4)^3,x)
```

output
```
((x^5*(28*a^2*c^3*d + 4*a^3*c^2*h + 6*b^4*c*d + 2*a*b^3*c*f - 49*a*b^2*c^2
*d + 28*a^2*b*c^2*f - 19*a^2*b^2*c*h))/(8*a^2*(b^4 + 16*a^2*c^2 - 8*a*b^2*
c)) - (x^2*(b^3*g - 10*a*c^2*e - 2*b^2*c*e - 5*a*b^2*i + 2*a^2*c*i + 5*a*b
*c*g))/(2*(b^4 + 16*a^2*c^2 - 8*a*b^2*c)) - (b^3*e + a*b^2*g + 8*a^2*c*g -
 6*a^2*b*i - 10*a*b*c*e)/(4*(b^4 + 16*a^2*c^2 - 8*a*b^2*c)) + (3*b*x^4*(6*
c^2*e + b^2*i - 3*b*c*g + 2*a*c*i))/(4*(b^4 + 16*a^2*c^2 - 8*a*b^2*c)) + (
c*x^6*(6*c^2*e + b^2*i - 3*b*c*g + 2*a*c*i))/(2*(b^4 + 16*a^2*c^2 - 8*a*b^
2*c)) + (x^3*(3*b^5*d + 36*a^3*c^2*f - 5*a^2*b^3*h + a*b^4*f - 20*a*b^3*c*
d - 16*a^3*b*c*h - 4*a^2*b*c^2*d + 5*a^2*b^2*c*f))/(8*a^2*(b^4 + 16*a^2*c^
2 - 8*a*b^2*c)) - (x*(3*a^2*b^2*h - 44*a^2*c^2*d - 5*b^4*d + a*b^3*f + 12*
a^3*c*h + 37*a*b^2*c*d - 16*a^2*b*c*f))/(8*a*(b^4 + 16*a^2*c^2 - 8*a*b^2*c
)) + (c*x^7*(20*a^2*c^2*f + 3*b^3*c*d - 24*a*b*c^2*d + a*b^2*c*f - 12*a^2*
b*c*h))/(8*a^2*(b^4 + 16*a^2*c^2 - 8*a*b^2*c)))/(x^4*(2*a*c + b^2) + a^2 +
 c^2*x^8 + 2*a*b*x^2 + 2*b*c*x^6) + symsum(log((10368*a*b^5*c^6*d^3 - 8000
*a^5*c^7*f^3 - 567*b^7*c^5*d^3 + 169344*a^3*b*c^8*d^3 + 193536*a^4*c^8*d*e
^2 - 141120*a^4*c^8*d^2*f + 1728*a^6*b*c^5*h^3 + 315*b^8*c^4*d^2*f + 27648
*a^5*c^7*e^2*h + 21504*a^6*c^6*d*i^2 - 135*b^9*c^3*d^2*h - 2880*a^6*c^6*f*
h^2 + 3072*a^7*c^5*h*i^2 - 67824*a^2*b^3*c^7*d^3 + 35*a^2*b^6*c^4*f^3 + 84
*a^3*b^4*c^5*f^3 - 12720*a^4*b^2*c^6*f^3 + 540*a^4*b^5*c^3*h^3 + 4320*a^5*
b^3*c^4*h^3 + 129024*a^5*c^7*d*e*i - 40320*a^5*c^7*d*f*h + 18432*a^6*c^...
```

3.56. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(a+bx^2+cx^4)^3}\,dx$

3.57 $\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^3}\,dx$

3.57.1 Optimal result

Integrand size = 55, antiderivative size = 1150

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^3}\,dx$$

$$= -\frac{bc(ce+aj)-ab^2l-2ac(cg-al)+(2c^3e-c^2(bg+2aj)-b^3l+bc(bj+3al))\,x^2}{4c^2(b^2-4ac)(a+bx^2+cx^4)^2}$$

$$-\frac{x(abc(cf+ak)-b^2(c^2d+a^2m)+2ac(c^2d-ach+a^2m)+(ab^2ck+2ac^2(cf-ak)-ab^3m-bc(c^2d+}{4ac^2(b^2-4ac)(a+bx^2+cx^4)^2}$$

$$+\frac{\frac{b^3j}{c}+2b(3ce+aj)-16a^2l-\frac{b^4l}{c^2}-b^2\left(3g-\frac{5al}{c}\right)+2(6c^2e-3bcg+b^2j+2acj-3abl)\,x^2}{4(b^2-4ac)^2(a+bx^2+cx^4)}$$

$$+\frac{x(4a^2bc^2(2cf+ak)+ab^3c(cf+2ak)-ab^2c(25c^2d+7ach-11a^2m)+4a^2c^2(7c^2d+ach-9a^2m)+b}{8a^2c^2(b^2-4a}$$

$$+\frac{\left(ab^2c(cf+3ak)+4a^2c^2(5cf+3ak)+b^3(3c^2d+a^2m)-4abc(6c^2d+3ach+4a^2m)+\frac{ab^3c(cf-3ak)-4a^2b}{}\right.}{8\sqrt{2}a^2c^{3/2}(b^2-4ac)^2\sqrt{b-}}$$

$$+\frac{\left(ab^2c(cf+3ak)+4a^2c^2(5cf+3ak)+b^3(3c^2d+a^2m)-4abc(6c^2d+3ach+4a^2m)-\frac{ab^3c(cf-3ak)-4a^2b}{}\right.}{8\sqrt{2}a^2c^{3/2}(b^2-4ac)^2\sqrt{b-}}$$

$$-\frac{(6c^2e-3bcg+b^2j+2acj-3abl)\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{5/2}}$$

output
```
1/4*(-b*c*(a*j+c*e)+a*b^2*l+2*a*c*(-a*l+c*g)-(2*c^3*e-c^2*(2*a*j+b*g)-b^3*
l+b*c*(3*a*l+b*j))*x^2)/c^2/(-4*a*c+b^2)/(c*x^4+b*x^2+a)^2-1/4*x*(a*b*c*(a
*k+c*f)-b^2*(a^2*m+c^2*d)+2*a*c*(a^2*m-a*c*h+c^2*d)+(a*b^2*c*k+2*a*c^2*(-a
*k+c*f)-a*b^3*m-b*c*(-3*a^2*m+a*c*h+c^2*d))*x^2)/a/c^2/(-4*a*c+b^2)/(c*x^4
+b*x^2+a)^2+1/4*(b^3*j/c+2*b*(a*j+3*c*e)-16*a^2*l-b^4*l/c^2-b^2*(3*g-5*a*l
/c)+2*(-3*a*b*l+2*a*c*j+b^2*j-3*b*c*g+6*c^2*e)*x^2)/(-4*a*c+b^2)^2/(c*x^4+
b*x^2+a)+1/8*x*(4*a^2*b*c^2*(a*k+2*c*f)+a*b^3*c*(2*a*k+c*f)-a*b^2*c*(-11*a
^2*m+7*a*c*h+25*c^2*d)+4*a^2*c^2*(-9*a^2*m+a*c*h+7*c^2*d)+b^4*(-2*a^2*m+3*
c^2*d)+c*(a*b^2*c*(3*a*k+c*f)+4*a^2*c^2*(3*a*k+5*c*f)+b^3*(a^2*m+3*c^2*d)-
4*a*b*c*(4*a^2*m+3*a*c*h+6*c^2*d))*x^2)/a^2/c^2/(-4*a*c+b^2)^2/(c*x^4+b*x^
2+a)-(-3*a*b*l+2*a*c*j+b^2*j-3*b*c*g+6*c^2*e)*arctanh((2*c*x^2+b)/(-4*a*c+
b^2)^(1/2))/(-4*a*c+b^2)^(5/2)+1/16*arctan(x*2^(1/2)*c^(1/2)/(b-(-4*a*c+b^
2)^(1/2))^(1/2))*(a*b^2*c*(3*a*k+c*f)+4*a^2*c^2*(3*a*k+5*c*f)+b^3*(a^2*m+3
*c^2*d)-4*a*b*c*(4*a^2*m+3*a*c*h+6*c^2*d)+(a*b^3*c*(-3*a*k+c*f)-4*a^2*b*c^
2*(9*a*k+13*c*f)-6*a*b^2*c*(-3*a^2*m-3*a*c*h+5*c^2*d)+b^4*(-a^2*m+3*c^2*d)
+8*a^2*c^2*(5*a^2*m+3*a*c*h+21*c^2*d))/(-4*a*c+b^2)^(1/2))/a^2/c^(3/2)/(-4
*a*c+b^2)^2*2^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2)+1/16*arctan(x*2^(1/2)*c^(
1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2))*(a*b^2*c*(3*a*k+c*f)+4*a^2*c^2*(3*a*k+5
*c*f)+b^3*(a^2*m+3*c^2*d)-4*a*b*c*(4*a^2*m+3*a*c*h+6*c^2*d)+(-a*b^3*c*(-3*
a*k+c*f)+4*a^2*b*c^2*(9*a*k+13*c*f)+6*a*b^2*c*(-3*a^2*m-3*a*c*h+5*c^2*d...
```

3.57.2 Mathematica [A] (verified)

Time = 6.85 (sec) , antiderivative size = 1590, normalized size of antiderivative = 1.38

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{\left(a+bx^2+cx^4\right)^3}\,dx$$

$$= \frac{abc^2e-2a^2c^2g+a^2bcj-a^2b^2l+2a^3cl-b^2c^2dx+2ac^3dx+abc^2fx-2a^2c^2hx+a^2bckx-a^2b^2mx+2a^3\cdots}{\cdots}$$

$$+\frac{12a^2bc^3e-6a^2b^2c^2g+2a^2b^3cj+4a^3bc^2j-2a^2b^4l+10a^3b^2cl-32a^4c^2l+3b^4c^2dx-25ab^2c^3dx+28a^2\cdots}{\cdots}$$

$$+\frac{\left(3b^4c^2d-30ab^2c^3d+168a^2c^4d+3b^3c^2\sqrt{b^2-4ac}d-24abc^3\sqrt{b^2-4ac}d+ab^3c^2f-52a^2bc^3f+ab^2c^2\sqrt{\cdots}\right.}{\cdots}$$

$$+\frac{\left(-3b^4c^2d+30ab^2c^3d-168a^2c^4d+3b^3c^2\sqrt{b^2-4ac}d-24abc^3\sqrt{b^2-4ac}d-ab^3c^2f+52a^2bc^3f+ab^2c^2\cdots\right.}{\cdots}$$

$$+\frac{\left(6c^2e-3bcg+b^2j+2acj-3abl\right)\log\left(-b+\sqrt{b^2-4ac}-2cx^2\right)}{2\left(b^2-4ac\right)^{5/2}}$$

$$+\frac{\left(-6c^2e+3bcg-b^2j-2acj+3abl\right)\log\left(b+\sqrt{b^2-4ac}+2cx^2\right)}{2\left(b^2-4ac\right)^{5/2}}$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4 + j*x^5 + k*x^6 + l*x^7 + m*x^8
)/(a + b*x^2 + c*x^4)^3,x]
```

output
```
(a*b*c^2*e - 2*a^2*c^2*g + a^2*b*c*j - a^2*b^2*l + 2*a^3*c*l - b^2*c^2*d*x
 + 2*a*c^3*d*x + a*b*c^2*f*x - 2*a^2*c^2*h*x + a^2*b*c*k*x - a^2*b^2*m*x +
 2*a^3*c*m*x + 2*a*c^3*e*x^2 - a*b*c^2*g*x^2 + a*b^2*c*j*x^2 - 2*a^2*c^2*j
*x^2 - a*b^3*l*x^2 + 3*a^2*b*c*l*x^2 - b*c^3*d*x^3 + 2*a*c^3*f*x^3 - a*b*c
^2*h*x^3 + a*b^2*c*k*x^3 - 2*a^2*c^2*k*x^3 - a*b^3*m*x^3 + 3*a^2*b*c*m*x^3
)/(4*a*c^2*(-b^2 + 4*a*c)*(a + b*x^2 + c*x^4)^2) + (12*a^2*b*c^3*e - 6*a^2
*b^2*c^2*g + 2*a^2*b^3*c*j + 4*a^3*b*c^2*j - 2*a^2*b^4*l + 10*a^3*b^2*c*l
- 32*a^4*c^2*l + 3*b^4*c^2*d*x - 25*a*b^2*c^3*d*x + 28*a^2*c^4*d*x + a*b^3
*c^2*f*x + 8*a^2*b*c^3*f*x - 7*a^2*b^2*c^2*h*x + 4*a^3*c^3*h*x + 2*a^2*b^3
*c*k*x + 4*a^3*b*c^2*k*x - 2*a^2*b^4*m*x + 11*a^3*b^2*c*m*x - 36*a^4*c^2*m
*x + 24*a^2*c^4*e*x^2 - 12*a^2*b*c^3*g*x^2 + 4*a^2*b^2*c^2*j*x^2 + 8*a^3*c
^3*j*x^2 - 12*a^3*b*c^2*l*x^2 + 3*b^3*c^3*d*x^3 - 24*a*b*c^4*d*x^3 + a*b^2
*c^3*f*x^3 + 20*a^2*c^4*f*x^3 - 12*a^2*b*c^3*h*x^3 + 3*a^2*b^2*c^2*k*x^3 +
 12*a^3*c^3*k*x^3 + a^2*b^3*c*m*x^3 - 16*a^3*b*c^2*m*x^3)/(8*a^2*c^2*(-b^2
 + 4*a*c)^2*(a + b*x^2 + c*x^4)) + ((3*b^4*c^2*d - 30*a*b^2*c^3*d + 168*a^
2*c^4*d + 3*b^3*c^2*Sqrt[b^2 - 4*a*c]*d - 24*a*b*c^3*Sqrt[b^2 - 4*a*c]*d +
 a*b^3*c^2*f - 52*a^2*b*c^3*f + a*b^2*c^2*Sqrt[b^2 - 4*a*c]*f + 20*a^2*c^3
*Sqrt[b^2 - 4*a*c]*f + 18*a^2*b^2*c^2*h + 24*a^3*c^3*h - 12*a^2*b*c^2*Sqrt
[b^2 - 4*a*c]*h - 3*a^2*b^3*c*k - 36*a^3*b*c^2*k + 3*a^2*b^2*c*Sqrt[b^2 -
4*a*c]*k + 12*a^3*c^2*Sqrt[b^2 - 4*a*c]*k - a^2*b^4*m + 18*a^3*b^2*c*m ...
```

3.57.3 Rubi [A] (verified)

Time = 2.87 (sec) , antiderivative size = 1172, normalized size of antiderivative = 1.02, number of steps used = 14, number of rules used = 13, $\frac{\text{number of rules}}{\text{integrand size}} = 0.236$, Rules used = {2202, 2194, 2191, 1159, 1083, 219, 2206, 25, 2206, 25, 27, 1480, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^3}dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{mx^8+kx^6+hx^4+fx^2+d}{(cx^4+bx^2+a)^3}dx+\int \frac{x(lx^6+jx^4+gx^2+e)}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{2194}$$

$$\int \frac{mx^8+kx^6+hx^4+fx^2+d}{(cx^4+bx^2+a)^3}dx+\frac{1}{2}\int \frac{lx^6+jx^4+gx^2+e}{(cx^4+bx^2+a)^3}dx^2$$

$\downarrow$ 2191

$$\frac{1}{2}\left(-\frac{\int \frac{-\frac{lb^3}{c^2}-3gb+\frac{(bj+al)b}{c}+2\left(4a-\frac{b^2}{c}\right)lx^2+6ce+2aj}{(cx^4+bx^2+a)^2}dx^2}{2(b^2-4ac)}-\frac{x^2\left(-c^2(2aj+bg)+bc(3al+bj)+b^3(-l)+2c^3e\right)-ab^2l+bc}{2c^2(b^2-4ac)(a+bx^2+cx^4)^2}\right.$$

$$\int \frac{mx^8+kx^6+hx^4+fx^2+d}{(cx^4+bx^2+a)^3}dx$$

$\downarrow$ 1159

$$\frac{1}{2}\left(-\frac{-\frac{2\left(-3abl+2acj+b^2j-3bcg+6c^2e\right)\int \frac{1}{cx^4+bx^2+a}dx^2}{b^2-4ac}-\frac{-16a^2l+2x^2\left(-3abl+2acj+b^2j-3bcg+6c^2e\right)-b^2\left(3g-\frac{5al}{c}\right)+2b(aj+3ce)-\frac{b^4l}{c^2}+b}{(b^2-4ac)(a+bx^2+cx^4)}}{2(b^2-4ac)}\right.$$

$$\int \frac{mx^8+kx^6+hx^4+fx^2+d}{(cx^4+bx^2+a)^3}dx$$

$\downarrow$ 1083

$$\frac{1}{2}\left(-\frac{\frac{4\left(-3abl+2acj+b^2j-3bcg+6c^2e\right)\int \frac{1}{-x^4+b^2-4ac}d(2cx^2+b)}{b^2-4ac}-\frac{-16a^2l+2x^2\left(-3abl+2acj+b^2j-3bcg+6c^2e\right)-b^2\left(3g-\frac{5al}{c}\right)+2b(aj+3ce)-}{(b^2-4ac)(a+bx^2+cx^4)}}{2(b^2-4ac)}\right.$$

$$\int \frac{mx^8+kx^6+hx^4+fx^2+d}{(cx^4+bx^2+a)^3}dx$$

$\downarrow$ 219

$$\int \frac{mx^8+kx^6+hx^4+fx^2+d}{(cx^4+bx^2+a)^3}dx+$$

$$\frac{1}{2}\left(-\frac{\frac{4\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-3abl+2acj+b^2j-3bcg+6c^2e\right)}{(b^2-4ac)^{3/2}}-\frac{-16a^2l+2x^2\left(-3abl+2acj+b^2j-3bcg+6c^2e\right)-b^2\left(3g-\frac{5al}{c}\right)+2b(aj+3ce)-\frac{b^4l}{c^2}}{(b^2-4ac)(a+bx^2+cx^4)}}{2(b^2-4ac)}\right.$$

$\downarrow$ 2206

3.57. $\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^3}\,dx$

$$-\frac{\int -\frac{-4a\left(4a-\frac{b^2}{c}\right)mx^4-\frac{\left(-amb^3+ackb^2-c\left(ma^2+5cha+5c^2d\right)b+2ac^2(5cf+3ak)\right)x^2}{c^2}+\frac{\left(3c^2d-a^2m\right)b^2+ac(cf+ak)b-2ac\left(-ma^2+cha+7c^2d\right)}{c^2}}{(cx^4+bx^2+a)^2}dx}{4a\left(b^2-4ac\right)}+$$

$$\frac{1}{2}\left(-\frac{\frac{4\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-3abl+2acj+b^2j-3bcg+6c^2e\right)}{(b^2-4ac)^{3/2}}-\frac{-16a^2l+2x^2\left(-3abl+2acj+b^2j-3bcg+6c^2e\right)-b^2\left(3g-\frac{5al}{c}\right)+2b(aj+3ce)-\frac{b^4l}{c^2}}{(b^2-4ac)(a+bx^2+cx^4)}}{2\left(b^2-4ac\right)}\right.$$

$$\frac{x\left(-\left(b^2\left(a^2m+c^2d\right)\right)+x^2\left(-bc\left(-3a^2m+ach+c^2d\right)-ab^3m+ab^2ck+2ac^2(cf-ak)\right)+2ac\left(a^2m-ach+c^2d\right)\right.}{4ac^2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$$\Big\downarrow \ 25$$

$$\frac{\int \frac{-4a\left(4a-\frac{b^2}{c}\right)mx^4-\frac{\left(-amb^3+ackb^2-c\left(ma^2+5cha+5c^2d\right)b+2ac^2(5cf+3ak)\right)x^2}{c^2}+\frac{\left(3c^2d-a^2m\right)b^2+ac(cf+ak)b-2ac\left(-ma^2+cha+7c^2d\right)}{c^2}}{(cx^4+bx^2+a)^2}dx}{4a\left(b^2-4ac\right)}+$$

$$\frac{1}{2}\left(-\frac{\frac{4\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-3abl+2acj+b^2j-3bcg+6c^2e\right)}{(b^2-4ac)^{3/2}}-\frac{-16a^2l+2x^2\left(-3abl+2acj+b^2j-3bcg+6c^2e\right)-b^2\left(3g-\frac{5al}{c}\right)+2b(aj+3ce)-\frac{b^4l}{c^2}}{(b^2-4ac)(a+bx^2+cx^4)}}{2\left(b^2-4ac\right)}\right.$$

$$\frac{x\left(-\left(b^2\left(a^2m+c^2d\right)\right)+x^2\left(-bc\left(-3a^2m+ach+c^2d\right)-ab^3m+ab^2ck+2ac^2(cf-ak)\right)+2ac\left(a^2m-ach+c^2d\right)\right.}{4ac^2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$$\Big\downarrow \ 2206$$

$$-\frac{x\left(-\left(\left(ma^2+c^2d\right)b^2\right)+ac(cf+ak)b+\left(-amb^3+ackb^2-c\left(-3ma^2+cha+c^2d\right)b+2ac^2(cf-ak)\right)x^2+2ac\right.}{4ac^2\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}$$

$$\frac{1}{2}\left(-\frac{-alb^2+c(ce+aj)b+\left(-lb^3+c(bj+3al)b+2c^3e-c^2(bg+2aj)\right)x^2-2ac(cg-al)}{2c^2\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}-\frac{4\left(jb^2-3cgb-3alb+6c^2e-\right.}{(b^2-}\right.$$

$$\frac{x\left(\left(\left(ma^2+3c^2d\right)b^3+ac(cf+3ak)b^2-4ac\left(4ma^2+3cha+6c^2d\right)b+4a^2c^2(5cf+3ak)\right)x^2+c\left(\left(3d-\frac{2a^2m}{c^2}\right)b^4+\frac{a(cf+2ak)b^3}{c}-a\left(-\frac{11ma^2}{c}+7ha+25cd\right)b\right.\right.}{2ac\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)}$$

$$\Big\downarrow \ 25$$

3.57. $\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^3}\,dx$

$$-\frac{x\left(-\left(\left(ma^2+c^2d\right)b^2\right)+ac(cf+ak)b+\left(-amb^3+ackb^2-c\left(-3ma^2+cha+c^2d\right)b+2ac^2(cf-ak)\right)x^2+2ac\right.}{4ac^2\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}$$

$$\frac{1}{2}\left(-\frac{-alb^2+c(ce+aj)b+\left(-lb^3+c(bj+3al)b+2c^3e-c^2(bg+2aj)\right)x^2-2ac(cg-al)}{2c^2\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}-\frac{\frac{4\left(jb^2-3cgb-3alb+6c^2e-\right.}{\left(b^2-\right.}}{}\right.$$

$$\frac{x\left(\left(\left(ma^2+3c^2d\right)b^3+ac(cf+3ak)b^2-4ac\left(4ma^2+3cha+6c^2d\right)b+4a^2c^2(5cf+3ak)\right)x^2+c\left(\left(3d-\frac{2a^2m}{c^2}\right)b^4+\frac{a(cf+2ak)b^3}{c}-a\left(-\frac{11ma^2}{c}+7ha+25cd\right)b\right.\right.}{2ac\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)}$$

$\downarrow$ 27

$$-\frac{x\left(-\left(\left(ma^2+c^2d\right)b^2\right)+ac(cf+ak)b+\left(-amb^3+ackb^2-c\left(-3ma^2+cha+c^2d\right)b+2ac^2(cf-ak)\right)x^2+2ac\right.}{4ac^2\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}$$

$$\frac{1}{2}\left(-\frac{-alb^2+c(ce+aj)b+\left(-lb^3+c(bj+3al)b+2c^3e-c^2(bg+2aj)\right)x^2-2ac(cg-al)}{2c^2\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}-\frac{\frac{4\left(jb^2-3cgb-3alb+6c^2e-\right.}{\left(b^2-\right.}}{}\right.$$

$$\frac{x\left(\left(\left(ma^2+3c^2d\right)b^3+ac(cf+3ak)b^2-4ac\left(4ma^2+3cha+6c^2d\right)b+4a^2c^2(5cf+3ak)\right)x^2+c\left(\left(3d-\frac{2a^2m}{c^2}\right)b^4+\frac{a(cf+2ak)b^3}{c}-a\left(-\frac{11ma^2}{c}+7ha+25cd\right)b\right.\right.}{2ac\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)}$$

$\downarrow$ 1480

$$-\frac{x\left(-\left(\left(ma^2+c^2d\right)b^2\right)+ac(cf+ak)b+\left(-amb^3+ackb^2-c\left(-3ma^2+cha+c^2d\right)b+2ac^2(cf-ak)\right)x^2+2ac\right.}{4ac^2\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}$$

$$\frac{1}{2}\left(-\frac{-alb^2+c(ce+aj)b+\left(-lb^3+c(bj+3al)b+2c^3e-c^2(bg+2aj)\right)x^2-2ac(cg-al)}{2c^2\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}-\frac{\frac{4\left(jb^2-3cgb-3alb+6c^2e-\right.}{\left(b^2-\right.}}{}\right.$$

$$\frac{x\left(\left(\left(ma^2+3c^2d\right)b^3+ac(cf+3ak)b^2-4ac\left(4ma^2+3cha+6c^2d\right)b+4a^2c^2(5cf+3ak)\right)x^2+c\left(\left(3d-\frac{2a^2m}{c^2}\right)b^4+\frac{a(cf+2ak)b^3}{c}-a\left(-\frac{11ma^2}{c}+7ha+25cd\right)b\right.\right.}{2ac\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)}$$

$\downarrow$ 218

3.57. $\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{\left(a+bx^2+cx^4\right)^3}\,dx$

$$-\frac{x\left(-\left(\left(ma^2+c^2d\right)b^2\right)+ac(cf+ak)b+\left(-amb^3+ackb^2-c\left(-3ma^2+cha+c^2d\right)b+2ac^2(cf-ak)\right)x^2+2ac}{4ac^2\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}$$

$$\frac{\frac{x\left(\left(\left(ma^2+3c^2d\right)b^3+ac(cf+3ak)b^2-4ac\left(4ma^2+3cha+6c^2d\right)b+4a^2c^2(5cf+3ak)\right)x^2+c\left(\left(3d-\frac{2a^2m}{c^2}\right)b^4+\frac{a(cf+2ak)b^3}{c}-a\left(-\frac{11ma^2}{c}+7ha+25cd\right)b\right.\right.}{2ac(b^2-4ac)(cx^4+bx^2+a)}}{}$$

$$\frac{1}{2}\left(-\frac{-alb^2+c(ce+aj)b+\left(-lb^3+c(bj+3al)b+2c^3e-c^2(bg+2aj)\right)x^2-2ac(cg-al)}{2c^2\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}-\frac{\frac{4\left(jb^2-3cgb-3alb+6c^2e-\right.}{}}{(b^2-}\right.$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4 + j*x^5 + k*x^6 + l*x^7 + m*x^8)/(a +
 b*x^2 + c*x^4)^3,x]
```

output
```
-1/4*(x*(a*b*c*(c*f + a*k) - b^2*(c^2*d + a^2*m) + 2*a*c*(c^2*d - a*c*h +
a^2*m) + (a*b^2*c*k + 2*a*c^2*(c*f - a*k) - a*b^3*m - b*c*(c^2*d + a*c*h -
 3*a^2*m))*x^2))/(a*c^2*(b^2 - 4*a*c)*(a + b*x^2 + c*x^4)^2) + ((x*(c*(4*a
^2*b*(2*c*f + a*k) + (a*b^3*(c*f + 2*a*k))/c + 4*a^2*(7*c^2*d + a*c*h - 9*
a^2*m) + b^4*(3*d - (2*a^2*m)/c^2) - a*b^2*(25*c*d + 7*a*h - (11*a^2*m)/c)
) + (a*b^2*c*(c*f + 3*a*k) + 4*a^2*c^2*(5*c*f + 3*a*k) + b^3*(3*c^2*d + a^
2*m) - 4*a*b*c*(6*c^2*d + 3*a*c*h + 4*a^2*m))*x^2))/(2*a*c*(b^2 - 4*a*c)*(
a + b*x^2 + c*x^4)) + (((a*b^2*c*(c*f + 3*a*k) + 4*a^2*c^2*(5*c*f + 3*a*k)
 + b^3*(3*c^2*d + a^2*m) - 4*a*b*c*(6*c^2*d + 3*a*c*h + 4*a^2*m) + (a*b^3*
c*(c*f - 3*a*k) - 4*a^2*b*c^2*(13*c*f + 9*a*k) - 6*a*b^2*c*(5*c^2*d - 3*a*
c*h - 3*a^2*m) + b^4*(3*c^2*d - a^2*m) + 8*a^2*c^2*(21*c^2*d + 3*a*c*h + 5
*a^2*m))/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[b^2 -
 4*a*c]]])/(Sqrt[2]*Sqrt[c]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + ((a*b^2*c*(c*f
+ 3*a*k) + 4*a^2*c^2*(5*c*f + 3*a*k) + b^3*(3*c^2*d + a^2*m) - 4*a*b*c*(6*
c^2*d + 3*a*c*h + 4*a^2*m) - (a*b^3*c*(c*f - 3*a*k) - 4*a^2*b*c^2*(13*c*f
+ 9*a*k) - 6*a*b^2*c*(5*c^2*d - 3*a*c*h - 3*a^2*m) + b^4*(3*c^2*d - a^2*m)
 + 8*a^2*c^2*(21*c^2*d + 3*a*c*h + 5*a^2*m))/Sqrt[b^2 - 4*a*c])*ArcTan[(Sq
rt[2]*Sqrt[c]*x)/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[c]*Sqrt[b + S
qrt[b^2 - 4*a*c]]))/(2*a*c*(b^2 - 4*a*c)))/(4*a*(b^2 - 4*a*c)) + (-1/2*(b*
c*(c*e + a*j) - a*b^2*l - 2*a*c*(c*g - a*l) + (2*c^3*e - c^2*(b*g + 2*a...
```

3.57. $\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{\left(a+bx^2+cx^4\right)^3}\,dx$

3.57.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1159
```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[((b*d - 2*a*e + (2*c*d - b*e)*x)/((p + 1)*(b^2 - 4*a*c)))*(a + b*
x + c*x^2)^(p + 1), x] - Simp[(2*p + 3)*((2*c*d - b*e)/((p + 1)*(b^2 - 4*a*
c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; FreeQ[{a, b, c, d, e}, x] &
& LtQ[p, -1] && NeQ[p, -3/2]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 2191
```
Int[(Pq_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> With[{Q =
PolynomialQuotient[Pq, a + b*x + c*x^2, x], f = Coeff[PolynomialRemainder[P
q, a + b*x + c*x^2, x], x, 0], g = Coeff[PolynomialRemainder[Pq, a + b*x +
c*x^2, x], x, 1]}, Simp[(b*f - 2*a*g + (2*c*f - b*g)*x)*((a + b*x + c*x^2)^
(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] + Simp[1/((p + 1)*(b^2 - 4*a*c))   Int
[(a + b*x + c*x^2)^(p + 1)*ExpandToSum[(p + 1)*(b^2 - 4*a*c)*Q - (2*p + 3)*
(2*c*f - b*g), x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Pq, x] && NeQ[b^
2 - 4*a*c, 0] && LtQ[p, -1]
```

rule 2194
```
Int[(Pq_)*(x_)^(m_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :
> Simp[1/2   Subst[Int[x^((m - 1)/2)*SubstFor[x^2, Pq, x]*(a + b*x + c*x^2)
^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pq, x^2] && IntegerQ
[(m - 1)/2]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2206
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> With[{d =
 Coeff[PolynomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 0], e = Coeff[Poly
nomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 2]}, Simp[x*(a + b*x^2 + c*x^
4)^(p + 1)*((a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)/(2*a*(p + 1)*(b
^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(a + b*x^2 + c
*x^4)^(p + 1)*ExpandToSum[2*a*(p + 1)*(b^2 - 4*a*c)*PolynomialQuotient[Px,
a + b*x^2 + c*x^4, x] + b^2*d*(2*p + 3) - 2*a*c*d*(4*p + 5) - a*b*e + c*(4*
p + 7)*(b*d - 2*a*e)*x^2, x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x
^2] && Expon[Px, x^2] > 1 && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1]
```

3.57. $\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^3} \, dx$

3.57.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 1.39 (sec) , antiderivative size = 1167, normalized size of antiderivative = 1.01

method	result	size
risch	Expression too large to display	1167
default	Expression too large to display	1987

input
```
int((m*x^8+l*x^7+k*x^6+j*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x,
method=_RETURNVERBOSE)
```

output
```
(-1/8*(16*a^3*b*c*m-12*a^3*c^2*k-a^2*b^3*m-3*a^2*b^2*c*k+12*a^2*b*c^2*h-20
*a^2*c^3*f-a*b^2*c^2*f+24*a*b*c^3*d-3*b^3*c^2*d)/a^2/(16*a^2*c^2-8*a*b^2*c
+b^4)*x^7-1/2*c*(3*a*b*l-2*a*c*j-b^2*j+3*b*c*g-6*c^2*e)/(16*a^2*c^2-8*a*b^
2*c+b^4)*x^6-1/8/a^2*(36*a^4*c^2*m+5*a^3*b^2*c*m-16*a^3*b*c^2*k-4*a^3*c^3*
h+a^2*b^4*m-5*a^2*b^3*c*k+19*a^2*b^2*c^2*h-28*a^2*b*c^3*f-28*a^2*c^4*d-2*a
*b^3*c^2*f+49*a*b^2*c^3*d-6*b^4*c^2*d)/(16*a^2*c^2-8*a*b^2*c+b^4)/c*x^5-1/
4*(16*a^2*c^2*l+a*b^2*c*l-6*a*b*c^2*j+b^4*l-3*b^3*c*j+9*b^2*c^2*g-18*b*c^3
*e)/(16*a^2*c^2-8*a*b^2*c+b^4)/c*x^4-1/8/c*(28*a^4*b*c*m+4*a^4*c^2*k+2*a^3
*b^3*m-19*a^3*b^2*c*k+16*a^3*b*c^2*h-36*a^3*c^3*f+5*a^2*b^3*c*h-5*a^2*b^2*
c^2*f+4*a^2*b*c^3*d-a*b^4*c*f+20*a*b^3*c^2*d-3*b^5*c*d)/a^2/(16*a^2*c^2-8*
a*b^2*c+b^4)*x^3-1/2/c*(5*a^2*b*c*l+2*a^2*c^2*j+a*b^3*l-5*a*b^2*c*j+5*a*b*
c^2*g-10*a*c^3*e+b^3*c*g-2*b^2*c^2*e)/(16*a^2*c^2-8*a*b^2*c+b^4)*x^2-1/8*(
20*a^4*c*m+a^3*b^2*m-12*a^3*b*c*k+12*a^3*c^2*h+3*a^2*b^2*c*h-16*a^2*b*c^2*
f-44*a^2*c^3*d+a*b^3*c*f+37*a*b^2*c^2*d-5*b^4*c*d)/(16*a^2*c^2-8*a*b^2*c+b
^4)/c/a*x-1/4/c*(8*a^3*c*l+a^2*b^2*l-6*a^2*b*c*j+8*a^2*c^2*g+a*b^2*c*g-10*
a*b*c^2*e+b^3*c*e)/(16*a^2*c^2-8*a*b^2*c+b^4))/(c*x^4+b*x^2+a)^2+1/16*sum(
(-(16*a^3*b*c*m-12*a^3*c^2*k-a^2*b^3*m-3*a^2*b^2*c*k+12*a^2*b*c^2*h-20*a^2
*c^3*f-a*b^2*c^2*f+24*a*b*c^3*d-3*b^3*c^2*d)/a^2/(16*a^2*c^2-8*a*b^2*c+b^4
)/c*_R^2-8*(3*a*b*l-2*a*c*j-b^2*j+3*b*c*g-6*c^2*e)/(16*a^2*c^2-8*a*b^2*c+b
^4)*_R+1/c*(20*a^4*c*m+a^3*b^2*m-12*a^3*b*c*k+12*a^3*c^2*h+3*a^2*b^2*c*...
```

3.57.5 Fricas [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^3}\,dx = \text{Timed out}$$

input
```
integrate((m*x^8+l*x^7+k*x^6+j*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a
)^3,x, algorithm="fricas")
```

output
```
Timed out
```

3.57.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^3}\,dx = \text{Timed out}$$

input
```
integrate((m*x**8+l*x**7+k*x**6+j*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(c*x**4
+b*x**2+a)**3,x)
```

output
```
Timed out
```

3.57.7 Maxima [F]

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^3}\,dx = \int \frac{mx^8+lx^7+kx^6+jx^5+hx^4+gx^3+fx^2+ex+d}{(cx^4+bx^2+a)^3}\,dx$$

input
```
integrate((m*x^8+l*x^7+k*x^6+j*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a
)^3,x, algorithm="maxima")
```

output
```
-1/8*((12*a^2*b*c^3*h - 3*(b^3*c^3 - 8*a*b*c^4)*d - (a*b^2*c^3 + 20*a^2*c^
4)*f - 3*(a^2*b^2*c^2 + 4*a^3*c^3)*k - (a^2*b^3*c - 16*a^3*b*c^2)*m)*x^7 -
 12*a^4*b*c*j - 4*(6*a^2*c^4*e - 3*a^2*b*c^3*g - 3*a^3*b*c^2*l + (a^2*b^2*
c^2 + 2*a^3*c^3)*j)*x^6 - ((6*b^4*c^2 - 49*a*b^2*c^3 + 28*a^2*c^4)*d + 2*(
a*b^3*c^2 + 14*a^2*b*c^3)*f - (19*a^2*b^2*c^2 - 4*a^3*c^3)*h + (5*a^2*b^3*
c + 16*a^3*b*c^2)*k - (a^2*b^4 + 5*a^3*b^2*c + 36*a^4*c^2)*m)*x^5 - 2*(18*
a^2*b*c^3*e - 9*a^2*b^2*c^2*g + 3*(a^2*b^3*c + 2*a^3*b*c^2)*j - (a^2*b^4 +
 a^3*b^2*c + 16*a^4*c^2)*l)*x^4 - ((3*b^5*c - 20*a*b^3*c^2 - 4*a^2*b*c^3)*
d + (a*b^4*c + 5*a^2*b^2*c^2 + 36*a^3*c^3)*f - (5*a^2*b^3*c + 16*a^3*b*c^2
)*h + (19*a^3*b^2*c - 4*a^4*c^2)*k - 2*(a^3*b^3 + 14*a^4*b*c)*m)*x^3 - 4*(
2*(a^2*b^2*c^2 + 5*a^3*c^3)*e - (a^2*b^3*c + 5*a^3*b*c^2)*g + (5*a^3*b^2*c
 - 2*a^4*c^2)*j - (a^3*b^3 + 5*a^4*b*c)*l)*x^2 + 2*(a^2*b^3*c - 10*a^3*b*c
^2)*e + 2*(a^3*b^2*c + 8*a^4*c^2)*g + 2*(a^4*b^2 + 8*a^5*c)*l - (12*a^4*b*
c*k + (5*a*b^4*c - 37*a^2*b^2*c^2 + 44*a^3*c^3)*d - (a^2*b^3*c - 16*a^3*b*
c^2)*f - 3*(a^3*b^2*c + 4*a^4*c^2)*h - (a^4*b^2 + 20*a^5*c)*m)*x)/(a^4*b^4
*c - 8*a^5*b^2*c^2 + 16*a^6*c^3 + (a^2*b^4*c^3 - 8*a^3*b^2*c^4 + 16*a^4*c^
5)*x^8 + 2*(a^2*b^5*c^2 - 8*a^3*b^3*c^3 + 16*a^4*b*c^4)*x^6 + (a^2*b^6*c -
 6*a^3*b^4*c^2 + 32*a^5*c^4)*x^4 + 2*(a^3*b^5*c - 8*a^4*b^3*c^2 + 16*a^5*b
*c^3)*x^2) - 1/8*integrate((12*a^3*b*c*k + (12*a^2*b*c^2*h - 3*(b^3*c^2 -
8*a*b*c^3)*d - (a*b^2*c^2 + 20*a^2*c^3)*f - 3*(a^2*b^2*c + 4*a^3*c^2)*k...
```

3.57.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 22429 vs. $2(1098) = 2196$.

Time = 3.50 (sec) , antiderivative size = 22429, normalized size of antiderivative = 19.50

$$\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^3}\,dx = \text{Too large to display}$$

input
```
integrate((m*x^8+l*x^7+k*x^6+j*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a
)^3,x, algorithm="giac")
```

output
```
1/64*(3*(2*b^5*c^4 - 24*a*b^3*c^5 + 64*a^2*b*c^6 - sqrt(2)*sqrt(b^2 - 4*a*
c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^5*c^2 + 12*sqrt(2)*sqrt(b^2 - 4*a*c)*
sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^3*c^3 + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sq
rt(b*c + sqrt(b^2 - 4*a*c)*c)*b^4*c^3 - 32*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(
b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b*c^4 - 16*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b
*c + sqrt(b^2 - 4*a*c)*c)*a*b^2*c^4 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c +
 sqrt(b^2 - 4*a*c)*c)*b^3*c^4 + 8*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqr
t(b^2 - 4*a*c)*c)*a*b*c^5 - 2*(b^2 - 4*a*c)*b^3*c^4 + 16*(b^2 - 4*a*c)*a*b
*c^5)*(a^2*b^4*c - 8*a^3*b^2*c^2 + 16*a^4*c^3)^2*d + (2*a*b^4*c^4 + 32*a^2
*b^2*c^5 - 160*a^3*c^6 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4
*a*c)*c)*a*b^4*c^2 - 16*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*
a*c)*c)*a^2*b^2*c^3 + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*
a*c)*c)*a*b^3*c^3 + 80*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a
*c)*c)*a^3*c^4 + 40*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)
*c)*a^2*b*c^4 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*
a*b^2*c^4 - 20*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a
^2*c^5 - 2*(b^2 - 4*a*c)*a*b^2*c^4 - 40*(b^2 - 4*a*c)*a^2*c^5)*(a^2*b^4*c
- 8*a^3*b^2*c^2 + 16*a^4*c^3)^2*f - 12*(2*a^2*b^3*c^4 - 8*a^3*b*c^5 - sqrt
(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^3*c^2 + 4*sqrt
(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*b*c^3 + 2*sqr...
```

3.57.9 Mupad [B] (verification not implemented)

Time = 32.04 (sec) , antiderivative size = 114377, normalized size of antiderivative = 99.46

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4 + jx^5 + kx^6 + lx^7 + mx^8}{(a + bx^2 + cx^4)^3} \, dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4 + j*x^5 + k*x^6 + l*x^7 + m*x^8)/(a +
 b*x^2 + c*x^4)^3,x)
```

output
```
symsum(log(root(56371445760*a^11*b^8*c^9*z^4 - 503316480*a^8*b^14*c^6*z^4
+ 47185920*a^7*b^16*c^5*z^4 - 2621440*a^6*b^18*c^4*z^4 + 65536*a^5*b^20*c^
3*z^4 - 171798691840*a^14*b^2*c^12*z^4 + 193273528320*a^13*b^4*c^11*z^4 -
128849018880*a^12*b^6*c^10*z^4 - 16911433728*a^10*b^10*c^8*z^4 + 352321536
0*a^9*b^12*c^7*z^4 + 68719476736*a^15*c^13*z^4 + 1536*a^5*b^16*c*k*m*z^2 +
 1536*a*b^18*c^3*d*f*z^2 - 2571632640*a^9*b^5*c^8*d*m*z^2 + 2548039680*a^9
*b^3*c^10*d*h*z^2 + 1509949440*a^10*b^3*c^9*e*l*z^2 + 1509949440*a^9*b^3*c
^10*e*g*z^2 - 1401421824*a^8*b^5*c^9*d*h*z^2 - 1321205760*a^9*b^2*c^11*d*f
*z^2 - 2793406464*a^11*b*c^10*d*m*z^2 + 890634240*a^8*b^7*c^7*d*m*z^2 - 75
4974720*a^10*b^4*c^8*g*l*z^2 - 754974720*a^9*b^5*c^8*e*l*z^2 + 719585280*a
^8*b^6*c^8*d*k*z^2 - 707788800*a^9*b^4*c^9*d*k*z^2 - 754974720*a^8*b^5*c^9
*e*g*z^2 + 603979776*a^11*b^2*c^9*g*l*z^2 - 581959680*a^10*b^4*c^8*f*m*z^2
 + 732168192*a^7*b^6*c^9*d*f*z^2 + 534773760*a^11*b^3*c^8*h*m*z^2 - 456130
560*a^11*b^4*c^7*k*m*z^2 - 603979776*a^10*b^2*c^10*e*j*z^2 + 534773760*a^1
0*b^3*c^9*f*k*z^2 + 384040960*a^9*b^6*c^7*f*m*z^2 + 377487360*a^9*b^6*c^7*
g*l*z^2 - 456130560*a^9*b^4*c^9*f*h*z^2 + 301989888*a^11*b^3*c^8*j*l*z^2 -
 415236096*a^10*b^2*c^10*d*k*z^2 + 254017536*a^10*b^6*c^6*k*m*z^2 - 330301
440*a^10*b^4*c^8*h*k*z^2 + 390463488*a^7*b^7*c^8*d*h*z^2 + 188743680*a^12*
b^2*c^8*k*m*z^2 + 301989888*a^10*b^3*c^9*g*j*z^2 - 297861120*a^7*b^8*c^7*d
*k*z^2 - 366280704*a^6*b^8*c^8*d*f*z^2 + 188743680*a^11*b^2*c^9*h*k*z^2...
```

3.57. $\int \frac{d+ex+fx^2+gx^3+hx^4+jx^5+kx^6+lx^7+mx^8}{(a+bx^2+cx^4)^3}\,dx$

3.58 $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^6+kx^7}{(a+bx^2+cx^4)^2}\,dx$

3.58.1 Optimal result

Integrand size = 50, antiderivative size = 645

$$\begin{aligned}
&\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^6+kx^7}{(a+bx^2+cx^4)^2}\,dx \\
&= \frac{x\left(c\left(b^2d-2a(cd-ah)-\frac{ab(cf+aj)}{c}\right)+\left(bc(cd+ah)-ab^2j-2ac(cf-aj)\right)x^2\right)}{2ac\,(b^2-4ac)\,(a+bx^2+cx^4)} \\
&\quad -\frac{bc(ce+ai)-ab^2k-2ac(cg-ak)+\left(2c^3e-c^2(bg+2ai)-b^3k+bc(bi+3ak)\right)x^2}{2c^2\,(b^2-4ac)\,(a+bx^2+cx^4)} \\
&\quad +\frac{\left(b(cd+ah)+\frac{ab^2j}{c}-2a(cf+3aj)+\frac{b^2c(cd-ah)-4ac^2(3cd+ah)-ab^3j+4abc(cf+2aj)}{c\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{2\sqrt{2}a\sqrt{c}\,(b^2-4ac)\sqrt{b-\sqrt{b^2-4ac}}} \\
&\quad +\frac{\left(b(cd+ah)+\frac{ab^2j}{c}-2a(cf+3aj)-\frac{b^2c(cd-ah)-4ac^2(3cd+ah)-ab^3j+4abc(cf+2aj)}{c\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{2\sqrt{2}a\sqrt{c}\,(b^2-4ac)\sqrt{b+\sqrt{b^2-4ac}}} \\
&\quad +\frac{\left(4c^3e-c^2(2bg-4ai)+b^3k-6abck\right)\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{2c^2\,(b^2-4ac)^{3/2}}+\frac{k\log\left(a+bx^2+cx^4\right)}{4c^2}
\end{aligned}$$

output
```
1/2*x*(c*(b^2*d-2*a*(-a*h+c*d)-a*b*(a*j+c*f)/c)+(b*c*(a*h+c*d)-a*b^2*j-2*a
*c*(-a*j+c*f))*x^2)/a/c/(-4*a*c+b^2)/(c*x^4+b*x^2+a)+1/2*(-b*c*(a*i+c*e)+a
*b^2*k+2*a*c*(-a*k+c*g)-(2*c^3*e-c^2*(2*a*i+b*g)-b^3*k+b*c*(3*a*k+b*i))*x^
2)/c^2/(-4*a*c+b^2)/(c*x^4+b*x^2+a)+1/2*(4*c^3*e-c^2*(-4*a*i+2*b*g)+b^3*k-
6*a*b*c*k)*arctanh((2*c*x^2+b)/(-4*a*c+b^2)^(1/2))/c^2/(-4*a*c+b^2)^(3/2)+
1/4*k*ln(c*x^4+b*x^2+a)/c^2+1/4*arctan(x*2^(1/2)*c^(1/2)/(b-(-4*a*c+b^2)^(
1/2))^(1/2))*(b*(a*h+c*d)+a*b^2*j/c-2*a*(3*a*j+c*f)+(b^2*c*(-a*h+c*d)-4*a*
c^2*(a*h+3*c*d)-a*b^3*j+4*a*b*c*(2*a*j+c*f))/c/(-4*a*c+b^2)^(1/2))/a/(-4*a
*c+b^2)*2^(1/2)/c^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2)+1/4*arctan(x*2^(1/2)*
c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2))*(b*(a*h+c*d)+a*b^2*j/c-2*a*(3*a*j+c*
f)+(-b^2*c*(-a*h+c*d)+4*a*c^2*(a*h+3*c*d)+a*b^3*j-4*a*b*c*(2*a*j+c*f))/c/(
-4*a*c+b^2)^(1/2))/a/(-4*a*c+b^2)*2^(1/2)/c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(
1/2)
```

3.58.2 Mathematica [A] (verified)

Time = 2.47 (sec) , antiderivative size = 775, normalized size of antiderivative = 1.20

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^6+kx^7}{(a+bx^2+cx^4)^2}\,dx$$

$$= \frac{\frac{2\left(2a^3ck-bc^2dx\left(b+cx^2\right)+a\left(-b^3kx^2+b^2cx^2(i+jx)+2c^3x(d+x(e+fx))+bc^2(e+x(f-x(g+hx)))\right)+a^2\left(-b^2k+bc(i+x(j+3kx))-2c^2(g+x(h+x(i+\right.\right.}{a(-b^2+4ac)(a+bx^2+cx^4)}}{}$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5 + j*x^6 + k*x^7)/(a + b
*x^2 + c*x^4)^2,x]
```

output
```
((2*(2*a^3*c*k - b*c^2*d*x*(b + c*x^2) + a*(-(b^3*k*x^2) + b^2*c*x^2*(i +
j*x) + 2*c^3*x*(d + x*(e + f*x)) + b*c^2*(e + x*(f - x*(g + h*x)))) + a^2*
(-(b^2*k) + b*c*(i + x*(j + 3*k*x)) - 2*c^2*(g + x*(h + x*(i + j*x))))))/(
a*(-b^2 + 4*a*c)*(a + b*x^2 + c*x^4)) - (Sqrt[2]*Sqrt[c]*(a*b^3*j - b*c*(c
*Sqrt[b^2 - 4*a*c]*d + 4*a*c*f + a*Sqrt[b^2 - 4*a*c]*h + 8*a^2*j) - b^2*(c
^2*d - a*c*h + a*Sqrt[b^2 - 4*a*c]*j) + 2*a*c*(6*c^2*d + c*Sqrt[b^2 - 4*a*
c]*f + 2*a*c*h + 3*a*Sqrt[b^2 - 4*a*c]*j))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt
[b - Sqrt[b^2 - 4*a*c]]])/(a*(b^2 - 4*a*c)^(3/2)*Sqrt[b - Sqrt[b^2 - 4*a*c
]]) + (Sqrt[2]*Sqrt[c]*(a*b^3*j + b*c*(c*Sqrt[b^2 - 4*a*c]*d - 4*a*c*f + a
*Sqrt[b^2 - 4*a*c]*h - 8*a^2*j) + 2*a*c*(6*c^2*d - c*Sqrt[b^2 - 4*a*c]*f +
 2*a*c*h - 3*a*Sqrt[b^2 - 4*a*c]*j) + b^2*(-(c^2*d) + a*c*h + a*Sqrt[b^2 -
 4*a*c]*j))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(a*(b
^2 - 4*a*c)^(3/2)*Sqrt[b + Sqrt[b^2 - 4*a*c]]) + ((-4*c^3*e + 2*c^2*(b*g -
 2*a*i) + b^2*(-b + Sqrt[b^2 - 4*a*c])*k + a*c*(6*b*k - 4*Sqrt[b^2 - 4*a*c
]*k))*Log[-b + Sqrt[b^2 - 4*a*c] - 2*c*x^2])/(b^2 - 4*a*c)^(3/2) + ((4*c^3
*e + c^2*(-2*b*g + 4*a*i) + b^2*(b + Sqrt[b^2 - 4*a*c])*k - 2*a*c*(3*b + 2
*Sqrt[b^2 - 4*a*c])*k)*Log[b + Sqrt[b^2 - 4*a*c] + 2*c*x^2])/(b^2 - 4*a*c)
^(3/2))/(4*c^2)
```

3.58.3 Rubi [A] (verified)

Time = 1.52 (sec) , antiderivative size = 653, normalized size of antiderivative = 1.01, number of steps used = 12, number of rules used = 11, $\frac{\text{number of rules}}{\text{integrand size}} = 0.220$, Rules used = {2202, 2194, 2191, 1142, 1083, 219, 1103, 2206, 25, 1480, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^6+kx^7}{(a+bx^2+cx^4)^2}dx$$

$$\Big\downarrow \; 2202$$

$$\int \frac{jx^6+hx^4+fx^2+d}{(cx^4+bx^2+a)^2}dx+\int \frac{x\left(kx^6+ix^4+gx^2+e\right)}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \; 2194$$

$$\int \frac{jx^6+hx^4+fx^2+d}{(cx^4+bx^2+a)^2}dx+\frac{1}{2}\int \frac{kx^6+ix^4+gx^2+e}{(cx^4+bx^2+a)^2}dx^2$$

$$\Big\downarrow \; 2191$$

$$\frac{1}{2}\left(-\frac{\int \frac{\left(4a-\frac{b^2}{c}\right)kx^2+2ce-bg+2ai-\frac{abk}{c}}{cx^4+bx^2+a}dx^2}{b^2-4ac}-\frac{x^2\left(-c^2(2ai+bg)+bc(3ak+bi)+b^3(-k)+2c^3e\right)-ab^2k+bc(ai+ce)}{c^2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right.$$

$$\int \frac{jx^6+hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^2}dx$$

$\downarrow$ 1142

$$\frac{1}{2}\left(-\frac{\frac{\left(-c^2(2bg-4ai)-6abck+b^3k+4c^3e\right)\int \frac{1}{cx^4+bx^2+a}dx^2}{2c^2}-\frac{k\left(b^2-4ac\right)\int \frac{2cx^2+b}{cx^4+bx^2+a}dx^2}{2c^2}}{b^2-4ac}-\frac{x^2\left(-c^2(2ai+bg)+bc(3ak+bi)+b^3\right.}{c^2\left(b^2-\right.}\right.$$

$$\int \frac{jx^6+hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^2}dx$$

$\downarrow$ 1083

$$\frac{1}{2}\left(-\frac{-\frac{k\left(b^2-4ac\right)\int \frac{2cx^2+b}{cx^4+bx^2+a}dx^2}{2c^2}-\frac{\left(-c^2(2bg-4ai)-6abck+b^3k+4c^3e\right)\int \frac{1}{-x^4+b^2-4ac}d\left(2cx^2+b\right)}{c^2}}{b^2-4ac}-\frac{x^2\left(-c^2(2ai+bg)+bc(3ak+\right.}{}\right.$$

$$\int \frac{jx^6+hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^2}dx$$

$\downarrow$ 219

$$\frac{1}{2}\left(-\frac{-\frac{k\left(b^2-4ac\right)\int \frac{2cx^2+b}{cx^4+bx^2+a}dx^2}{2c^2}-\frac{\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-c^2(2bg-4ai)-6abck+b^3k+4c^3e\right)}{c^2\sqrt{b^2-4ac}}}{b^2-4ac}-\frac{x^2\left(-c^2(2ai+bg)+bc(3ak+bi\right.}{c^2}\right.$$

$$\int \frac{jx^6+hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^2}dx$$

$\downarrow$ 1103

$$\int \frac{jx^6+hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^2}dx+$$

$$\frac{1}{2}\left(-\frac{-\frac{\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-c^2(2bg-4ai)-6abck+b^3k+4c^3e\right)}{c^2\sqrt{b^2-4ac}}-\frac{k\left(b^2-4ac\right)\log\left(a+bx^2+cx^4\right)}{2c^2}}{b^2-4ac}-\frac{x^2\left(-c^2(2ai+bg)+bc(3ak+bi\right.}{c^2}\right.$$

$\downarrow$ 2206

3.58. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^6+kx^7}{\left(a+bx^2+cx^4\right)^2}\,dx$

$$-\frac{\int -\frac{db^2+\frac{a(cf+aj)b}{c}+\left(\frac{ajb^2}{c}+(cd+ah)b-2a(cf+3aj)\right)x^2-2a(3cd+ah)}{cx^4+bx^2+a}dx}{2a\left(b^2-4ac\right)}+$$

$$\frac{1}{2}\left(-\frac{-\frac{\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-c^2(2bg-4ai)-6abck+b^3k+4c^3e\right)}{c^2\sqrt{b^2-4ac}}-\frac{k\left(b^2-4ac\right)\log\left(a+bx^2+cx^4\right)}{2c^2}}{b^2-4ac}-\frac{x^2\left(-c^2(2ai+bg)+bc(3ak+b\right.}{c^2}\right.$$

$$\frac{x\left(x^2\left(-ab^2j+bc(ah+cd)-2ac(cf-aj)\right)+c\left(-\frac{ab(aj+cf)}{c}-2a(cd-ah)+b^2d\right)\right)}{2ac\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

$\big\downarrow$ 25

$$\frac{\int \frac{db^2+\frac{a(cf+aj)b}{c}+\left(\frac{ajb^2}{c}+(cd+ah)b-2a(cf+3aj)\right)x^2-2a(3cd+ah)}{cx^4+bx^2+a}dx}{2a\left(b^2-4ac\right)}+$$

$$\frac{1}{2}\left(-\frac{-\frac{\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-c^2(2bg-4ai)-6abck+b^3k+4c^3e\right)}{c^2\sqrt{b^2-4ac}}-\frac{k\left(b^2-4ac\right)\log\left(a+bx^2+cx^4\right)}{2c^2}}{b^2-4ac}-\frac{x^2\left(-c^2(2ai+bg)+bc(3ak+b\right.}{c^2}\right.$$

$$\frac{x\left(x^2\left(-ab^2j+bc(ah+cd)-2ac(cf-aj)\right)+c\left(-\frac{ab(aj+cf)}{c}-2a(cd-ah)+b^2d\right)\right)}{2ac\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

$\big\downarrow$ 1480

$$\frac{\frac{1}{2}\left(\frac{ab^2j}{c}+\frac{-ab^3j+b^2c(cd-ah)+4abc(2aj+cf)-4ac^2(ah+3cd)}{c\sqrt{b^2-4ac}}+b(ah+cd)-2a(3aj+cf)\right)\int\frac{1}{cx^2+\frac{1}{2}\left(b-\sqrt{b^2-4ac}\right)}dx+\frac{1}{2}\left(\frac{ab^2j}{c}\right.}{2a\left(b^2-4ac\right)}$$

$$\frac{1}{2}\left(-\frac{-\frac{\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-c^2(2bg-4ai)-6abck+b^3k+4c^3e\right)}{c^2\sqrt{b^2-4ac}}-\frac{k\left(b^2-4ac\right)\log\left(a+bx^2+cx^4\right)}{2c^2}}{b^2-4ac}-\frac{x^2\left(-c^2(2ai+bg)+bc(3ak+b\right.}{c^2}\right.$$

$$\frac{x\left(x^2\left(-ab^2j+bc(ah+cd)-2ac(cf-aj)\right)+c\left(-\frac{ab(aj+cf)}{c}-2a(cd-ah)+b^2d\right)\right)}{2ac\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

$\big\downarrow$ 218

3.58. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^6+kx^7}{\left(a+bx^2+cx^4\right)^2}\,dx$

$$\frac{\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{ab^2j}{c}+\frac{-ab^3j+b^2c(cd-ah)+4abc(2aj+cf)-4ac^2(ah+3cd)}{c\sqrt{b^2-4ac}}+b(ah+cd)-2a(3aj+cf)\right)}{\sqrt{2}\sqrt{c}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(\frac{ab^2j}{c}-\frac{-a\ldots}{}\right.}{}}{2a\left(b^2-4ac\right)}$$

$$\frac{1}{2}\left(-\frac{-\frac{\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)\left(-c^2(2bg-4ai)-6abck+b^3k+4c^3e\right)}{c^2\sqrt{b^2-4ac}}-\frac{k(b^2-4ac)\log(a+bx^2+cx^4)}{2c^2}}{b^2-4ac}-\frac{x^2\left(-c^2(2ai+bg)+bc(3ak+b\ldots\right.}{c^2\ldots}\right.$$

$$\frac{x\left(x^2\left(-ab^2j+bc(ah+cd)-2ac(cf-aj)\right)+c\left(-\frac{ab(aj+cf)}{c}-2a(cd-ah)+b^2d\right)\right)}{2ac\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5 + j*x^6 + k*x^7)/(a + b*x^2 +
 c*x^4)^2,x]
```

output
```
(x*(c*(b^2*d - 2*a*(c*d - a*h) - (a*b*(c*f + a*j))/c) + (b*c*(c*d + a*h) -
 a*b^2*j - 2*a*c*(c*f - a*j))*x^2))/(2*a*c*(b^2 - 4*a*c)*(a + b*x^2 + c*x^
4)) + (((b*(c*d + a*h) + (a*b^2*j)/c - 2*a*(c*f + 3*a*j) + (b^2*c*(c*d - a
*h) - 4*a*c^2*(3*c*d + a*h) - a*b^3*j + 4*a*b*c*(c*f + 2*a*j))/(c*Sqrt[b^2
 - 4*a*c]))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]])/(Sqrt
[2]*Sqrt[c]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + ((b*(c*d + a*h) + (a*b^2*j)/c -
 2*a*(c*f + 3*a*j) - (b^2*c*(c*d - a*h) - 4*a*c^2*(3*c*d + a*h) - a*b^3*j
+ 4*a*b*c*(c*f + 2*a*j))/(c*Sqrt[b^2 - 4*a*c]))*ArcTan[(Sqrt[2]*Sqrt[c]*x)
/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[c]*Sqrt[b + Sqrt[b^2 - 4*a*c]
]))/(2*a*(b^2 - 4*a*c)) + (-((b*c*(c*e + a*i) - a*b^2*k - 2*a*c*(c*g - a*k
) + (2*c^3*e - c^2*(b*g + 2*a*i) - b^3*k + b*c*(b*i + 3*a*k))*x^2)/(c^2*(b
^2 - 4*a*c)*(a + b*x^2 + c*x^4))) - (-(((4*c^3*e - c^2*(2*b*g - 4*a*i) + b
^3*k - 6*a*b*c*k)*ArcTanh[(b + 2*c*x^2)/Sqrt[b^2 - 4*a*c]])/(c^2*Sqrt[b^2
- 4*a*c])) - ((b^2 - 4*a*c)*k*Log[a + b*x^2 + c*x^4])/(2*c^2))/(b^2 - 4*a*
c))/2
```

3.58.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 2191
```
Int[(Pq_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> With[{Q =
PolynomialQuotient[Pq, a + b*x + c*x^2, x], f = Coeff[PolynomialRemainder[P
q, a + b*x + c*x^2, x], x, 0], g = Coeff[PolynomialRemainder[Pq, a + b*x +
c*x^2, x], x, 1]}, Simp[(b*f - 2*a*g + (2*c*f - b*g)*x)*((a + b*x + c*x^2)^
(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] + Simp[1/((p + 1)*(b^2 - 4*a*c))   Int
[(a + b*x + c*x^2)^(p + 1)*ExpandToSum[(p + 1)*(b^2 - 4*a*c)*Q - (2*p + 3)*
(2*c*f - b*g), x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Pq, x] && NeQ[b^
2 - 4*a*c, 0] && LtQ[p, -1]
```

rule 2194
```
Int[(Pq_)*(x_)^(m_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :
> Simp[1/2   Subst[Int[x^((m - 1)/2)*SubstFor[x^2, Pq, x]*(a + b*x + c*x^2)
^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pq, x^2] && IntegerQ
[(m - 1)/2]
```

3.58. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^6+kx^7}{(a+bx^2+cx^4)^2}\,dx$

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2206
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> With[{d =
 Coeff[PolynomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 0], e = Coeff[Poly
nomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 2]}, Simp[x*(a + b*x^2 + c*x^
4)^(p + 1)*((a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)/(2*a*(p + 1)*(b
^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(a + b*x^2 + c
*x^4)^(p + 1)*ExpandToSum[2*a*(p + 1)*(b^2 - 4*a*c)*PolynomialQuotient[Px,
a + b*x^2 + c*x^4, x] + b^2*d*(2*p + 3) - 2*a*c*d*(4*p + 5) - a*b*e + c*(4*
p + 7)*(b*d - 2*a*e)*x^2, x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x
^2] && Expon[Px, x^2] > 1 && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1]
```

3.58.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.58 (sec) , antiderivative size = 422, normalized size of antiderivative = 0.65

method	result
risch	$\frac{-\frac{\left(2a^2cj-a\,b^2j+abch-2a\,c^2f+b\,c^2d\right)x^3}{2a\left(4ac-b^2\right)c}+\frac{\left(3abck-2a\,c^2i-b^3k+b^2ci-b\,c^2g+2c^3e\right)x^2}{2\left(4ac-b^2\right)c^2}+\frac{\left(a^2bj-2a^2ch+abcf+2a\,c^2d-b^2cd\right)x}{2ac\left(4ac-b^2\right)}+\frac{2a^2ck-a\,b^2k+abc}{2\left(4ac-\right.}}{c\,x^4+b\,x^2+a}$
default	Expression too large to display

input
```
int((k*x^7+j*x^6+i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x,method
=_RETURNVERBOSE)
```

output
```
(-1/2/a*(2*a^2*c*j-a*b^2*j+a*b*c*h-2*a*c^2*f+b*c^2*d)/(4*a*c-b^2)/c*x^3+1/
2*(3*a*b*c*k-2*a*c^2*i-b^3*k+b^2*c*i-b*c^2*g+2*c^3*e)/(4*a*c-b^2)/c^2*x^2+
1/2*(a^2*b*j-2*a^2*c*h+a*b*c*f+2*a*c^2*d-b^2*c*d)/a/c/(4*a*c-b^2)*x+1/2*(2
*a^2*c*k-a*b^2*k+a*b*c*i-2*a*c^2*g+b*c^2*e)/(4*a*c-b^2)/c^2)/(c*x^4+b*x^2+
a)+1/4/c*sum((2*k*_R^3+1/a*(6*a^2*c*j-a*b^2*j-a*b*c*h+2*a*c^2*f-b*c^2*d)/(
4*a*c-b^2)*_R^2-2*(a*b*k-2*a*c*i+b*c*g-2*c^2*e)/(4*a*c-b^2)*_R-1/a*(a^2*b*
j-2*a^2*c*h+a*b*c*f-6*a*c^2*d+b^2*c*d)/(4*a*c-b^2))/(2*_R^3*c+_R*b)*ln(x-_
R),_R=RootOf(_Z^4*c+_Z^2*b+a))
```

3.58.5 Fricas [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^6+kx^7}{(a+bx^2+cx^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((k*x^7+j*x^6+i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x,
 algorithm="fricas")
```

output
```
Timed out
```

3.58.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^6+kx^7}{(a+bx^2+cx^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((k*x**7+j*x**6+i*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(c*x**4+b*x**2
+a)**2,x)
```

output
```
Timed out
```

3.58.7 Maxima [F]

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4 + ix^5 + jx^6 + kx^7}{(a + bx^2 + cx^4)^2} dx = \int \frac{kx^7 + jx^6 + ix^5 + hx^4 + gx^3 + fx^2 + ex + d}{(cx^4 + bx^2 + a)^2} dx$$

input
```
integrate((k*x^7+j*x^6+i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x,
 algorithm="maxima")
```

output
```
-1/2*(a*b*c^2*e - 2*a^2*c^2*g + a^2*b*c*i - (b*c^3*d - 2*a*c^3*f + a*b*c^2
*h - (a*b^2*c - 2*a^2*c^2)*j)*x^3 + (2*a*c^3*e - a*b*c^2*g + (a*b^2*c - 2*
a^2*c^2)*i - (a*b^3 - 3*a^2*b*c)*k)*x^2 - (a^2*b^2 - 2*a^3*c)*k + (a*b*c^2
*f - 2*a^2*c^2*h + a^2*b*c*j - (b^2*c^2 - 2*a*c^3)*d)*x)/(a^2*b^2*c^2 - 4*
a^3*c^3 + (a*b^2*c^3 - 4*a^2*c^4)*x^4 + (a*b^3*c^2 - 4*a^2*b*c^3)*x^2) - 1
/2*integrate(-(2*(a*b^2 - 4*a^2*c)*k*x^3 + a*b*c*f - 2*a^2*c*h + a^2*b*j +
 (b*c^2*d - 2*a*c^2*f + a*b*c*h + (a*b^2 - 6*a^2*c)*j)*x^2 + (b^2*c - 6*a*
c^2)*d - 2*(2*a*c^2*e - a*b*c*g + 2*a^2*c*i - a^2*b*k)*x)/(c*x^4 + b*x^2 +
 a), x)/(a*b^2*c - 4*a^2*c^2)
```

3.58.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 16214 vs. $2(594) = 1188$.

Time = 3.25 (sec) , antiderivative size = 16214, normalized size of antiderivative = 25.14

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4 + ix^5 + jx^6 + kx^7}{(a + bx^2 + cx^4)^2} dx = \text{Too large to display}$$

input
```
integrate((k*x^7+j*x^6+i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^2,x,
 algorithm="giac")
```

output
```
1/4*k*log(abs(c*x^4 + b*x^2 + a))/c^2 + 1/16*((a^2*b^4*c^3 - 8*a^3*b^2*c^4
 + 16*a^4*c^5)^2*(2*b^3*c^4 - 8*a*b*c^5 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b
*c + sqrt(b^2 - 4*a*c)*c)*b^3*c^2 + 4*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c +
 sqrt(b^2 - 4*a*c)*c)*a*b*c^3 + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqr
t(b^2 - 4*a*c)*c)*b^2*c^3 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2
- 4*a*c)*c)*b*c^4 - 2*(b^2 - 4*a*c)*b*c^4)*d - 2*(a^2*b^4*c^3 - 8*a^3*b^2*
c^4 + 16*a^4*c^5)^2*(2*a*b^2*c^4 - 8*a^2*c^5 - sqrt(2)*sqrt(b^2 - 4*a*c)*s
qrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^2*c^2 + 4*sqrt(2)*sqrt(b^2 - 4*a*c)*sqr
t(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*c^3 + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*
c + sqrt(b^2 - 4*a*c)*c)*a*b*c^3 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sq
rt(b^2 - 4*a*c)*c)*a*c^4 - 2*(b^2 - 4*a*c)*a*c^4)*f + (a^2*b^4*c^3 - 8*a^3
*b^2*c^4 + 16*a^4*c^5)^2*(2*a*b^3*c^3 - 8*a^2*b*c^4 - sqrt(2)*sqrt(b^2 - 4
*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^3*c + 4*sqrt(2)*sqrt(b^2 - 4*a*c
)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b*c^2 + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*
sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^2*c^2 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt
(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c^3 - 2*(b^2 - 4*a*c)*a*b*c^3)*h + (a^2*b^
4*c^3 - 8*a^3*b^2*c^4 + 16*a^4*c^5)^2*(2*a*b^4*c^2 - 20*a^2*b^2*c^3 + 48*a
^3*c^4 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^4 +
 10*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^2*c +
2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^3*c - 2...
```

3.58.9 Mupad [B] (verification not implemented)

Time = 13.37 (sec) , antiderivative size = 53538, normalized size of antiderivative = 83.00

$$\int \frac{d + ex + fx^2 + gx^3 + hx^4 + ix^5 + jx^6 + kx^7}{(a + bx^2 + cx^4)^2} \, dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5 + j*x^6 + k*x^7)/(a + b*x^2 +
 c*x^4)^2,x)
```

output
```
((b*c^2*e - 2*a*c^2*g - a*b^2*k + 2*a^2*c*k + a*b*c*i)/(2*c^2*(4*a*c - b^2
)) + (x^2*(2*c^3*e - b^3*k - b*c^2*g - 2*a*c^2*i + b^2*c*i + 3*a*b*c*k))/(
2*c^2*(4*a*c - b^2)) + (x*(2*a*c^2*d - b^2*c*d - 2*a^2*c*h + a^2*b*j + a*b
*c*f))/(2*a*c*(4*a*c - b^2)) - (x^3*(b*c^2*d - 2*a*c^2*f - a*b^2*j + 2*a^2
*c*j + a*b*c*h))/(2*a*c*(4*a*c - b^2)))/(a + b*x^2 + c*x^4) + symsum(log(r
oot(1572864*a^8*b^2*c^9*z^4 - 983040*a^7*b^4*c^8*z^4 + 327680*a^6*b^6*c^7*
z^4 - 61440*a^5*b^8*c^6*z^4 + 6144*a^4*b^10*c^5*z^4 - 256*a^3*b^12*c^4*z^4
 - 1048576*a^9*c^10*z^4 - 1572864*a^8*b^2*c^7*k*z^3 + 983040*a^7*b^4*c^6*k
*z^3 - 327680*a^6*b^6*c^5*k*z^3 + 61440*a^5*b^8*c^4*k*z^3 - 6144*a^4*b^10*
c^3*k*z^3 + 256*a^3*b^12*c^2*k*z^3 + 1048576*a^9*c^8*k*z^3 + 98304*a^8*b*c
^6*i*k*z^2 + 98304*a^7*b*c^7*e*k*z^2 + 57344*a^7*b*c^7*f*j*z^2 + 32768*a^7
*b*c^7*g*i*z^2 + 57344*a^6*b*c^8*d*h*z^2 + 32768*a^6*b*c^8*e*g*z^2 - 32*a*
b^10*c^4*d*f*z^2 - 90112*a^7*b^3*c^5*i*k*z^2 + 30720*a^6*b^5*c^4*i*k*z^2 -
 4608*a^5*b^7*c^3*i*k*z^2 + 256*a^4*b^9*c^2*i*k*z^2 - 49152*a^7*b^2*c^6*g*
k*z^2 + 45056*a^6*b^4*c^5*g*k*z^2 + 24576*a^7*b^2*c^6*h*j*z^2 - 15360*a^5*
b^6*c^4*g*k*z^2 - 3072*a^5*b^6*c^4*h*j*z^2 + 2304*a^4*b^8*c^3*g*k*z^2 + 20
48*a^6*b^4*c^5*h*j*z^2 + 576*a^4*b^8*c^3*h*j*z^2 - 128*a^3*b^10*c^2*g*k*z^
2 - 32*a^3*b^10*c^2*h*j*z^2 - 90112*a^6*b^3*c^6*e*k*z^2 - 49152*a^6*b^3*c^
6*f*j*z^2 + 30720*a^5*b^5*c^5*e*k*z^2 - 24576*a^6*b^3*c^6*g*i*z^2 + 15360*
a^5*b^5*c^5*f*j*z^2 + 6144*a^5*b^5*c^5*g*i*z^2 - 4608*a^4*b^7*c^4*e*k*z...
```

3.58. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^6+kx^7}{(a+bx^2+cx^4)^2}\,dx$

3.59 $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^8+kx^{11}}{(a+bx^2+cx^4)^3}\,dx$

3.59.1 Optimal result

Integrand size = 50, antiderivative size = 1177

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^8+kx^{11}}{(a+bx^2+cx^4)^3}\,dx =$$

$$-\frac{x\left(c^2\left(abf-b^2\left(d+\frac{a^2j}{c^2}\right)+2a\left(cd-ah+\frac{a^2j}{c}\right)\right)+(2ac^3f-ab^3j-bc(c^2d+ach-3a^2j))\,x^2\right)}{4ac^2\,(b^2-4ac)\,(a+bx^2+cx^4)^2}$$

$$-\frac{bc^3(ce+ai)-ab^4k+4a^2b^2ck-2ac^2(c^2g+a^2k)+(2c^5e+b^2c^3i-c^4(bg+2ai)-b^5k+5ab^3ck-5a^2b \cdots}{4c^4\,(b^2-4ac)\,(a+bx^2+cx^4)^2}$$

$$+\frac{x\left(c\left(ab^3f+8a^2bcf+4a^2(7c^2d+ach-9a^2j)+b^4\left(3d-\frac{2a^2j}{c^2}\right)-ab^2\left(25cd+7ah-\frac{11a^2j}{c}\right)\right)+(ab^2c^2f \cdots}{8a^2c\,(b^2-4ac)^2\,(a+bx^2+cx^4)}$$

$$+\frac{b^3c^2i+2bc^3(3ce+ai)+11ab^4k-\frac{b^6k}{c}+32a^3c^2k-3b^2(c^3g+13a^2ck)+2(6c^5e+b^2c^3i-c^4(3bg-2ai) \cdots}{4c^3\,(b^2-4ac)^2\,(a+bx^2+cx^4)}$$

$$+\frac{\left(ab^2c^2f+20a^2c^3f+b^3(3c^2d+a^2j)-4abc(6c^2d+3ach+4a^2j)+\frac{ab^3c^2f-52a^2bc^3f-6ab^2c(5c^2d-3ach-3a^2j)+b \cdots}{\sqrt{b^2-4 \cdots}} \cdots}{8\sqrt{2}a^2c^{3/2}\,(b^2-4ac)^2\,\sqrt{b-\sqrt{b^2-4ac}}}$$

$$+\frac{\left(ab^2c^2f+20a^2c^3f+b^3(3c^2d+a^2j)-4abc(6c^2d+3ach+4a^2j)-\frac{ab^3c^2f-52a^2bc^3f-6ab^2c(5c^2d-3ach-3a^2j)+b \cdots}{\sqrt{b^2-4 \cdots}} \cdots}{8\sqrt{2}a^2c^{3/2}\,(b^2-4ac)^2\,\sqrt{b+\sqrt{b^2-4ac}}}$$

$$-\frac{(12c^5e+2b^2c^3i-c^4(6bg-4ai)-b^5k+10ab^3ck-30a^2bc^2k)\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{2c^3\,(b^2-4ac)^{5/2}}$$

$$+\frac{k\log(a+bx^2+cx^4)}{4c^3}$$

output
```
-1/4*x*(c^2*(a*b*f-b^2*(d+a^2*j/c^2)+2*a*(c*d-a*h+a^2*j/c))+(2*a*c^3*f-a*b
^3*j-b*c*(-3*a^2*j+a*c*h+c^2*d))*x^2)/a/c^2/(-4*a*c+b^2)/(c*x^4+b*x^2+a)^2
+1/4*(-b*c^3*(a*i+c*e)+a*b^4*k-4*a^2*b^2*c*k+2*a*c^2*(a^2*k+c^2*g)-(2*c^5*
e+b^2*c^3*i-c^4*(2*a*i+b*g)-b^5*k+5*a*b^3*c*k-5*a^2*b*c^2*k)*x^2)/c^4/(-4*
a*c+b^2)/(c*x^4+b*x^2+a)^2+1/8*x*(c*(a*b^3*f+8*a^2*b*c*f+4*a^2*(-9*a^2*j+a
*c*h+7*c^2*d)+b^4*(3*d-2*a^2*j/c^2)-a*b^2*(25*c*d+7*a*h-11*a^2*j/c))+(a*b^
2*c^2*f+20*a^2*c^3*f+b^3*(a^2*j+3*c^2*d)-4*a*b*c*(4*a^2*j+3*a*c*h+6*c^2*d)
)*x^2)/a^2/c/(-4*a*c+b^2)^2/(c*x^4+b*x^2+a)+1/4*(b^3*c^2*i+2*b*c^3*(a*i+3*
c*e)+11*a*b^4*k-b^6*k/c+32*a^3*c^2*k-3*b^2*(13*a^2*c*k+c^3*g)+2*(6*c^5*e+b
^2*c^3*i-c^4*(-2*a*i+3*b*g)+2*b^5*k-15*a*b^3*c*k+25*a^2*b*c^2*k)*x^2)/c^3/
(-4*a*c+b^2)^2/(c*x^4+b*x^2+a)-1/2*(12*c^5*e+2*b^2*c^3*i-c^4*(-4*a*i+6*b*g
)-b^5*k+10*a*b^3*c*k-30*a^2*b*c^2*k)*arctanh((2*c*x^2+b)/(-4*a*c+b^2)^(1/2
))/c^3/(-4*a*c+b^2)^(5/2)+1/4*k*ln(c*x^4+b*x^2+a)/c^3+1/16*arctan(x*2^(1/2
)*c^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2))*(a*b^2*c^2*f+20*a^2*c^3*f+b^3*(a^2
*j+3*c^2*d)-4*a*b*c*(4*a^2*j+3*a*c*h+6*c^2*d)+(a*b^3*c^2*f-52*a^2*b*c^3*f-
6*a*b^2*c*(-3*a^2*j-3*a*c*h+5*c^2*d)+b^4*(-a^2*j+3*c^2*d)+8*a^2*c^2*(5*a^2
*j+3*a*c*h+21*c^2*d))/(-4*a*c+b^2)^(1/2))/a^2/c^(3/2)/(-4*a*c+b^2)^2*2^(1/
2)/(b-(-4*a*c+b^2)^(1/2))^(1/2)+1/16*arctan(x*2^(1/2)*c^(1/2)/(b+(-4*a*c+b
^2)^(1/2))^(1/2))*(a*b^2*c^2*f+20*a^2*c^3*f+b^3*(a^2*j+3*c^2*d)-4*a*b*c*(4
*a^2*j+3*a*c*h+6*c^2*d)+(-a*b^3*c^2*f+52*a^2*b*c^3*f+6*a*b^2*c*(-3*a^2*...
```

3.59.2 Mathematica [A] (verified)

Time = 6.80 (sec) , antiderivative size = 1649, normalized size of antiderivative = 1.40

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^8+kx^{11}}{(a+bx^2+cx^4)^3}\,dx$$

$$= \frac{abc^4e - 2a^2c^4g + a^2bc^3i - a^2b^4k + 4a^3b^2ck - 2a^4c^2k - b^2c^4dx + 2ac^5dx + abc^4fx - 2a^2c^4hx - a^2b^2c^2jx + \cdots}{\cdots}$$

$$+ \frac{12a^2bc^5e - 6a^2b^2c^4g + 2a^2b^3c^3i + 4a^3bc^4i - 2a^2b^6k + 22a^3b^4ck - 78a^4b^2c^2k + 64a^5c^3k + 3b^4c^4dx - 25a\cdots}{\cdots}$$

$$+ \frac{\left(3b^4c^2d - 30ab^2c^3d + 168a^2c^4d + 3b^3c^2\sqrt{b^2-4ac}\,d - 24abc^3\sqrt{b^2-4ac}\,d + ab^3c^2f - 52a^2bc^3f + ab^2c^2\sqrt{\cdots}\right.}{\cdots}$$

$$+ \frac{\left(-3b^4c^2d + 30ab^2c^3d - 168a^2c^4d + 3b^3c^2\sqrt{b^2-4ac}\,d - 24abc^3\sqrt{b^2-4ac}\,d - ab^3c^2f + 52a^2bc^3f + ab^2c^2\cdots\right.}{\cdots}$$

$$+ \frac{\left(12c^5e - 6bc^4g + 2b^2c^3i + 4ac^4i - b^5k + 10ab^3ck - 30a^2bc^2k + b^4\sqrt{b^2-4ac}\,k - 8ab^2c\sqrt{b^2-4ac}\,k + 16a\cdots\right.}{4c^3\left(b^2-4ac\right)^{5/2}}$$

$$+ \frac{\left(-12c^5e + 6bc^4g - 2b^2c^3i - 4ac^4i + b^5k - 10ab^3ck + 30a^2bc^2k + b^4\sqrt{b^2-4ac}\,k - 8ab^2c\sqrt{b^2-4ac}\,k + 1\cdots\right.}{4c^3\left(b^2-4ac\right)^{5/2}}$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5 + j*x^8 + k*x^11)/(a +
b*x^2 + c*x^4)^3,x]
```

output
```
(a*b*c^4*e - 2*a^2*c^4*g + a^2*b*c^3*i - a^2*b^4*k + 4*a^3*b^2*c*k - 2*a^4
*c^2*k - b^2*c^4*d*x + 2*a*c^5*d*x + a*b*c^4*f*x - 2*a^2*c^4*h*x - a^2*b^2
*c^2*j*x + 2*a^3*c^3*j*x + 2*a*c^5*e*x^2 - a*b*c^4*g*x^2 + a*b^2*c^3*i*x^2
 - 2*a^2*c^4*i*x^2 - a*b^5*k*x^2 + 5*a^2*b^3*c*k*x^2 - 5*a^3*b*c^2*k*x^2 -
 b*c^5*d*x^3 + 2*a*c^5*f*x^3 - a*b*c^4*h*x^3 - a*b^3*c^2*j*x^3 + 3*a^2*b*c
^3*j*x^3)/(4*a*c^4*(-b^2 + 4*a*c)*(a + b*x^2 + c*x^4)^2) + (12*a^2*b*c^5*e
 - 6*a^2*b^2*c^4*g + 2*a^2*b^3*c^3*i + 4*a^3*b*c^4*i - 2*a^2*b^6*k + 22*a^
3*b^4*c*k - 78*a^4*b^2*c^2*k + 64*a^5*c^3*k + 3*b^4*c^4*d*x - 25*a*b^2*c^5
*d*x + 28*a^2*c^6*d*x + a*b^3*c^4*f*x + 8*a^2*b*c^5*f*x - 7*a^2*b^2*c^4*h*
x + 4*a^3*c^5*h*x - 2*a^2*b^4*c^2*j*x + 11*a^3*b^2*c^3*j*x - 36*a^4*c^4*j*
x + 24*a^2*c^6*e*x^2 - 12*a^2*b*c^5*g*x^2 + 4*a^2*b^2*c^4*i*x^2 + 8*a^3*c^
5*i*x^2 + 8*a^2*b^5*c*k*x^2 - 60*a^3*b^3*c^2*k*x^2 + 100*a^4*b*c^3*k*x^2 +
 3*b^3*c^5*d*x^3 - 24*a*b*c^6*d*x^3 + a*b^2*c^5*f*x^3 + 20*a^2*c^6*f*x^3 -
 12*a^2*b*c^5*h*x^3 + a^2*b^3*c^3*j*x^3 - 16*a^3*b*c^4*j*x^3)/(8*a^2*c^4*(
-b^2 + 4*a*c)^2*(a + b*x^2 + c*x^4)) + ((3*b^4*c^2*d - 30*a*b^2*c^3*d + 16
8*a^2*c^4*d + 3*b^3*c^2*Sqrt[b^2 - 4*a*c]*d - 24*a*b*c^3*Sqrt[b^2 - 4*a*c]
*d + a*b^3*c^2*f - 52*a^2*b*c^3*f + a*b^2*c^2*Sqrt[b^2 - 4*a*c]*f + 20*a^2
*c^3*Sqrt[b^2 - 4*a*c]*f + 18*a^2*b^2*c^2*h + 24*a^3*c^3*h - 12*a^2*b*c^2*
Sqrt[b^2 - 4*a*c]*h - a^2*b^4*j + 18*a^3*b^2*c*j + 40*a^4*c^2*j + a^2*b^3*
Sqrt[b^2 - 4*a*c]*j - 16*a^3*b*c*Sqrt[b^2 - 4*a*c]*j)*ArcTan[(Sqrt[2]*S...
```

3.59.3 Rubi [A] (verified)

Time = 3.03 (sec) , antiderivative size = 1236, normalized size of antiderivative = 1.05, number of steps used = 17, number of rules used = 16, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.320, Rules used = {2202, 2194, 2191, 2191, 27, 1142, 1083, 219, 1103, 2206, 25, 2206, 25, 27, 1480, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^8+kx^{11}}{\left(a+bx^2+cx^4\right)^3}dx$$

$$\Big\downarrow \ 2202$$

$$\int \frac{jx^8+hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^3}dx+\int \frac{x\left(kx^{10}+ix^4+gx^2+e\right)}{\left(cx^4+bx^2+a\right)^3}dx$$

$$\Big\downarrow \text{2194}$$

$$\int \frac{jx^8+hx^4+fx^2+d}{(cx^4+bx^2+a)^3}dx+\frac{1}{2}\int \frac{kx^{10}+ix^4+gx^2+e}{(cx^4+bx^2+a)^3}dx^2$$

$$\Big\downarrow \text{2191}$$

$$\frac{1}{2}\left(\frac{c^4\left(\frac{2a^3k}{c^2}-\frac{4a^2b^2k}{c^3}+\frac{ab^4k}{c^4}-b\left(\frac{ai}{c}+e\right)+2ag\right)-x^2\left(-5a^2bc^2k+5ab^3ck-c^4(2ai+bg)+b^5(-k)+b^2c^3i+2c^5e\right)}{2c^4\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right.$$

$$\int \frac{jx^8+hx^4+fx^2+d}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{2191}$$

$$\frac{1}{2}\left(\frac{c^4\left(\frac{2a^3k}{c^2}-\frac{4a^2b^2k}{c^3}+\frac{ab^4k}{c^4}-b\left(\frac{ai}{c}+e\right)+2ag\right)-x^2\left(-5a^2bc^2k+5ab^3ck-c^4(2ai+bg)+b^5(-k)+b^2c^3i+2c^5e\right)}{2c^4\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right.$$

$$\int \frac{jx^8+hx^4+fx^2+d}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{27}$$

$$\frac{1}{2}\left(\frac{c^4\left(\frac{2a^3k}{c^2}-\frac{4a^2b^2k}{c^3}+\frac{ab^4k}{c^4}-b\left(\frac{ai}{c}+e\right)+2ag\right)-x^2\left(-5a^2bc^2k+5ab^3ck-c^4(2ai+bg)+b^5(-k)+b^2c^3i+2c^5e\right)}{2c^4\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right.$$

$$\int \frac{jx^8+hx^4+fx^2+d}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{1142}$$

$$\frac{1}{2}\left(\frac{c^4\left(\frac{2a^3k}{c^2}-\frac{4a^2b^2k}{c^3}+\frac{ab^4k}{c^4}-b\left(\frac{ai}{c}+e\right)+2ag\right)-x^2\left(-5a^2bc^2k+5ab^3ck-c^4(2ai+bg)+b^5(-k)+b^2c^3i+2c^5e\right)}{2c^4\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right.$$

$$\int \frac{jx^8+hx^4+fx^2+d}{(cx^4+bx^2+a)^3}dx$$

3.59. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^8+kx^{11}}{(a+bx^2+cx^4)^3}\,dx$

$$\Big\downarrow \ 1083$$

$$\frac{1}{2}\left(\frac{c^4\left(\frac{2a^3k}{c^2}-\frac{4a^2b^2k}{c^3}+\frac{ab^4k}{c^4}-b\left(\frac{ai}{c}+e\right)+2ag\right)-x^2\left(-5a^2bc^2k+5ab^3ck-c^4(2ai+bg)+b^5(-k)+b^2c^3i+2c^5e\right)}{2c^4\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right.$$

$$\int \frac{jx^8+hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^3}dx$$

$$\Big\downarrow \ 219$$

$$\frac{1}{2}\left(\frac{c^4\left(\frac{2a^3k}{c^2}-\frac{4a^2b^2k}{c^3}+\frac{ab^4k}{c^4}-b\left(\frac{ai}{c}+e\right)+2ag\right)-x^2\left(-5a^2bc^2k+5ab^3ck-c^4(2ai+bg)+b^5(-k)+b^2c^3i+2c^5e\right)}{2c^4\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right.$$

$$\int \frac{jx^8+hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^3}dx$$

$$\Big\downarrow \ 1103$$

$$\int \frac{jx^8+hx^4+fx^2+d}{\left(cx^4+bx^2+a\right)^3}dx+$$

$$\frac{1}{2}\left(\frac{c^4\left(\frac{2a^3k}{c^2}-\frac{4a^2b^2k}{c^3}+\frac{ab^4k}{c^4}-b\left(\frac{ai}{c}+e\right)+2ag\right)-x^2\left(-5a^2bc^2k+5ab^3ck-c^4(2ai+bg)+b^5(-k)+b^2c^3i+2c^5e\right)}{2c^4\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right.$$

$$\Big\downarrow \ 2206$$

3.59. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^8+kx^{11}}{\left(a+bx^2+cx^4\right)^3}\,dx$

$$-\frac{\int -\frac{-4a\left(4a-\frac{b^2}{c}\right)jx^4-\frac{\left(-ajb^3-c\left(ja^2+5cha+5c^2d\right)b+10ac^3f\right)x^2}{c^2}+abf+b^2\left(3d-\frac{a^2j}{c^2}\right)-2a\left(-\frac{ja^2}{c}+ha+7cd\right)}{(cx^4+bx^2+a)^2}dx}{4a\left(b^2-4ac\right)}-$$

$$\frac{x\left(x^2\left(-bc\left(-3a^2j+ach+c^2d\right)-ab^3j+2ac^3f\right)+c^2\left(-\left(b^2\left(\frac{a^2j}{c^2}+d\right)\right)+2a\left(\frac{a^2j}{c}-ah+cd\right)+abf\right)\right)}{4ac^2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}+$$

$$\frac{1}{2}\left(\frac{c^4\left(\frac{2a^3k}{c^2}-\frac{4a^2b^2k}{c^3}+\frac{ab^4k}{c^4}-b\left(\frac{ai}{c}+e\right)+2ag\right)-x^2\left(-5a^2bc^2k+5ab^3ck-c^4(2ai+bg)+b^5(-k)+b^2c^3i+2c^5e\right)}{2c^4\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right.$$

$\downarrow$ 25

$$\frac{\int \frac{-4a\left(4a-\frac{b^2}{c}\right)jx^4-\frac{\left(-ajb^3-c\left(ja^2+5cha+5c^2d\right)b+10ac^3f\right)x^2}{c^2}+abf+b^2\left(3d-\frac{a^2j}{c^2}\right)-2a\left(-\frac{ja^2}{c}+ha+7cd\right)}{(cx^4+bx^2+a)^2}dx}{4a\left(b^2-4ac\right)}-$$

$$\frac{x\left(x^2\left(-bc\left(-3a^2j+ach+c^2d\right)-ab^3j+2ac^3f\right)+c^2\left(-\left(b^2\left(\frac{a^2j}{c^2}+d\right)\right)+2a\left(\frac{a^2j}{c}-ah+cd\right)+abf\right)\right)}{4ac^2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}+$$

$$\frac{1}{2}\left(\frac{c^4\left(\frac{2a^3k}{c^2}-\frac{4a^2b^2k}{c^3}+\frac{ab^4k}{c^4}-b\left(\frac{ai}{c}+e\right)+2ag\right)-x^2\left(-5a^2bc^2k+5ab^3ck-c^4(2ai+bg)+b^5(-k)+b^2c^3i+2c^5e\right)}{2c^4\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right.$$

$\downarrow$ 2206

$$-\frac{x\left(\left(-\left(\left(\frac{ja^2}{c^2}+d\right)b^2\right)+afb+2a\left(\frac{ja^2}{c}-ha+cd\right)\right)c^2+\left(-ajb^3-c\left(-3ja^2+cha+c^2d\right)b+2ac^3f\right)x^2\right)}{4ac^2\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}+$$

$$\frac{1}{2}\left(\frac{c^4\left(\frac{akb^4}{c^4}-\frac{4a^2kb^2}{c^3}-\left(e+\frac{ai}{c}\right)b+2ag+\frac{2a^3k}{c^2}\right)-\left(-kb^5+5ackb^3+c^3ib^2-5a^2c^2kb+2c^5e-c^4(bg+2ai)\right)x^2}{2c^4\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}-\right.$$

$$\frac{\frac{x\left(\left(\left(ja^2+3c^2d\right)b^3+ac^2fb^2-4ac\left(4ja^2+3cha+6c^2d\right)b+20a^2c^3f\right)x^2+c\left(\left(3d-\frac{2a^2j}{c^2}\right)b^4+afb^3-a\left(-\frac{11ja^2}{c}+7ha+25cd\right)b^2+8a^2cfb+4a^2\left(-9ja^2+c\right.\right.\right.}{2ac\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)}}{4a}$$

$\downarrow$ 25

3.59. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^8+kx^{11}}{\left(a+bx^2+cx^4\right)^3}dx$

$$-\frac{x\left(\left(-\left(\left(\frac{ja^2}{c^2}+d\right)b^2\right)+afb+2a\left(\frac{ja^2}{c}-ha+cd\right)\right)c^2+\left(-ajb^3-c\left(-3ja^2+cha+c^2d\right)b+2ac^3f\right)x^2\right)}{4ac^2\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}+$$

$$\frac{1}{2}\left(\frac{c^4\left(\frac{akb^4}{c^4}-\frac{4a^2kb^2}{c^3}-\left(e+\frac{ai}{c}\right)b+2ag+\frac{2a^3k}{c^2}\right)-\left(-kb^5+5ackb^3+c^3ib^2-5a^2c^2kb+2c^5e-c^4(bg+2ai)\right)x^2}{2c^4\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}-\right.$$

$$\frac{\frac{x\left(\left(\left(ja^2+3c^2d\right)b^3+ac^2fb^2-4ac\left(4ja^2+3cha+6c^2d\right)b+20a^2c^3f\right)x^2+c\left(\left(3d-\frac{2a^2j}{c^2}\right)b^4+afb^3-a\left(-\frac{11ja^2}{c}+7ha+25cd\right)b^2+8a^2cfb+4a^2\left(-9ja^2+c\ldots\right.\right.\right.}{2ac\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)}}{4a\left(\ldots\right.}$$

$$\Big\downarrow \; 27$$

$$-\frac{x\left(\left(-\left(\left(\frac{ja^2}{c^2}+d\right)b^2\right)+afb+2a\left(\frac{ja^2}{c}-ha+cd\right)\right)c^2+\left(-ajb^3-c\left(-3ja^2+cha+c^2d\right)b+2ac^3f\right)x^2\right)}{4ac^2\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}+$$

$$\frac{1}{2}\left(\frac{c^4\left(\frac{akb^4}{c^4}-\frac{4a^2kb^2}{c^3}-\left(e+\frac{ai}{c}\right)b+2ag+\frac{2a^3k}{c^2}\right)-\left(-kb^5+5ackb^3+c^3ib^2-5a^2c^2kb+2c^5e-c^4(bg+2ai)\right)x^2}{2c^4\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}-\right.$$

$$\frac{\frac{x\left(\left(\left(ja^2+3c^2d\right)b^3+ac^2fb^2-4ac\left(4ja^2+3cha+6c^2d\right)b+20a^2c^3f\right)x^2+c\left(\left(3d-\frac{2a^2j}{c^2}\right)b^4+afb^3-a\left(-\frac{11ja^2}{c}+7ha+25cd\right)b^2+8a^2cfb+4a^2\left(-9ja^2+c\ldots\right.\right.\right.}{2ac\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)}}{4a\left(\ldots\right.}$$

$$\Big\downarrow \; 1480$$

$$-\frac{x\left(\left(-\left(\left(\frac{ja^2}{c^2}+d\right)b^2\right)+afb+2a\left(\frac{ja^2}{c}-ha+cd\right)\right)c^2+\left(-ajb^3-c\left(-3ja^2+cha+c^2d\right)b+2ac^3f\right)x^2\right)}{4ac^2\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}+$$

$$\frac{1}{2}\left(\frac{c^4\left(\frac{akb^4}{c^4}-\frac{4a^2kb^2}{c^3}-\left(e+\frac{ai}{c}\right)b+2ag+\frac{2a^3k}{c^2}\right)-\left(-kb^5+5ackb^3+c^3ib^2-5a^2c^2kb+2c^5e-c^4(bg+2ai)\right)x^2}{2c^4\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}-\right.$$

$$\frac{\frac{x\left(\left(\left(ja^2+3c^2d\right)b^3+ac^2fb^2-4ac\left(4ja^2+3cha+6c^2d\right)b+20a^2c^3f\right)x^2+c\left(\left(3d-\frac{2a^2j}{c^2}\right)b^4+afb^3-a\left(-\frac{11ja^2}{c}+7ha+25cd\right)b^2+8a^2cfb+4a^2\left(-9ja^2+c\ldots\right.\right.\right.}{2ac\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)}}{\ldots}$$

3.59. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^8+kx^{11}}{\left(a+bx^2+cx^4\right)^3}\,dx$

$\Big\downarrow$ 218

$$-\frac{x\left(\left(-\left(\left(\frac{ja^2}{c^2}+d\right)b^2\right)+afb+2a\left(\frac{ja^2}{c}-ha+cd\right)\right)c^2+\left(-ajb^3-c\left(-3ja^2+cha+c^2d\right)b+2ac^3f\right)x^2\right)}{4ac^2\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}+$$

$$\frac{x\left(\left(\left(ja^2+3c^2d\right)b^3+ac^2fb^2-4ac\left(4ja^2+3cha+6c^2d\right)b+20a^2c^3f\right)x^2+c\left(\left(3d-\frac{2a^2j}{c^2}\right)b^4+afb^3-a\left(-\frac{11ja^2}{c}+7ha+25cd\right)b^2+8a^2cfb+4a^2\left(-9ja^2+c\right.\right.\right.}{2ac\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)}$$

$$\frac{1}{2}\left(\frac{c^4\left(\frac{akb^4}{c^4}-\frac{4a^2kb^2}{c^3}-\left(e+\frac{ai}{c}\right)b+2ag+\frac{2a^3k}{c^2}\right)-\left(-kb^5+5ackb^3+c^3ib^2-5a^2c^2kb+2c^5e-c^4(bg+2ai)\right)x^2}{2c^4\left(b^2-4ac\right)\left(cx^4+bx^2+a\right)^2}-\right.$$

input
```
Int[(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5 + j*x^8 + k*x^11)/(a + b*x^2
+ c*x^4)^3,x]
```

output
```
-1/4*(x*(c^2*(a*b*f - b^2*(d + (a^2*j)/c^2) + 2*a*(c*d - a*h + (a^2*j)/c))
 + (2*a*c^3*f - a*b^3*j - b*c*(c^2*d + a*c*h - 3*a^2*j))*x^2))/(a*c^2*(b^2
 - 4*a*c)*(a + b*x^2 + c*x^4)^2) + ((x*(c*(a*b^3*f + 8*a^2*b*c*f + 4*a^2*(
7*c^2*d + a*c*h - 9*a^2*j) + b^4*(3*d - (2*a^2*j)/c^2) - a*b^2*(25*c*d + 7
*a*h - (11*a^2*j)/c)) + (a*b^2*c^2*f + 20*a^2*c^3*f + b^3*(3*c^2*d + a^2*j
) - 4*a*b*c*(6*c^2*d + 3*a*c*h + 4*a^2*j))*x^2))/(2*a*c*(b^2 - 4*a*c)*(a +
 b*x^2 + c*x^4)) + (((a*b^2*c^2*f + 20*a^2*c^3*f + b^3*(3*c^2*d + a^2*j) -
 4*a*b*c*(6*c^2*d + 3*a*c*h + 4*a^2*j) + (a*b^3*c^2*f - 52*a^2*b*c^3*f - 6
*a*b^2*c*(5*c^2*d - 3*a*c*h - 3*a^2*j) + b^4*(3*c^2*d - a^2*j) + 8*a^2*c^2
*(21*c^2*d + 3*a*c*h + 5*a^2*j))/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c
]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[c]*Sqrt[b - Sqrt[b^2 - 4*
a*c]]) + ((a*b^2*c^2*f + 20*a^2*c^3*f + b^3*(3*c^2*d + a^2*j) - 4*a*b*c*(6
*c^2*d + 3*a*c*h + 4*a^2*j) - (a*b^3*c^2*f - 52*a^2*b*c^3*f - 6*a*b^2*c*(5
*c^2*d - 3*a*c*h - 3*a^2*j) + b^4*(3*c^2*d - a^2*j) + 8*a^2*c^2*(21*c^2*d
+ 3*a*c*h + 5*a^2*j))/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b
 + Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[c]*Sqrt[b + Sqrt[b^2 - 4*a*c]]))/(2*
a*c*(b^2 - 4*a*c)))/(4*a*(b^2 - 4*a*c)) + ((c^4*(2*a*g - b*(e + (a*i)/c) +
 (a*b^4*k)/c^4 - (4*a^2*b^2*k)/c^3 + (2*a^3*k)/c^2) - (2*c^5*e + b^2*c^3*i
 - c^4*(b*g + 2*a*i) - b^5*k + 5*a*b^3*c*k - 5*a^2*b*c^2*k)*x^2)/(2*c^4*(b
^2 - 4*a*c)*(a + b*x^2 + c*x^4)^2) - (-((c^3*((b^3*i)/c + 2*b*(3*c*e + ...
```

3.59. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^8+kx^{11}}{(a+bx^2+cx^4)^3}\,dx$

3.59.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1103
```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

rule 1142
```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[(2*c*d - b*e)/(2*c)   Int[1/(a + b*x + c*x^2), x], x] + Simp[e/(2*c)
Int[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

3.59. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^8+kx^{11}}{(a+bx^2+cx^4)^3} \, dx$

rule 2191
```
Int[(Pq_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> With[{Q =
PolynomialQuotient[Pq, a + b*x + c*x^2, x], f = Coeff[PolynomialRemainder[P
q, a + b*x + c*x^2, x], x, 0], g = Coeff[PolynomialRemainder[Pq, a + b*x +
c*x^2, x], x, 1]}, Simp[(b*f - 2*a*g + (2*c*f - b*g)*x)*((a + b*x + c*x^2)^
(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] + Simp[1/((p + 1)*(b^2 - 4*a*c))   Int
[(a + b*x + c*x^2)^(p + 1)*ExpandToSum[(p + 1)*(b^2 - 4*a*c)*Q - (2*p + 3)*
(2*c*f - b*g), x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Pq, x] && NeQ[b^
2 - 4*a*c, 0] && LtQ[p, -1]
```

rule 2194
```
Int[(Pq_)*(x_)^(m_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :
> Simp[1/2   Subst[Int[x^((m - 1)/2)*SubstFor[x^2, Pq, x]*(a + b*x + c*x^2)
^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pq, x^2] && IntegerQ
[(m - 1)/2]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

rule 2206
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> With[{d =
 Coeff[PolynomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 0], e = Coeff[Poly
nomialRemainder[Px, a + b*x^2 + c*x^4, x], x, 2]}, Simp[x*(a + b*x^2 + c*x^
4)^(p + 1)*((a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)/(2*a*(p + 1)*(b
^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2 - 4*a*c))   Int[(a + b*x^2 + c
*x^4)^(p + 1)*ExpandToSum[2*a*(p + 1)*(b^2 - 4*a*c)*PolynomialQuotient[Px,
a + b*x^2 + c*x^4, x] + b^2*d*(2*p + 3) - 2*a*c*d*(4*p + 5) - a*b*e + c*(4*
p + 7)*(b*d - 2*a*e)*x^2, x], x], x]] /; FreeQ[{a, b, c}, x] && PolyQ[Px, x
^2] && Expon[Px, x^2] > 1 && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1]
```

3.59. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^8+kx^{11}}{(a+bx^2+cx^4)^3}\,dx$

3.59.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 1.14 (sec) , antiderivative size = 1182, normalized size of antiderivative = 1.00

method	result	size
risch	Expression too large to display	1182
default	Expression too large to display	2058

input
```
int((k*x^11+j*x^8+i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x,metho
d=_RETURNVERBOSE)
```

output
```
(-1/8*(16*a^3*b*c*j-a^2*b^3*j+12*a^2*b*c^2*h-20*a^2*c^3*f-a*b^2*c^2*f+24*a
*b*c^3*d-3*b^3*c^2*d)/a^2/(16*a^2*c^2-8*a*b^2*c+b^4)*x^7+1/2*(25*a^2*b*c^2
*k-15*a*b^3*c*k+2*a*c^4*i+2*b^5*k+b^2*c^3*i-3*b*c^4*g+6*c^5*e)/c^2/(16*a^2
*c^2-8*a*b^2*c+b^4)*x^6-1/8/a^2*(36*a^4*c^2*j+5*a^3*b^2*c*j-4*a^3*c^3*h+a^
2*b^4*j+19*a^2*b^2*c^2*h-28*a^2*b*c^3*f-28*a^2*c^4*d-2*a*b^3*c^2*f+49*a*b^
2*c^3*d-6*b^4*c^2*d)/(16*a^2*c^2-8*a*b^2*c+b^4)/c*x^5+1/4*(32*a^3*c^3*k+11
*a^2*b^2*c^2*k-19*a*b^4*c*k+6*a*b*c^4*i+3*b^6*k+3*b^3*c^3*i-9*b^2*c^4*g+18
*b*c^5*e)/(16*a^2*c^2-8*a*b^2*c+b^4)/c^3*x^4-1/8/c*(28*a^4*b*c*j+2*a^3*b^3
*j+16*a^3*b*c^2*h-36*a^3*c^3*f+5*a^2*b^3*c*h-5*a^2*b^2*c^2*f+4*a^2*b*c^3*d
-a*b^4*c*f+20*a*b^3*c^2*d-3*b^5*c*d)/a^2/(16*a^2*c^2-8*a*b^2*c+b^4)*x^3+1/
2/c^3*(31*a^3*b*c^2*k-22*a^2*b^3*c*k-2*a^2*c^4*i+3*a*b^5*k+5*a*b^2*c^3*i-5
*a*b*c^4*g+10*a*c^5*e-b^3*c^3*g+2*b^2*c^4*e)/(16*a^2*c^2-8*a*b^2*c+b^4)*x^
2-1/8*(20*a^4*c*j+a^3*b^2*j+12*a^3*c^2*h+3*a^2*b^2*c*h-16*a^2*b*c^2*f-44*a
^2*c^3*d+a*b^3*c*f+37*a*b^2*c^2*d-5*b^4*c*d)/c/(16*a^2*c^2-8*a*b^2*c+b^4)/
a*x+1/4*(24*a^4*c^2*k-21*a^3*b^2*c*k+3*a^2*b^4*k+6*a^2*b*c^3*i-8*a^2*c^4*g
-a*b^2*c^3*g+10*a*b*c^4*e-b^3*c^3*e)/(16*a^2*c^2-8*a*b^2*c+b^4)/c^3)/(c*x^
4+b*x^2+a)^2+1/16/c*sum((8/c*k*_R^3-1/a^2*(16*a^3*b*c*j-a^2*b^3*j+12*a^2*b
*c^2*h-20*a^2*c^3*f-a*b^2*c^2*f+24*a*b*c^3*d-3*b^3*c^2*d)/(16*a^2*c^2-8*a*
b^2*c+b^4)*_R^2-8/c*(7*a^2*b*c*k-a*b^3*k-2*a*c^3*i-b^2*c^2*i+3*b*c^3*g-6*c
^4*e)/(16*a^2*c^2-8*a*b^2*c+b^4)*_R+1/a^2*(20*a^4*c*j+a^3*b^2*j+12*a^3*...
```

3.59.5 Fricas [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^8+kx^{11}}{(a+bx^2+cx^4)^3}\,dx = \text{Timed out}$$

input
```
integrate((k*x^11+j*x^8+i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x
, algorithm="fricas")
```

output
```
Timed out
```

3.59.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^8+kx^{11}}{(a+bx^2+cx^4)^3}\,dx = \text{Timed out}$$

input
```
integrate((k*x**11+j*x**8+i*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(c*x**4+b*x**
2+a)**3,x)
```

output
```
Timed out
```

3.59.7 Maxima [F]

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^8+kx^{11}}{(a+bx^2+cx^4)^3}\,dx$$
$$= \int \frac{kx^{11}+jx^8+ix^5+hx^4+gx^3+fx^2+ex+d}{(cx^4+bx^2+a)^3}\,dx$$

input
```
integrate((k*x^11+j*x^8+i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x
, algorithm="maxima")
```

output
```
1/8*(12*a^4*b*c^3*i - (12*a^2*b*c^5*h - 3*(b^3*c^5 - 8*a*b*c^6)*d - (a*b^2
*c^5 + 20*a^2*c^6)*f - (a^2*b^3*c^3 - 16*a^3*b*c^4)*j)*x^7 + 4*(6*a^2*c^6*
e - 3*a^2*b*c^5*g + (a^2*b^2*c^4 + 2*a^3*c^5)*i + (2*a^2*b^5*c - 15*a^3*b^
3*c^2 + 25*a^4*b*c^3)*k)*x^6 + ((6*b^4*c^4 - 49*a*b^2*c^5 + 28*a^2*c^6)*d
+ 2*(a*b^3*c^4 + 14*a^2*b*c^5)*f - (19*a^2*b^2*c^4 - 4*a^3*c^5)*h - (a^2*b
^4*c^2 + 5*a^3*b^2*c^3 + 36*a^4*c^4)*j)*x^5 + 2*(18*a^2*b*c^5*e - 9*a^2*b^
2*c^4*g + 3*(a^2*b^3*c^3 + 2*a^3*b*c^4)*i + (3*a^2*b^6 - 19*a^3*b^4*c + 11
*a^4*b^2*c^2 + 32*a^5*c^3)*k)*x^4 + ((3*b^5*c^3 - 20*a*b^3*c^4 - 4*a^2*b*c
^5)*d + (a*b^4*c^3 + 5*a^2*b^2*c^4 + 36*a^3*c^5)*f - (5*a^2*b^3*c^3 + 16*a
^3*b*c^4)*h - 2*(a^3*b^3*c^2 + 14*a^4*b*c^3)*j)*x^3 + 4*(2*(a^2*b^2*c^4 +
5*a^3*c^5)*e - (a^2*b^3*c^3 + 5*a^3*b*c^4)*g + (5*a^3*b^2*c^3 - 2*a^4*c^4)
*i + (3*a^3*b^5 - 22*a^4*b^3*c + 31*a^5*b*c^2)*k)*x^2 - 2*(a^2*b^3*c^3 - 1
0*a^3*b*c^4)*e - 2*(a^3*b^2*c^3 + 8*a^4*c^4)*g + 6*(a^4*b^4 - 7*a^5*b^2*c
+ 8*a^6*c^2)*k + ((5*a*b^4*c^3 - 37*a^2*b^2*c^4 + 44*a^3*c^5)*d - (a^2*b^3
*c^3 - 16*a^3*b*c^4)*f - 3*(a^3*b^2*c^3 + 4*a^4*c^4)*h - (a^4*b^2*c^2 + 20
*a^5*c^3)*j)*x)/(a^4*b^4*c^3 - 8*a^5*b^2*c^4 + 16*a^6*c^5 + (a^2*b^4*c^5 -
 8*a^3*b^2*c^6 + 16*a^4*c^7)*x^8 + 2*(a^2*b^5*c^4 - 8*a^3*b^3*c^5 + 16*a^4
*b*c^6)*x^6 + (a^2*b^6*c^3 - 6*a^3*b^4*c^4 + 32*a^5*c^6)*x^4 + 2*(a^3*b^5*
c^3 - 8*a^4*b^3*c^4 + 16*a^5*b*c^5)*x^2) + 1/8*integrate((8*(a^2*b^4 - 8*a
^3*b^2*c + 16*a^4*c^2)*k*x^3 - (12*a^2*b*c^3*h - 3*(b^3*c^3 - 8*a*b*c^4...
```

3.59.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 29142 vs. 2(1123) = 2246.

Time = 4.87 (sec) , antiderivative size = 29142, normalized size of antiderivative = 24.76

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^8+kx^{11}}{(a+bx^2+cx^4)^3}\,dx = \text{Too large to display}$$

input
```
integrate((k*x^11+j*x^8+i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^3,x
, algorithm="giac")
```

output
```
1/64*(3*(a^4*b^8*c^5 - 16*a^5*b^6*c^6 + 96*a^6*b^4*c^7 - 256*a^7*b^2*c^8 +
 256*a^8*c^9)^2*(2*b^5*c^4 - 24*a*b^3*c^5 + 64*a^2*b*c^6 - sqrt(2)*sqrt(b^
2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^5*c^2 + 12*sqrt(2)*sqrt(b^2 -
 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^3*c^3 + 2*sqrt(2)*sqrt(b^2 - 4
*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^4*c^3 - 32*sqrt(2)*sqrt(b^2 - 4*a*
c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b*c^4 - 16*sqrt(2)*sqrt(b^2 - 4*a*c
)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^2*c^4 - sqrt(2)*sqrt(b^2 - 4*a*c)*sq
rt(b*c + sqrt(b^2 - 4*a*c)*c)*b^3*c^4 + 8*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b
*c + sqrt(b^2 - 4*a*c)*c)*a*b*c^5 - 2*(b^2 - 4*a*c)*b^3*c^4 + 16*(b^2 - 4*
a*c)*a*b*c^5)*d + (a^4*b^8*c^5 - 16*a^5*b^6*c^6 + 96*a^6*b^4*c^7 - 256*a^7
*b^2*c^8 + 256*a^8*c^9)^2*(2*a*b^4*c^4 + 32*a^2*b^2*c^5 - 160*a^3*c^6 - sq
rt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^4*c^2 - 16*sqr
t(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^2*c^3 + 2*sqr
t(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^3*c^3 + 80*sqrt
(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*c^4 + 40*sqrt(2)
*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b*c^4 - sqrt(2)*sqr
t(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^2*c^4 - 20*sqrt(2)*sqrt
(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*c^5 - 2*(b^2 - 4*a*c)*a*
b^2*c^4 - 40*(b^2 - 4*a*c)*a^2*c^5)*f - 12*(a^4*b^8*c^5 - 16*a^5*b^6*c^6 +
 96*a^6*b^4*c^7 - 256*a^7*b^2*c^8 + 256*a^8*c^9)^2*(2*a^2*b^3*c^4 - 8*a...
```

3.59.9 Mupad [B] (verification not implemented)

Time = 27.14 (sec) , antiderivative size = 97905, normalized size of antiderivative = 83.18

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^8+kx^{11}}{(a+bx^2+cx^4)^3}\,dx = \text{Too large to display}$$

input
```
int((d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5 + j*x^8 + k*x^11)/(a + b*x^2
+ c*x^4)^3,x)
```

output
```
((x^7*(3*b^3*c^2*d + 20*a^2*c^3*f + a^2*b^3*j - 24*a*b*c^3*d - 16*a^3*b*c*
j + a*b^2*c^2*f - 12*a^2*b*c^2*h))/(8*a^2*(b^4 + 16*a^2*c^2 - 8*a*b^2*c))
- (b^3*c^3*e + 8*a^2*c^4*g - 3*a^2*b^4*k - 24*a^4*c^2*k - 10*a*b*c^4*e + a
*b^2*c^3*g - 6*a^2*b*c^3*i + 21*a^3*b^2*c*k)/(4*c^3*(b^4 + 16*a^2*c^2 - 8*
a*b^2*c)) + (x^4*(3*b^6*k - 9*b^2*c^4*g + 3*b^3*c^3*i + 32*a^3*c^3*k + 18*
b*c^5*e + 11*a^2*b^2*c^2*k + 6*a*b*c^4*i - 19*a*b^4*c*k))/(4*c^3*(b^4 + 16
*a^2*c^2 - 8*a*b^2*c)) + (x^2*(2*b^2*c^4*e - b^3*c^3*g - 2*a^2*c^4*i + 10*
a*c^5*e + 3*a*b^5*k - 5*a*b*c^4*g + 5*a*b^2*c^3*i - 22*a^2*b^3*c*k + 31*a^
3*b*c^2*k))/(2*c^3*(b^4 + 16*a^2*c^2 - 8*a*b^2*c)) + (x^6*(6*c^5*e + 2*b^5
*k + b^2*c^3*i - 3*b*c^4*g + 2*a*c^4*i - 15*a*b^3*c*k + 25*a^2*b*c^2*k))/(
2*c^2*(b^4 + 16*a^2*c^2 - 8*a*b^2*c)) - (x^3*(2*a^3*b^3*j - 36*a^3*c^3*f -
 3*b^5*c*d - 5*a^2*b^2*c^2*f - a*b^4*c*f + 28*a^4*b*c*j + 20*a*b^3*c^2*d +
 4*a^2*b*c^3*d + 5*a^2*b^3*c*h + 16*a^3*b*c^2*h))/(8*a^2*c*(b^4 + 16*a^2*c
^2 - 8*a*b^2*c)) + (x^5*(28*a^2*c^4*d + 6*b^4*c^2*d + 4*a^3*c^3*h - a^2*b^
4*j - 36*a^4*c^2*j - 19*a^2*b^2*c^2*h - 49*a*b^2*c^3*d + 2*a*b^3*c^2*f + 2
8*a^2*b*c^3*f - 5*a^3*b^2*c*j))/(8*a^2*c*(b^4 + 16*a^2*c^2 - 8*a*b^2*c)) -
 (x*(12*a^3*c^2*h - 44*a^2*c^3*d + a^3*b^2*j - 5*b^4*c*d + 20*a^4*c*j + a*
b^3*c*f + 37*a*b^2*c^2*d - 16*a^2*b*c^2*f + 3*a^2*b^2*c*h))/(8*a*c*(b^4 +
16*a^2*c^2 - 8*a*b^2*c)))/(x^4*(2*a*c + b^2) + a^2 + c^2*x^8 + 2*a*b*x^2 +
 2*b*c*x^6) + symsum(log((10368*a*b^5*c^10*d^3 - 8000*a^5*c^11*f^3 - 56...
```

3.59. $\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5+jx^8+kx^{11}}{(a+bx^2+cx^4)^3}\,dx$

3.60 $\int (a+bx^2+cx^4)^3 (ad+aex+(bd+af)x^2+bex^3+$

3.60.1 Optimal result

Integrand size = 63, antiderivative size = 416

$$\begin{aligned}
&\int (a+bx^2+cx^4)^3 \left(ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6\right)\,dx \\
&= a^4dx+\frac{1}{2}a^4ex^2+\frac{1}{3}a^3(4bd+af)x^3+a^3bex^4+\frac{2}{5}a^2\left(3b^2d+2acd+2abf\right)x^5 \\
&+\frac{1}{3}a^2\left(3b^2+2ac\right)ex^6+\frac{2}{7}a\left(2b^3d+6abcd+3ab^2f+2a^2cf\right)x^7+\frac{1}{2}ab\left(b^2+3ac\right)ex^8 \\
&+\frac{1}{9}\left(b^4d+12ab^2cd+6a^2c^2d+4ab^3f+12a^2bcf\right)x^9+\frac{1}{10}\left(b^4+12ab^2c+6a^2c^2\right)ex^{10} \\
&+\frac{1}{11}\left(4b^3cd+12abc^2d+b^4f+12ab^2cf+6a^2c^2f\right)x^{11}+\frac{1}{3}bc\left(b^2+3ac\right)ex^{12} \\
&+\frac{2}{13}c\left(3b^2cd+2ac^2d+2b^3f+6abcf\right)x^{13}+\frac{1}{7}c^2\left(3b^2+2ac\right)ex^{14} \\
&+\frac{2}{15}c^2\left(2bcd+3b^2f+2acf\right)x^{15}+\frac{1}{4}bc^3ex^{16}+\frac{1}{17}c^3(cd+4bf)x^{17}+\frac{1}{18}c^4ex^{18}+\frac{1}{19}c^4fx^{19}
\end{aligned}$$

output
```
a^4*d*x+1/2*a^4*e*x^2+1/3*a^3*(a*f+4*b*d)*x^3+a^3*b*e*x^4+2/5*a^2*(2*a*b*f
+2*a*c*d+3*b^2*d)*x^5+1/3*a^2*(2*a*c+3*b^2)*e*x^6+2/7*a*(2*a^2*c*f+3*a*b^2
*f+6*a*b*c*d+2*b^3*d)*x^7+1/2*a*b*(3*a*c+b^2)*e*x^8+1/9*(12*a^2*b*c*f+6*a^
2*c^2*d+4*a*b^3*f+12*a*b^2*c*d+b^4*d)*x^9+1/10*(6*a^2*c^2+12*a*b^2*c+b^4)*
e*x^10+1/11*(6*a^2*c^2*f+12*a*b^2*c*f+12*a*b*c^2*d+b^4*f+4*b^3*c*d)*x^11+1
/3*b*c*(3*a*c+b^2)*e*x^12+2/13*c*(6*a*b*c*f+2*a*c^2*d+2*b^3*f+3*b^2*c*d)*x
^13+1/7*c^2*(2*a*c+3*b^2)*e*x^14+2/15*c^2*(2*a*c*f+3*b^2*f+2*b*c*d)*x^15+1
/4*b*c^3*e*x^16+1/17*c^3*(4*b*f+c*d)*x^17+1/18*c^4*e*x^18+1/19*c^4*f*x^19
```

3.60.2 Mathematica [A] (verified)

Time = 0.08 (sec) , antiderivative size = 416, normalized size of antiderivative = 1.00

$$\int (a+bx^2+cx^4)^3 \left(ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6\right) dx$$
$$= a^4dx+\frac{1}{2}a^4ex^2+\frac{1}{3}a^3(4bd+af)x^3+a^3bex^4+\frac{2}{5}a^2\left(3b^2d+2acd+2abf\right)x^5$$
$$+\frac{1}{3}a^2\left(3b^2+2ac\right)ex^6+\frac{2}{7}a\left(2b^3d+6abcd+3ab^2f+2a^2cf\right)x^7+\frac{1}{2}ab\left(b^2+3ac\right)ex^8$$
$$+\frac{1}{9}\left(b^4d+12ab^2cd+6a^2c^2d+4ab^3f+12a^2bcf\right)x^9+\frac{1}{10}\left(b^4+12ab^2c+6a^2c^2\right)ex^{10}$$
$$+\frac{1}{11}\left(4b^3cd+12abc^2d+b^4f+12ab^2cf+6a^2c^2f\right)x^{11}+\frac{1}{3}bc\left(b^2+3ac\right)ex^{12}$$
$$+\frac{2}{13}c\left(3b^2cd+2ac^2d+2b^3f+6abcf\right)x^{13}+\frac{1}{7}c^2\left(3b^2+2ac\right)ex^{14}$$
$$+\frac{2}{15}c^2\left(2bcd+3b^2f+2acf\right)x^{15}+\frac{1}{4}bc^3ex^{16}+\frac{1}{17}c^3(cd+4bf)x^{17}+\frac{1}{18}c^4ex^{18}+\frac{1}{19}c^4fx^{19}$$

input
```
Integrate[(a + b*x^2 + c*x^4)^3*(a*d + a*e*x + (b*d + a*f)*x^2 + b*e*x^3 +
 (c*d + b*f)*x^4 + c*e*x^5 + c*f*x^6),x]
```

output
```
a^4*d*x + (a^4*e*x^2)/2 + (a^3*(4*b*d + a*f)*x^3)/3 + a^3*b*e*x^4 + (2*a^2
*(3*b^2*d + 2*a*c*d + 2*a*b*f)*x^5)/5 + (a^2*(3*b^2 + 2*a*c)*e*x^6)/3 + (2
*a*(2*b^3*d + 6*a*b*c*d + 3*a*b^2*f + 2*a^2*c*f)*x^7)/7 + (a*b*(b^2 + 3*a*
c)*e*x^8)/2 + ((b^4*d + 12*a*b^2*c*d + 6*a^2*c^2*d + 4*a*b^3*f + 12*a^2*b*
c*f)*x^9)/9 + ((b^4 + 12*a*b^2*c + 6*a^2*c^2)*e*x^10)/10 + ((4*b^3*c*d + 1
2*a*b*c^2*d + b^4*f + 12*a*b^2*c*f + 6*a^2*c^2*f)*x^11)/11 + (b*c*(b^2 + 3
*a*c)*e*x^12)/3 + (2*c*(3*b^2*c*d + 2*a*c^2*d + 2*b^3*f + 6*a*b*c*f)*x^13)
/13 + (c^2*(3*b^2 + 2*a*c)*e*x^14)/7 + (2*c^2*(2*b*c*d + 3*b^2*f + 2*a*c*f
)*x^15)/15 + (b*c^3*e*x^16)/4 + (c^3*(c*d + 4*b*f)*x^17)/17 + (c^4*e*x^18)
/18 + (c^4*f*x^19)/19
```

3.60.3 Rubi [A] (verified)

Time = 0.91 (sec) , antiderivative size = 416, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.032, Rules used = {2200, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

3.60. $\int (a+bx^2+cx^4)^3 \left(ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6\right) dx$

$$\int \left(a+bx^2+cx^4\right)^3 \left(x^2(af+bd)+ad+aex+x^4(bf+cd)+bex^3+cex^5+cfx^6\right) dx$$

$$\Big\downarrow \text{2200}$$

$$\int \left(a^4d+a^4ex+a^3x^2(af+4bd)+4a^3bex^3+2a^2x^4\left(2abf+2acd+3b^2d\right)+2a^2ex^5\left(2ac+3b^2\right)+ex^9\left(6a^2c^2+12\right.\right.$$

$$\Big\downarrow \text{2009}$$

$$a^4dx+\frac{1}{2}a^4ex^2+\frac{1}{3}a^3x^3(af+4bd)+a^3bex^4+\frac{2}{5}a^2x^5\left(2abf+2acd+3b^2d\right)+\frac{1}{3}a^2ex^6\left(2ac+3b^2\right)+\frac{1}{10}ex^{10}\left(6a^2c^2+12ab^2c+b^4\right)+\frac{2}{7}ax^7\left(2a^2cf+3ab^2f+6abcd+2b^3d\right)+\frac{1}{11}x^{11}\left(6a^2c^2f+12ab^2cf+12abc^2d+b^4f+4b^3cd\right)+\frac{1}{9}x^9\left(12a^2bcf+6a^2c^2d+4ab^3f+12ab^2cd+b^4d\right)+\frac{2}{15}c^2x^{15}\left(2acf+3b^2f+2bcd\right)+\frac{1}{7}c^2ex^{14}\left(2ac+3b^2\right)+\frac{1}{3}bcex^{12}\left(3ac+b^2\right)+\frac{1}{2}abex^8\left(3ac+b^2\right)+\frac{2}{13}cx^{13}\left(6abcf+2ac^2d+2b^3f+3b^2cd\right)+\frac{1}{17}c^3x^{17}(4bf+cd)+\frac{1}{4}bc^3ex^{16}+\frac{1}{18}c^4ex^{18}+\frac{1}{19}c^4fx^{19}$$

input
```
Int[(a + b*x^2 + c*x^4)^3*(a*d + a*e*x + (b*d + a*f)*x^2 + b*e*x^3 + (c*d
+ b*f)*x^4 + c*e*x^5 + c*f*x^6),x]
```

output
```
a^4*d*x + (a^4*e*x^2)/2 + (a^3*(4*b*d + a*f)*x^3)/3 + a^3*b*e*x^4 + (2*a^2
*(3*b^2*d + 2*a*c*d + 2*a*b*f)*x^5)/5 + (a^2*(3*b^2 + 2*a*c)*e*x^6)/3 + (2
*a*(2*b^3*d + 6*a*b*c*d + 3*a*b^2*f + 2*a^2*c*f)*x^7)/7 + (a*b*(b^2 + 3*a*
c)*e*x^8)/2 + ((b^4*d + 12*a*b^2*c*d + 6*a^2*c^2*d + 4*a*b^3*f + 12*a^2*b*
c*f)*x^9)/9 + ((b^4 + 12*a*b^2*c + 6*a^2*c^2)*e*x^10)/10 + ((4*b^3*c*d + 1
2*a*b*c^2*d + b^4*f + 12*a*b^2*c*f + 6*a^2*c^2*f)*x^11)/11 + (b*c*(b^2 + 3
*a*c)*e*x^12)/3 + (2*c*(3*b^2*c*d + 2*a*c^2*d + 2*b^3*f + 6*a*b*c*f)*x^13)
/13 + (c^2*(3*b^2 + 2*a*c)*e*x^14)/7 + (2*c^2*(2*b*c*d + 3*b^2*f + 2*a*c*f
)*x^15)/15 + (b*c^3*e*x^16)/4 + (c^3*(c*d + 4*b*f)*x^17)/17 + (c^4*e*x^18)
/18 + (c^4*f*x^19)/19
```

3.60.
$\int \left(a+bx^2+cx^4\right)^3 \left(ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6\right) dx$

3.60.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2200
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_.), x_Symbol] :> Int[Expa
ndIntegrand[Px*(a + b*x^2 + c*x^4)^p, x], x] /; FreeQ[{a, b, c}, x] && Poly
Q[Px, x] && IGtQ[p, 0]
```

3.60.4 Maple [A] (verified)

Time = 28.81 (sec) , antiderivative size = 412, normalized size of antiderivative = 0.99

method	result
norman	$\left(\frac{1}{3}f\,a^4+\frac{4}{3}d\,a^3b\right)x^3+\left(\frac{2}{7}a\,c^3e+\frac{3}{7}b^2c^2e\right)x^{14}+\left(\frac{2}{3}a^3ce+a^2b^2e\right)x^6+\left(\frac{4}{17}b\,c^3f+\frac{1}{17}c^4d\right)x^{17}+$
risch	$\frac{1}{2}a^4e\,x^2+\frac{1}{18}c^4e\,x^{18}+\frac{1}{19}c^4f\,x^{19}+\frac{2}{3}x^6a^3ce+x^6a^2b^2e+\frac{4}{5}x^5f\,a^3b+\frac{4}{5}x^5a^3cd+\frac{6}{5}x^5a^2b^2d+\frac{4}{3}$
parallelrisch	$\frac{1}{2}a^4e\,x^2+\frac{1}{18}c^4e\,x^{18}+\frac{1}{19}c^4f\,x^{19}+\frac{2}{3}x^6a^3ce+x^6a^2b^2e+\frac{4}{5}x^5f\,a^3b+\frac{4}{5}x^5a^3cd+\frac{6}{5}x^5a^2b^2d+\frac{4}{3}$
gosper	$x(3063060f\,c^4x^{18}+3233230c^4e\,x^{17}+13693680b\,c^3f\,x^{16}+3423420c^4d\,x^{16}+145495535b\,c^3e\,x^{15}+15519504a\,c^3f\,x^{14}+23279256b^2$
default	$\frac{c^4f\,x^{19}}{19}+\frac{c^4e\,x^{18}}{18}+\frac{(3b\,c^3f+c^3(bf+cd))x^{17}}{17}+\frac{b\,c^3e\,x^{16}}{4}+\frac{((a\,c^2+2b^2c+c(2ac+b^2))cf+3b\,c^2(bf+cd)+c^3(af+bd))x^{15}}{15}$

input
```
int((c*x^4+b*x^2+a)^3*(d*a+a*e*x+(a*f+b*d)*x^2+e*x^3*b+(b*f+c*d)*x^4+c*e*x
^5+c*f*x^6),x,method=_RETURNVERBOSE)
```

output
```
(1/3*f*a^4+4/3*d*a^3*b)*x^3+(2/7*a*c^3*e+3/7*b^2*c^2*e)*x^14+(2/3*a^3*c*e+
a^2*b^2*e)*x^6+(4/17*b*c^3*f+1/17*c^4*d)*x^17+(a*b*c^2*e+1/3*b^3*c*e)*x^12
+(3/2*a^2*b*c*e+1/2*a*b^3*e)*x^8+(4/15*a*c^3*f+2/5*b^2*c^2*f+4/15*b*c^3*d)
*x^15+(3/5*a^2*c^2*e+6/5*a*b^2*c*e+1/10*b^4*e)*x^10+(4/5*f*a^3*b+4/5*a^3*c
*d+6/5*a^2*b^2*d)*x^5+(4/7*a^3*c*f+6/7*a^2*b^2*f+12/7*a^2*b*c*d+4/7*a*b^3*
d)*x^7+(12/13*a*b*c^2*f+4/13*a*c^3*d+4/13*b^3*c*f+6/13*b^2*c^2*d)*x^13+(6/
11*a^2*c^2*f+12/11*a*b^2*c*f+12/11*a*b*c^2*d+1/11*b^4*f+4/11*b^3*c*d)*x^11
+(4/3*a^2*b*c*f+2/3*a^2*c^2*d+4/9*a*b^3*f+4/3*a*b^2*c*d+1/9*d*b^4)*x^9+a^4
*d*x+a^3*b*e*x^4+1/2*a^4*e*x^2+1/18*c^4*e*x^18+1/19*c^4*f*x^19+1/4*b*c^3*e
*x^16
```

3.60.5 Fricas [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 418, normalized size of antiderivative = 1.00

$$\int (a+bx^2+cx^4)^3 (ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6) \, dx$$
$$= \frac{1}{19}c^4fx^{19}+\frac{1}{18}c^4ex^{18}+\frac{1}{4}bc^3ex^{16}+\frac{1}{17}(c^4d+4bc^3f)x^{17}+\frac{1}{7}(3b^2c^2+2ac^3)ex^{14}$$
$$+\frac{2}{15}(2bc^3d+(3b^2c^2+2ac^3)f)x^{15}+\frac{1}{3}(b^3c+3abc^2)ex^{12}$$
$$+\frac{2}{13}((3b^2c^2+2ac^3)d+2(b^3c+3abc^2)f)x^{13}+\frac{1}{10}(b^4+12ab^2c+6a^2c^2)ex^{10}$$
$$+\frac{1}{11}(4(b^3c+3abc^2)d+(b^4+12ab^2c+6a^2c^2)f)x^{11}+\frac{1}{2}(ab^3+3a^2bc)ex^8$$
$$+\frac{1}{9}((b^4+12ab^2c+6a^2c^2)d+4(ab^3+3a^2bc)f)x^9+a^3bex^4$$
$$+\frac{1}{3}(3a^2b^2+2a^3c)ex^6+\frac{2}{7}(2(ab^3+3a^2bc)d+(3a^2b^2+2a^3c)f)x^7$$
$$+\frac{1}{2}a^4ex^2+a^4dx+\frac{2}{5}(2a^3bf+(3a^2b^2+2a^3c)d)x^5+\frac{1}{3}(4a^3bd+a^4f)x^3$$

input
```
integrate((c*x^4+b*x^2+a)^3*(a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4
+c*e*x^5+c*f*x^6),x, algorithm="fricas")
```

output
```
1/19*c^4*f*x^19 + 1/18*c^4*e*x^18 + 1/4*b*c^3*e*x^16 + 1/17*(c^4*d + 4*b*c
^3*f)*x^17 + 1/7*(3*b^2*c^2 + 2*a*c^3)*e*x^14 + 2/15*(2*b*c^3*d + (3*b^2*c
^2 + 2*a*c^3)*f)*x^15 + 1/3*(b^3*c + 3*a*b*c^2)*e*x^12 + 2/13*((3*b^2*c^2
+ 2*a*c^3)*d + 2*(b^3*c + 3*a*b*c^2)*f)*x^13 + 1/10*(b^4 + 12*a*b^2*c + 6*
a^2*c^2)*e*x^10 + 1/11*(4*(b^3*c + 3*a*b*c^2)*d + (b^4 + 12*a*b^2*c + 6*a^
2*c^2)*f)*x^11 + 1/2*(a*b^3 + 3*a^2*b*c)*e*x^8 + 1/9*((b^4 + 12*a*b^2*c +
6*a^2*c^2)*d + 4*(a*b^3 + 3*a^2*b*c)*f)*x^9 + a^3*b*e*x^4 + 1/3*(3*a^2*b^2
 + 2*a^3*c)*e*x^6 + 2/7*(2*(a*b^3 + 3*a^2*b*c)*d + (3*a^2*b^2 + 2*a^3*c)*f
)*x^7 + 1/2*a^4*e*x^2 + a^4*d*x + 2/5*(2*a^3*b*f + (3*a^2*b^2 + 2*a^3*c)*d
)*x^5 + 1/3*(4*a^3*b*d + a^4*f)*x^3
```

3.60.6 Sympy [A] (verification not implemented)

Time = 0.05 (sec) , antiderivative size = 503, normalized size of antiderivative = 1.21

$$\int \left(a+bx^2+cx^4\right)^3 \left(ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6\right)\,dx$$
$$= a^4dx + \frac{a^4ex^2}{2} + a^3bex^4 + \frac{bc^3ex^{16}}{4} + \frac{c^4ex^{18}}{18} + \frac{c^4fx^{19}}{19} + x^{17}\cdot\left(\frac{4bc^3f}{17} + \frac{c^4d}{17}\right) + x^{15}\cdot\left(\frac{4ac^3f}{15} + \frac{2b^2c^2f}{5} + \frac{4bc^3d}{15}\right) + x^{14}\cdot\left(\frac{2ac^3e}{7} + \frac{3b^2c^2e}{7}\right) + x^{13}\cdot\left(\frac{12abc^2f}{13} + \frac{4ac^3d}{13} + \frac{4b^3cf}{13} + \frac{6b^2c^2d}{13}\right) + x^{12}\left(abc^2e + \frac{b^3ce}{3}\right) + x^{11}\cdot\left(\frac{6a^2c^2f}{11} + \frac{12ab^2cf}{11} + \frac{12abc^2d}{11} + \frac{b^4f}{11} + \frac{4b^3cd}{11}\right) + x^{10}\cdot\left(\frac{3a^2c^2e}{5} + \frac{6ab^2ce}{5} + \frac{b^4e}{10}\right) + x^9\cdot\left(\frac{4a^2bcf}{3} + \frac{2a^2c^2d}{3} + \frac{4ab^3f}{9} + \frac{4ab^2cd}{3} + \frac{b^4d}{9}\right) + x^8\cdot\left(\frac{3a^2bce}{2} + \frac{ab^3e}{2}\right) + x^7\cdot\left(\frac{4a^3cf}{7} + \frac{6a^2b^2f}{7} + \frac{12a^2bcd}{7} + \frac{4ab^3d}{7}\right) + x^6\cdot\left(\frac{2a^3ce}{3} + a^2b^2e\right) + x^5\cdot\left(\frac{4a^3bf}{5} + \frac{4a^3cd}{5} + \frac{6a^2b^2d}{5}\right) + x^3\left(\frac{a^4f}{3} + \frac{4a^3bd}{3}\right)$$

input
```
integrate((c*x**4+b*x**2+a)**3*(a*d+a*e*x+(a*f+b*d)*x**2+b*e*x**3+(b*f+c*d
)*x**4+c*e*x**5+c*f*x**6),x)
```

output
```
a**4*d*x + a**4*e*x**2/2 + a**3*b*e*x**4 + b*c**3*e*x**16/4 + c**4*e*x**18
/18 + c**4*f*x**19/19 + x**17*(4*b*c**3*f/17 + c**4*d/17) + x**15*(4*a*c**
3*f/15 + 2*b**2*c**2*f/5 + 4*b*c**3*d/15) + x**14*(2*a*c**3*e/7 + 3*b**2*c
**2*e/7) + x**13*(12*a*b*c**2*f/13 + 4*a*c**3*d/13 + 4*b**3*c*f/13 + 6*b**
2*c**2*d/13) + x**12*(a*b*c**2*e + b**3*c*e/3) + x**11*(6*a**2*c**2*f/11 +
 12*a*b**2*c*f/11 + 12*a*b*c**2*d/11 + b**4*f/11 + 4*b**3*c*d/11) + x**10*
(3*a**2*c**2*e/5 + 6*a*b**2*c*e/5 + b**4*e/10) + x**9*(4*a**2*b*c*f/3 + 2*
a**2*c**2*d/3 + 4*a*b**3*f/9 + 4*a*b**2*c*d/3 + b**4*d/9) + x**8*(3*a**2*b
*c*e/2 + a*b**3*e/2) + x**7*(4*a**3*c*f/7 + 6*a**2*b**2*f/7 + 12*a**2*b*c*
d/7 + 4*a*b**3*d/7) + x**6*(2*a**3*c*e/3 + a**2*b**2*e) + x**5*(4*a**3*b*f
/5 + 4*a**3*c*d/5 + 6*a**2*b**2*d/5) + x**3*(a**4*f/3 + 4*a**3*b*d/3)
```

3.60.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 418, normalized size of antiderivative = 1.00

$$\int \left(a+bx^2+cx^4\right)^3 \left(ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6\right)\,dx$$
$$= \frac{1}{19}\,c^4fx^{19}+\frac{1}{18}\,c^4ex^{18}+\frac{1}{4}\,bc^3ex^{16}+\frac{1}{17}\,\left(c^4d+4\,bc^3f\right)x^{17}+\frac{1}{7}\,\left(3\,b^2c^2+2\,ac^3\right)ex^{14}$$
$$+\frac{2}{15}\,\left(2\,bc^3d+\left(3\,b^2c^2+2\,ac^3\right)f\right)x^{15}+\frac{1}{3}\,\left(b^3c+3\,abc^2\right)ex^{12}$$
$$+\frac{2}{13}\,\left(\left(3\,b^2c^2+2\,ac^3\right)d+2\,\left(b^3c+3\,abc^2\right)f\right)x^{13}+\frac{1}{10}\,\left(b^4+12\,ab^2c+6\,a^2c^2\right)ex^{10}$$
$$+\frac{1}{11}\,\left(4\,\left(b^3c+3\,abc^2\right)d+\left(b^4+12\,ab^2c+6\,a^2c^2\right)f\right)x^{11}+\frac{1}{2}\,\left(ab^3+3\,a^2bc\right)ex^8$$
$$+\frac{1}{9}\,\left(\left(b^4+12\,ab^2c+6\,a^2c^2\right)d+4\,\left(ab^3+3\,a^2bc\right)f\right)x^9+a^3bex^4$$
$$+\frac{1}{3}\,\left(3\,a^2b^2+2\,a^3c\right)ex^6+\frac{2}{7}\,\left(2\,\left(ab^3+3\,a^2bc\right)d+\left(3\,a^2b^2+2\,a^3c\right)f\right)x^7$$
$$+\frac{1}{2}\,a^4ex^2+a^4dx+\frac{2}{5}\,\left(2\,a^3bf+\left(3\,a^2b^2+2\,a^3c\right)d\right)x^5+\frac{1}{3}\,\left(4\,a^3bd+a^4f\right)x^3$$

input
```
integrate((c*x^4+b*x^2+a)^3*(a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4
+c*e*x^5+c*f*x^6),x, algorithm="maxima")
```

output
```
1/19*c^4*f*x^19 + 1/18*c^4*e*x^18 + 1/4*b*c^3*e*x^16 + 1/17*(c^4*d + 4*b*c
^3*f)*x^17 + 1/7*(3*b^2*c^2 + 2*a*c^3)*e*x^14 + 2/15*(2*b*c^3*d + (3*b^2*c
^2 + 2*a*c^3)*f)*x^15 + 1/3*(b^3*c + 3*a*b*c^2)*e*x^12 + 2/13*((3*b^2*c^2
+ 2*a*c^3)*d + 2*(b^3*c + 3*a*b*c^2)*f)*x^13 + 1/10*(b^4 + 12*a*b^2*c + 6*
a^2*c^2)*e*x^10 + 1/11*(4*(b^3*c + 3*a*b*c^2)*d + (b^4 + 12*a*b^2*c + 6*a^
2*c^2)*f)*x^11 + 1/2*(a*b^3 + 3*a^2*b*c)*e*x^8 + 1/9*((b^4 + 12*a*b^2*c +
6*a^2*c^2)*d + 4*(a*b^3 + 3*a^2*b*c)*f)*x^9 + a^3*b*e*x^4 + 1/3*(3*a^2*b^2
 + 2*a^3*c)*e*x^6 + 2/7*(2*(a*b^3 + 3*a^2*b*c)*d + (3*a^2*b^2 + 2*a^3*c)*f
)*x^7 + 1/2*a^4*e*x^2 + a^4*d*x + 2/5*(2*a^3*b*f + (3*a^2*b^2 + 2*a^3*c)*d
)*x^5 + 1/3*(4*a^3*b*d + a^4*f)*x^3
```

3.60.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 463, normalized size of antiderivative = 1.11

$$\int \left(a+bx^2+cx^4\right)^3\left(ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6\right)\,dx$$
$$=\frac{1}{19}c^4fx^{19}+\frac{1}{18}c^4ex^{18}+\frac{1}{17}c^4dx^{17}+\frac{4}{17}bc^3fx^{17}+\frac{1}{4}bc^3ex^{16}+\frac{4}{15}bc^3dx^{15}$$
$$+\frac{2}{5}b^2c^2fx^{15}+\frac{4}{15}ac^3fx^{15}+\frac{3}{7}b^2c^2ex^{14}+\frac{2}{7}ac^3ex^{14}+\frac{6}{13}b^2c^2dx^{13}+\frac{4}{13}ac^3dx^{13}$$
$$+\frac{4}{13}b^3cfx^{13}+\frac{12}{13}abc^2fx^{13}+\frac{1}{3}b^3cex^{12}+abc^2ex^{12}+\frac{4}{11}b^3cdx^{11}+\frac{12}{11}abc^2dx^{11}$$
$$+\frac{1}{11}b^4fx^{11}+\frac{12}{11}ab^2cfx^{11}+\frac{6}{11}a^2c^2fx^{11}+\frac{1}{10}b^4ex^{10}+\frac{6}{5}ab^2cex^{10}+\frac{3}{5}a^2c^2ex^{10}$$
$$+\frac{1}{9}b^4dx^9+\frac{4}{3}ab^2cdx^9+\frac{2}{3}a^2c^2dx^9+\frac{4}{9}ab^3fx^9+\frac{4}{3}a^2bcfx^9+\frac{1}{2}ab^3ex^8+\frac{3}{2}a^2bcex^8$$
$$+\frac{4}{7}ab^3dx^7+\frac{12}{7}a^2bcdx^7+\frac{6}{7}a^2b^2fx^7+\frac{4}{7}a^3cfx^7+a^2b^2ex^6+\frac{2}{3}a^3cex^6+\frac{6}{5}a^2b^2dx^5$$
$$+\frac{4}{5}a^3cdx^5+\frac{4}{5}a^3bfx^5+a^3bex^4+\frac{4}{3}a^3bdx^3+\frac{1}{3}a^4fx^3+\frac{1}{2}a^4ex^2+a^4dx$$

input
```
integrate((c*x^4+b*x^2+a)^3*(a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4
+c*e*x^5+c*f*x^6),x, algorithm="giac")
```

output
```
1/19*c^4*f*x^19 + 1/18*c^4*e*x^18 + 1/17*c^4*d*x^17 + 4/17*b*c^3*f*x^17 +
1/4*b*c^3*e*x^16 + 4/15*b*c^3*d*x^15 + 2/5*b^2*c^2*f*x^15 + 4/15*a*c^3*f*x
^15 + 3/7*b^2*c^2*e*x^14 + 2/7*a*c^3*e*x^14 + 6/13*b^2*c^2*d*x^13 + 4/13*a
*c^3*d*x^13 + 4/13*b^3*c*f*x^13 + 12/13*a*b*c^2*f*x^13 + 1/3*b^3*c*e*x^12
+ a*b*c^2*e*x^12 + 4/11*b^3*c*d*x^11 + 12/11*a*b*c^2*d*x^11 + 1/11*b^4*f*x
^11 + 12/11*a*b^2*c*f*x^11 + 6/11*a^2*c^2*f*x^11 + 1/10*b^4*e*x^10 + 6/5*a
*b^2*c*e*x^10 + 3/5*a^2*c^2*e*x^10 + 1/9*b^4*d*x^9 + 4/3*a*b^2*c*d*x^9 + 2
/3*a^2*c^2*d*x^9 + 4/9*a*b^3*f*x^9 + 4/3*a^2*b*c*f*x^9 + 1/2*a*b^3*e*x^8 +
 3/2*a^2*b*c*e*x^8 + 4/7*a*b^3*d*x^7 + 12/7*a^2*b*c*d*x^7 + 6/7*a^2*b^2*f*
x^7 + 4/7*a^3*c*f*x^7 + a^2*b^2*e*x^6 + 2/3*a^3*c*e*x^6 + 6/5*a^2*b^2*d*x^
5 + 4/5*a^3*c*d*x^5 + 4/5*a^3*b*f*x^5 + a^3*b*e*x^4 + 4/3*a^3*b*d*x^3 + 1/
3*a^4*f*x^3 + 1/2*a^4*e*x^2 + a^4*d*x
```

3.60.9 Mupad [B] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 398, normalized size of antiderivative = 0.96

$$\int \left(a+bx^2+cx^4\right)^3 \left(ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6\right)\,dx$$
$$= x^3\left(\frac{f\,a^4}{3}+\frac{4\,b\,d\,a^3}{3}\right)+x^{17}\left(\frac{d\,c^4}{17}+\frac{4\,b\,f\,c^3}{17}\right)+x^5\left(\frac{4\,f\,a^3\,b}{5}+\frac{4\,c\,d\,a^3}{5}+\frac{6\,d\,a^2\,b^2}{5}\right)+x^{15}\left(\frac{2\,f\,b^2\,c^2}{5}+\frac{4\,d\,b\,c^3}{15}+\frac{4\,a\,f\,c^3}{15}\right)+x^9\left(\frac{4\,f\,a^2\,b\,c}{3}+\frac{2\,d\,a^2\,c^2}{3}+\frac{4\,f\,a\,b^3}{9}+\frac{4\,d\,a\,b^2\,c}{3}+\frac{d\,b^4}{9}\right)+x^{11}\left(\frac{6\,f\,a^2\,c^2}{11}+\frac{12\,f\,a\,b^2\,c}{11}+\frac{12\,d\,a\,b\,c^2}{11}+\frac{f\,b^4}{11}+\frac{4\,d\,b^3\,c}{11}\right)+x^7\left(\frac{4\,c\,f\,a^3}{7}+\frac{6\,f\,a^2\,b^2}{7}+\frac{12\,c\,d\,a^2\,b}{7}+\frac{4\,d\,a\,b^3}{7}\right)+x^{13}\left(\frac{4\,f\,b^3\,c}{13}+\frac{6\,d\,b^2\,c^2}{13}+\frac{12\,a\,f\,b\,c^2}{13}+\frac{4\,a\,d\,c^3}{13}\right)+\frac{a^4\,e\,x^2}{2}+\frac{c^4\,e\,x^{18}}{18}+\frac{c^4\,f\,x^{19}}{19}+\frac{e\,x^{10}\,\left(6\,a^2\,c^2+12\,a\,b^2\,c+b^4\right)}{10}+a^4\,d\,x+\frac{a^2\,e\,x^6\,\left(3\,b^2+2\,a\,c\right)}{3}+\frac{c^2\,e\,x^{14}\,\left(3\,b^2+2\,a\,c\right)}{7}+a^3\,b\,e\,x^4+\frac{b\,c^3\,e\,x^{16}}{4}+\frac{a\,b\,e\,x^8\,\left(b^2+3\,a\,c\right)}{2}+\frac{b\,c\,e\,x^{12}\,\left(b^2+3\,a\,c\right)}{3}$$

input
```
int((a + b*x^2 + c*x^4)^3*(a*d + x^2*(b*d + a*f) + x^4*(c*d + b*f) + a*e*x
 + b*e*x^3 + c*e*x^5 + c*f*x^6),x)
```

output
```
x^3*((a^4*f)/3 + (4*a^3*b*d)/3) + x^17*((c^4*d)/17 + (4*b*c^3*f)/17) + x^5
*((6*a^2*b^2*d)/5 + (4*a^3*c*d)/5 + (4*a^3*b*f)/5) + x^15*((2*b^2*c^2*f)/5
 + (4*b*c^3*d)/15 + (4*a*c^3*f)/15) + x^9*((b^4*d)/9 + (2*a^2*c^2*d)/3 + (
4*a*b^3*f)/9 + (4*a*b^2*c*d)/3 + (4*a^2*b*c*f)/3) + x^11*((b^4*f)/11 + (6*
a^2*c^2*f)/11 + (4*b^3*c*d)/11 + (12*a*b*c^2*d)/11 + (12*a*b^2*c*f)/11) +
x^7*((6*a^2*b^2*f)/7 + (4*a*b^3*d)/7 + (4*a^3*c*f)/7 + (12*a^2*b*c*d)/7) +
 x^13*((6*b^2*c^2*d)/13 + (4*a*c^3*d)/13 + (4*b^3*c*f)/13 + (12*a*b*c^2*f)
/13) + (a^4*e*x^2)/2 + (c^4*e*x^18)/18 + (c^4*f*x^19)/19 + (e*x^10*(b^4 +
6*a^2*c^2 + 12*a*b^2*c))/10 + a^4*d*x + (a^2*e*x^6*(2*a*c + 3*b^2))/3 + (c
^2*e*x^14*(2*a*c + 3*b^2))/7 + a^3*b*e*x^4 + (b*c^3*e*x^16)/4 + (a*b*e*x^8
*(3*a*c + b^2))/2 + (b*c*e*x^12*(3*a*c + b^2))/3
```

3.61 $\int (a+bx^2+cx^4)^2 (ad+aex+(bd+af)x^2+bex^3+$

3.61.1 Optimal result

Integrand size = 63, antiderivative size = 259

$$\begin{aligned}
&\int (a+bx^2+cx^4)^2 \left(ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6\right) dx \\
&= a^3dx+\frac{1}{2}a^3ex^2+\frac{1}{3}a^2(3bd+af)x^3+\frac{3}{4}a^2bex^4+\frac{3}{5}a\left(b^2d+acd+abf\right)x^5 \\
&\quad +\frac{1}{2}a\left(b^2+ac\right)ex^6+\frac{1}{7}\left(b^3d+6abcd+3ab^2f+3a^2cf\right)x^7+\frac{1}{8}b\left(b^2+6ac\right)ex^8 \\
&\quad +\frac{1}{9}\left(3b^2cd+3ac^2d+b^3f+6abcf\right)x^9+\frac{3}{10}c\left(b^2+ac\right)ex^{10} \\
&\quad +\frac{3}{11}c\left(bcd+b^2f+acf\right)x^{11}+\frac{1}{4}bc^2ex^{12}+\frac{1}{13}c^2(cd+3bf)x^{13}+\frac{1}{14}c^3ex^{14}+\frac{1}{15}c^3fx^{15}
\end{aligned}$$

output
```
a^3*d*x+1/2*a^3*e*x^2+1/3*a^2*(a*f+3*b*d)*x^3+3/4*a^2*b*e*x^4+3/5*a*(a*b*f
+a*c*d+b^2*d)*x^5+1/2*a*(a*c+b^2)*e*x^6+1/7*(3*a^2*c*f+3*a*b^2*f+6*a*b*c*d
+b^3*d)*x^7+1/8*b*(6*a*c+b^2)*e*x^8+1/9*(6*a*b*c*f+3*a*c^2*d+b^3*f+3*b^2*c
*d)*x^9+3/10*c*(a*c+b^2)*e*x^10+3/11*c*(a*c*f+b^2*f+b*c*d)*x^11+1/4*b*c^2*
e*x^12+1/13*c^2*(3*b*f+c*d)*x^13+1/14*c^3*e*x^14+1/15*c^3*f*x^15
```

3.61.2 Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 259, normalized size of antiderivative = 1.00

$$\int \left(a+bx^2+cx^4\right)^2 \left(ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6\right) \, dx$$
$$= a^3dx+\frac{1}{2}a^3ex^2+\frac{1}{3}a^2(3bd+af)x^3+\frac{3}{4}a^2bex^4+\frac{3}{5}a\left(b^2d+acd+abf\right)x^5$$
$$+\frac{1}{2}a\left(b^2+ac\right)ex^6+\frac{1}{7}\left(b^3d+6abcd+3ab^2f+3a^2cf\right)x^7+\frac{1}{8}b\left(b^2+6ac\right)ex^8$$
$$+\frac{1}{9}\left(3b^2cd+3ac^2d+b^3f+6abcf\right)x^9+\frac{3}{10}c\left(b^2+ac\right)ex^{10}$$
$$+\frac{3}{11}c\left(bcd+b^2f+acf\right)x^{11}+\frac{1}{4}bc^2ex^{12}+\frac{1}{13}c^2(cd+3bf)x^{13}+\frac{1}{14}c^3ex^{14}+\frac{1}{15}c^3fx^{15}$$

input
```
Integrate[(a + b*x^2 + c*x^4)^2*(a*d + a*e*x + (b*d + a*f)*x^2 + b*e*x^3 +
 (c*d + b*f)*x^4 + c*e*x^5 + c*f*x^6),x]
```

output
```
a^3*d*x + (a^3*e*x^2)/2 + (a^2*(3*b*d + a*f)*x^3)/3 + (3*a^2*b*e*x^4)/4 +
(3*a*(b^2*d + a*c*d + a*b*f)*x^5)/5 + (a*(b^2 + a*c)*e*x^6)/2 + ((b^3*d +
6*a*b*c*d + 3*a*b^2*f + 3*a^2*c*f)*x^7)/7 + (b*(b^2 + 6*a*c)*e*x^8)/8 + ((
3*b^2*c*d + 3*a*c^2*d + b^3*f + 6*a*b*c*f)*x^9)/9 + (3*c*(b^2 + a*c)*e*x^1
0)/10 + (3*c*(b*c*d + b^2*f + a*c*f)*x^11)/11 + (b*c^2*e*x^12)/4 + (c^2*(c
*d + 3*b*f)*x^13)/13 + (c^3*e*x^14)/14 + (c^3*f*x^15)/15
```

3.61.3 Rubi [A] (verified)

Time = 0.60 (sec) , antiderivative size = 259, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.032, Rules used = {2200, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \left(a+bx^2+cx^4\right)^2 \left(x^2(af+bd)+ad+aex+x^4(bf+cd)+bex^3+cex^5+cfx^6\right) \, dx$$

$$\Big\downarrow \text{2200}$$

$$\int \left(a^3d+a^3ex+x^6\left(3a^2cf+3ab^2f+6abcd+b^3d\right)+a^2x^2(af+3bd)+3a^2bex^3+3cx^{10}\left(acf+b^2f+bcd\right)+3ax$$

$$\Big\downarrow \text{2009}$$

3.61.
$\int \left(a+bx^2+cx^4\right)^2 \left(ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6\right) \, dx$

$$a^3dx + \frac{1}{2}a^3ex^2 + \frac{1}{7}x^7(3a^2cf + 3ab^2f + 6abcd + b^3d) + \frac{1}{3}a^2x^3(af + 3bd) + \frac{3}{4}a^2bex^4 + \frac{3}{11}cx^{11}(acf + b^2f + bcd) + \frac{3}{5}ax^5(abf + acd + b^2d) + \frac{3}{10}cex^{10}(ac + b^2) + \frac{1}{8}bex^8(6ac + b^2) + \frac{1}{2}aex^6(ac + b^2) + \frac{1}{9}x^9(6abcf + 3ac^2d + b^3f + 3b^2cd) + \frac{1}{13}c^2x^{13}(3bf + cd) + \frac{1}{4}bc^2ex^{12} + \frac{1}{14}c^3ex^{14} + \frac{1}{15}c^3fx^{15}$$

input
```
Int[(a + b*x^2 + c*x^4)^2*(a*d + a*e*x + (b*d + a*f)*x^2 + b*e*x^3 + (c*d
+ b*f)*x^4 + c*e*x^5 + c*f*x^6),x]
```

output
```
a^3*d*x + (a^3*e*x^2)/2 + (a^2*(3*b*d + a*f)*x^3)/3 + (3*a^2*b*e*x^4)/4 +
(3*a*(b^2*d + a*c*d + a*b*f)*x^5)/5 + (a*(b^2 + a*c)*e*x^6)/2 + ((b^3*d +
6*a*b*c*d + 3*a*b^2*f + 3*a^2*c*f)*x^7)/7 + (b*(b^2 + 6*a*c)*e*x^8)/8 + ((
3*b^2*c*d + 3*a*c^2*d + b^3*f + 6*a*b*c*f)*x^9)/9 + (3*c*(b^2 + a*c)*e*x^1
0)/10 + (3*c*(b*c*d + b^2*f + a*c*f)*x^11)/11 + (b*c^2*e*x^12)/4 + (c^2*(c
*d + 3*b*f)*x^13)/13 + (c^3*e*x^14)/14 + (c^3*f*x^15)/15
```

3.61.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2200
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_.), x_Symbol] :> Int[Expa
ndIntegrand[Px*(a + b*x^2 + c*x^4)^p, x], x] /; FreeQ[{a, b, c}, x] && Poly
Q[Px, x] && IGtQ[p, 0]
```

3.61.4 Maple [A] (verified)

Time = 27.40 (sec) , antiderivative size = 259, normalized size of antiderivative = 1.00

method	result
norman	$\left(\frac{1}{3}a^3f + a^2bd\right)x^3 + \left(\frac{3}{10}a\,c^2e + \frac{3}{10}b^2ce\right)x^{10} + \left(\frac{1}{2}a^2ce + \frac{1}{2}a\,b^2e\right)x^6 + \left(\frac{3}{13}b\,c^2f + \frac{1}{13}c^3d\right)x^{13} +$
risch	$a^3dx + \frac{3}{4}a^2be\,x^4 + \frac{1}{2}a\,b^2e\,x^6 + \frac{1}{8}b^3e\,x^8 + \frac{1}{4}b\,c^2e\,x^{12} + \frac{1}{2}a^3e\,x^2 + \frac{1}{14}c^3e\,x^{14} + \frac{1}{15}c^3f\,x^{15} + \frac{3}{13}x^{1}$
parallelrisch	$a^3dx + \frac{3}{4}a^2be\,x^4 + \frac{1}{2}a\,b^2e\,x^6 + \frac{1}{8}b^3e\,x^8 + \frac{1}{4}b\,c^2e\,x^{12} + \frac{1}{2}a^3e\,x^2 + \frac{1}{14}c^3e\,x^{14} + \frac{1}{15}c^3f\,x^{15} + \frac{3}{13}x^{1}$
gosper	$x(24024c^3f\,x^{14}+25740c^3e\,x^{13}+83160b\,c^2f\,x^{12}+27720c^3d\,x^{12}+90090b\,c^2e\,x^{11}+98280a\,c^2f\,x^{10}+98280b^2cf\,x^{10}+98280b\,c^2d\,x$
default	$\frac{c^3f\,x^{15}}{15} + \frac{c^3e\,x^{14}}{14} + \frac{(2b\,c^2f+c^2(bf+cd))x^{13}}{13} + \frac{b\,c^2e\,x^{12}}{4} + \frac{((2ac+b^2)cf+2bc(bf+cd)+c^2(af+bd))x^{11}}{11} + \frac{((2ac+b^2)e}{}$

input
```
int((c*x^4+b*x^2+a)^2*(d*a+a*e*x+(a*f+b*d)*x^2+e*x^3*b+(b*f+c*d)*x^4+c*e*x
^5+c*f*x^6),x,method=_RETURNVERBOSE)
```

output
```
(1/3*a^3*f+a^2*b*d)*x^3+(3/10*a*c^2*e+3/10*b^2*c*e)*x^10+(1/2*a^2*c*e+1/2*
a*b^2*e)*x^6+(3/13*b*c^2*f+1/13*c^3*d)*x^13+(3/4*a*b*c*e+1/8*b^3*e)*x^8+(3
/11*a*c^2*f+3/11*b^2*c*f+3/11*b*c^2*d)*x^11+(3/5*a^2*b*f+3/5*a^2*c*d+3/5*a
*b^2*d)*x^5+(3/7*a^2*c*f+3/7*a*b^2*f+6/7*a*b*c*d+1/7*b^3*d)*x^7+(2/3*a*b*c
*f+1/3*a*c^2*d+1/9*b^3*f+1/3*b^2*c*d)*x^9+a^3*d*x+1/2*a^3*e*x^2+1/14*c^3*e
*x^14+1/15*c^3*f*x^15+3/4*a^2*b*e*x^4+1/4*b*c^2*e*x^12
```

3.61.5 Fricas [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 251, normalized size of antiderivative = 0.97

$$\begin{aligned}\int (a+bx^2+cx^4)^2 \left(ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6\right) dx\\
&= \frac{1}{15}c^3fx^{15}+\frac{1}{14}c^3ex^{14}+\frac{1}{4}bc^2ex^{12}+\frac{1}{13}\left(c^3d+3\,bc^2f\right)x^{13}\\
&\quad+\frac{3}{10}\left(b^2c+ac^2\right)ex^{10}+\frac{3}{11}\left(bc^2d+\left(b^2c+ac^2\right)f\right)x^{11}\\
&\quad+\frac{1}{8}\left(b^3+6\,abc\right)ex^8+\frac{1}{9}\left(3\left(b^2c+ac^2\right)d+\left(b^3+6\,abc\right)f\right)x^9\\
&\quad+\frac{3}{4}a^2bex^4+\frac{1}{2}\left(ab^2+a^2c\right)ex^6+\frac{1}{7}\left(\left(b^3+6\,abc\right)d+3\left(ab^2+a^2c\right)f\right)x^7\\
&\quad+\frac{1}{2}a^3ex^2+\frac{3}{5}\left(a^2bf+\left(ab^2+a^2c\right)d\right)x^5+a^3dx+\frac{1}{3}\left(3\,a^2bd+a^3f\right)x^3\end{aligned}$$

input
```
integrate((c*x^4+b*x^2+a)^2*(a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4
+c*e*x^5+c*f*x^6),x, algorithm="fricas")
```

output
```
1/15*c^3*f*x^15 + 1/14*c^3*e*x^14 + 1/4*b*c^2*e*x^12 + 1/13*(c^3*d + 3*b*c
^2*f)*x^13 + 3/10*(b^2*c + a*c^2)*e*x^10 + 3/11*(b*c^2*d + (b^2*c + a*c^2)
*f)*x^11 + 1/8*(b^3 + 6*a*b*c)*e*x^8 + 1/9*(3*(b^2*c + a*c^2)*d + (b^3 + 6
*a*b*c)*f)*x^9 + 3/4*a^2*b*e*x^4 + 1/2*(a*b^2 + a^2*c)*e*x^6 + 1/7*((b^3 +
 6*a*b*c)*d + 3*(a*b^2 + a^2*c)*f)*x^7 + 1/2*a^3*e*x^2 + 3/5*(a^2*b*f + (a
*b^2 + a^2*c)*d)*x^5 + a^3*d*x + 1/3*(3*a^2*b*d + a^3*f)*x^3
```

3.61.6 Sympy [A] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 309, normalized size of antiderivative = 1.19

$$\int (a+bx^2+cx^4)^2 (ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6)\,dx$$
$$= a^3dx+\frac{a^3ex^2}{2}+\frac{3a^2bex^4}{4}+\frac{bc^2ex^{12}}{4}+\frac{c^3ex^{14}}{14}+\frac{c^3fx^{15}}{15}+x^{13}\cdot\left(\frac{3bc^2f}{13}+\frac{c^3d}{13}\right)+x^{11}\cdot\left(\frac{3ac^2f}{11}+\frac{3b^2cf}{11}+\frac{3bc^2d}{11}\right)+x^{10}\cdot\left(\frac{3ac^2e}{10}+\frac{3b^2ce}{10}\right)+x^9\cdot\left(\frac{2abcf}{3}+\frac{ac^2d}{3}+\frac{b^3f}{9}+\frac{b^2cd}{3}\right)+x^8\cdot\left(\frac{3abce}{4}+\frac{b^3e}{8}\right)+x^7\cdot\left(\frac{3a^2cf}{7}+\frac{3ab^2f}{7}+\frac{6abcd}{7}+\frac{b^3d}{7}\right)+x^6\left(\frac{a^2ce}{2}+\frac{ab^2e}{2}\right)+x^5\cdot\left(\frac{3a^2bf}{5}+\frac{3a^2cd}{5}+\frac{3ab^2d}{5}\right)+x^3\left(\frac{a^3f}{3}+a^2bd\right)$$

input
```
integrate((c*x**4+b*x**2+a)**2*(a*d+a*e*x+(a*f+b*d)*x**2+b*e*x**3+(b*f+c*d
)*x**4+c*e*x**5+c*f*x**6),x)
```

output
```
a**3*d*x + a**3*e*x**2/2 + 3*a**2*b*e*x**4/4 + b*c**2*e*x**12/4 + c**3*e*x
**14/14 + c**3*f*x**15/15 + x**13*(3*b*c**2*f/13 + c**3*d/13) + x**11*(3*a
*c**2*f/11 + 3*b**2*c*f/11 + 3*b*c**2*d/11) + x**10*(3*a*c**2*e/10 + 3*b**
2*c*e/10) + x**9*(2*a*b*c*f/3 + a*c**2*d/3 + b**3*f/9 + b**2*c*d/3) + x**8
*(3*a*b*c*e/4 + b**3*e/8) + x**7*(3*a**2*c*f/7 + 3*a*b**2*f/7 + 6*a*b*c*d/
7 + b**3*d/7) + x**6*(a**2*c*e/2 + a*b**2*e/2) + x**5*(3*a**2*b*f/5 + 3*a*
*2*c*d/5 + 3*a*b**2*d/5) + x**3*(a**3*f/3 + a**2*b*d)
```

3.61.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 251, normalized size of antiderivative = 0.97

$$\int (a+bx^2+cx^4)^2 (ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6)\,dx$$
$$= \frac{1}{15}c^3fx^{15}+\frac{1}{14}c^3ex^{14}+\frac{1}{4}bc^2ex^{12}+\frac{1}{13}(c^3d+3bc^2f)x^{13}+\frac{3}{10}(b^2c+ac^2)ex^{10}+\frac{3}{11}(bc^2d+(b^2c+ac^2)f)x^{11}+\frac{1}{8}(b^3+6abc)ex^8+\frac{1}{9}(3(b^2c+ac^2)d+(b^3+6abc)f)x^9+\frac{3}{4}a^2bex^4+\frac{1}{2}(ab^2+a^2c)ex^6+\frac{1}{7}((b^3+6abc)d+3(ab^2+a^2c)f)x^7+\frac{1}{2}a^3ex^2+\frac{3}{5}(a^2bf+(ab^2+a^2c)d)x^5+a^3dx+\frac{1}{3}(3a^2bd+a^3f)x^3$$

input
```
integrate((c*x^4+b*x^2+a)^2*(a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4
+c*e*x^5+c*f*x^6),x, algorithm="maxima")
```

output
```
1/15*c^3*f*x^15 + 1/14*c^3*e*x^14 + 1/4*b*c^2*e*x^12 + 1/13*(c^3*d + 3*b*c
^2*f)*x^13 + 3/10*(b^2*c + a*c^2)*e*x^10 + 3/11*(b*c^2*d + (b^2*c + a*c^2)
*f)*x^11 + 1/8*(b^3 + 6*a*b*c)*e*x^8 + 1/9*(3*(b^2*c + a*c^2)*d + (b^3 + 6
*a*b*c)*f)*x^9 + 3/4*a^2*b*e*x^4 + 1/2*(a*b^2 + a^2*c)*e*x^6 + 1/7*((b^3 +
 6*a*b*c)*d + 3*(a*b^2 + a^2*c)*f)*x^7 + 1/2*a^3*e*x^2 + 3/5*(a^2*b*f + (a
*b^2 + a^2*c)*d)*x^5 + a^3*d*x + 1/3*(3*a^2*b*d + a^3*f)*x^3
```

3.61.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 285, normalized size of antiderivative = 1.10

$$\int (a+bx^2+cx^4)^2 \left(ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6\right) dx$$
$$= \frac{1}{15}c^3fx^{15}+\frac{1}{14}c^3ex^{14}+\frac{1}{13}c^3dx^{13}+\frac{3}{13}bc^2fx^{13}+\frac{1}{4}bc^2ex^{12}+\frac{3}{11}bc^2dx^{11}+\frac{3}{11}b^2cfx^{11}+\frac{3}{11}ac^2fx^{11}+\frac{3}{10}b^2cex^{10}+\frac{3}{10}ac^2ex^{10}+\frac{1}{3}b^2cdx^9+\frac{1}{3}ac^2dx^9+\frac{1}{9}b^3fx^9+\frac{2}{3}abcfx^9+\frac{1}{8}b^3ex^8+\frac{3}{4}abcex^8+\frac{1}{7}b^3dx^7+\frac{6}{7}abcdx^7+\frac{3}{7}ab^2fx^7+\frac{3}{7}a^2cfx^7+\frac{1}{2}ab^2ex^6+\frac{1}{2}a^2cex^6+\frac{3}{5}ab^2dx^5+\frac{3}{5}a^2cdx^5+\frac{3}{5}a^2bfx^5+\frac{3}{4}a^2bex^4+a^2bdx^3+\frac{1}{3}a^3fx^3+\frac{1}{2}a^3ex^2+a^3dx$$

input
```
integrate((c*x^4+b*x^2+a)^2*(a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4
+c*e*x^5+c*f*x^6),x, algorithm="giac")
```

output
```
1/15*c^3*f*x^15 + 1/14*c^3*e*x^14 + 1/13*c^3*d*x^13 + 3/13*b*c^2*f*x^13 +
1/4*b*c^2*e*x^12 + 3/11*b*c^2*d*x^11 + 3/11*b^2*c*f*x^11 + 3/11*a*c^2*f*x^
11 + 3/10*b^2*c*e*x^10 + 3/10*a*c^2*e*x^10 + 1/3*b^2*c*d*x^9 + 1/3*a*c^2*d
*x^9 + 1/9*b^3*f*x^9 + 2/3*a*b*c*f*x^9 + 1/8*b^3*e*x^8 + 3/4*a*b*c*e*x^8 +
 1/7*b^3*d*x^7 + 6/7*a*b*c*d*x^7 + 3/7*a*b^2*f*x^7 + 3/7*a^2*c*f*x^7 + 1/2
*a*b^2*e*x^6 + 1/2*a^2*c*e*x^6 + 3/5*a*b^2*d*x^5 + 3/5*a^2*c*d*x^5 + 3/5*a
^2*b*f*x^5 + 3/4*a^2*b*e*x^4 + a^2*b*d*x^3 + 1/3*a^3*f*x^3 + 1/2*a^3*e*x^2
 + a^3*d*x
```

3.61.9 Mupad [B] (verification not implemented)

Time = 8.11 (sec) , antiderivative size = 246, normalized size of antiderivative = 0.95

$$\int \left(a+bx^2+cx^4\right)^2 \left(ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6\right)\,dx$$
$$=x^3\left(\frac{f\,a^3}{3}+b\,d\,a^2\right)+x^{13}\left(\frac{d\,c^3}{13}+\frac{3\,b\,f\,c^2}{13}\right)+x^5\left(\frac{3\,f\,a^2\,b}{5}+\frac{3\,c\,d\,a^2}{5}+\frac{3\,d\,a\,b^2}{5}\right)$$
$$+x^{11}\left(\frac{3\,f\,b^2\,c}{11}+\frac{3\,d\,b\,c^2}{11}+\frac{3\,a\,f\,c^2}{11}\right)+x^7\left(\frac{3\,c\,f\,a^2}{7}+\frac{3\,f\,a\,b^2}{7}+\frac{6\,c\,d\,a\,b}{7}+\frac{d\,b^3}{7}\right)$$
$$+x^9\left(\frac{f\,b^3}{9}+\frac{d\,b^2\,c}{3}+\frac{2\,a\,f\,b\,c}{3}+\frac{a\,d\,c^2}{3}\right)+\frac{a^3\,e\,x^2}{2}+\frac{c^3\,e\,x^{14}}{14}+\frac{c^3\,f\,x^{15}}{15}+a^3\,d\,x$$
$$+\frac{a\,e\,x^6\,(b^2+a\,c)}{2}+\frac{b\,e\,x^8\,(b^2+6\,a\,c)}{8}+\frac{3\,c\,e\,x^{10}\,(b^2+a\,c)}{10}+\frac{3\,a^2\,b\,e\,x^4}{4}+\frac{b\,c^2\,e\,x^{12}}{4}$$

input
```
int((a + b*x^2 + c*x^4)^2*(a*d + x^2*(b*d + a*f) + x^4*(c*d + b*f) + a*e*x
 + b*e*x^3 + c*e*x^5 + c*f*x^6),x)
```

output
```
x^3*((a^3*f)/3 + a^2*b*d) + x^13*((c^3*d)/13 + (3*b*c^2*f)/13) + x^5*((3*a
*b^2*d)/5 + (3*a^2*c*d)/5 + (3*a^2*b*f)/5) + x^11*((3*b*c^2*d)/11 + (3*a*c
^2*f)/11 + (3*b^2*c*f)/11) + x^7*((b^3*d)/7 + (3*a*b^2*f)/7 + (3*a^2*c*f)/
7 + (6*a*b*c*d)/7) + x^9*((b^3*f)/9 + (a*c^2*d)/3 + (b^2*c*d)/3 + (2*a*b*c
*f)/3) + (a^3*e*x^2)/2 + (c^3*e*x^14)/14 + (c^3*f*x^15)/15 + a^3*d*x + (a*
e*x^6*(a*c + b^2))/2 + (b*e*x^8*(6*a*c + b^2))/8 + (3*c*e*x^10*(a*c + b^2)
)/10 + (3*a^2*b*e*x^4)/4 + (b*c^2*e*x^12)/4
```

3.62 $\int (a+bx^2+cx^4)(ad+aex+(bd+af)x^2+bex^3+($

3.62.1 Optimal result

Integrand size = 61, antiderivative size = 154

$$\int (a+bx^2+cx^4)\left(ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6\right)dx$$
$$=a^2dx+\frac{1}{2}a^2ex^2+\frac{1}{3}a(2bd+af)x^3+\frac{1}{2}abex^4+\frac{1}{5}\left(b^2d+2acd+2abf\right)x^5+\frac{1}{6}\left(b^2+2ac\right)ex^6$$
$$+\frac{1}{7}\left(2bcd+b^2f+2acf\right)x^7+\frac{1}{4}bcex^8+\frac{1}{9}c(cd+2bf)x^9+\frac{1}{10}c^2ex^{10}+\frac{1}{11}c^2fx^{11}$$

output
```
a^2*d*x+1/2*a^2*e*x^2+1/3*a*(a*f+2*b*d)*x^3+1/2*a*b*e*x^4+1/5*(2*a*b*f+2*a
*c*d+b^2*d)*x^5+1/6*(2*a*c+b^2)*e*x^6+1/7*(2*a*c*f+b^2*f+2*b*c*d)*x^7+1/4*
b*c*e*x^8+1/9*c*(2*b*f+c*d)*x^9+1/10*c^2*e*x^10+1/11*c^2*f*x^11
```

3.62.2 Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 154, normalized size of antiderivative = 1.00

$$\int (a+bx^2+cx^4)\left(ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6\right)dx$$
$$=a^2dx+\frac{1}{2}a^2ex^2+\frac{1}{3}a(2bd+af)x^3+\frac{1}{2}abex^4+\frac{1}{5}\left(b^2d+2acd+2abf\right)x^5+\frac{1}{6}\left(b^2+2ac\right)ex^6$$
$$+\frac{1}{7}\left(2bcd+b^2f+2acf\right)x^7+\frac{1}{4}bcex^8+\frac{1}{9}c(cd+2bf)x^9+\frac{1}{10}c^2ex^{10}+\frac{1}{11}c^2fx^{11}$$

input
```
Integrate[(a + b*x^2 + c*x^4)*(a*d + a*e*x + (b*d + a*f)*x^2 + b*e*x^3 + (
c*d + b*f)*x^4 + c*e*x^5 + c*f*x^6),x]
```

output
```
a^2*d*x + (a^2*e*x^2)/2 + (a*(2*b*d + a*f)*x^3)/3 + (a*b*e*x^4)/2 + ((b^2*
d + 2*a*c*d + 2*a*b*f)*x^5)/5 + ((b^2 + 2*a*c)*e*x^6)/6 + ((2*b*c*d + b^2*
f + 2*a*c*f)*x^7)/7 + (b*c*e*x^8)/4 + (c*(c*d + 2*b*f)*x^9)/9 + (c^2*e*x^1
0)/10 + (c^2*f*x^11)/11
```

3.62.3 Rubi [A] (verified)

Time = 0.36 (sec) , antiderivative size = 154, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}} = 0.033$, Rules used = {2200, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \left(a+bx^2+cx^4\right)\left(x^2(af+bd)+ad+aex+x^4(bf+cd)+bex^3+cex^5+cfx^6\right)\,dx$$

$$\Big\downarrow \; 2200$$

$$\int \left(a^2d+a^2ex+x^6\left(2acf+b^2f+2bcd\right)+x^4\left(2abf+2acd+b^2d\right)+ex^5\left(2ac+b^2\right)+ax^2(af+2bd)+2abex^3+$$

$$\Big\downarrow \; 2009$$

$$a^2dx+\frac{1}{2}a^2ex^2+\frac{1}{7}x^7\left(2acf+b^2f+2bcd\right)+\frac{1}{5}x^5\left(2abf+2acd+b^2d\right)+\frac{1}{6}ex^6\left(2ac+b^2\right)+\frac{1}{3}ax^3(af+2bd)+\frac{1}{2}abex^4+\frac{1}{9}cx^9(2bf+cd)+\frac{1}{4}bcex^8+\frac{1}{10}c^2ex^{10}+\frac{1}{11}c^2fx^{11}$$

input
```
Int[(a + b*x^2 + c*x^4)*(a*d + a*e*x + (b*d + a*f)*x^2 + b*e*x^3 + (c*d +
b*f)*x^4 + c*e*x^5 + c*f*x^6),x]
```

output
```
a^2*d*x + (a^2*e*x^2)/2 + (a*(2*b*d + a*f)*x^3)/3 + (a*b*e*x^4)/2 + ((b^2*
d + 2*a*c*d + 2*a*b*f)*x^5)/5 + ((b^2 + 2*a*c)*e*x^6)/6 + ((2*b*c*d + b^2*
f + 2*a*c*f)*x^7)/7 + (b*c*e*x^8)/4 + (c*(c*d + 2*b*f)*x^9)/9 + (c^2*e*x^1
0)/10 + (c^2*f*x^11)/11
```

3.62.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2200
```
Int[(Px_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_.), x_Symbol] :> Int[Expa
ndIntegrand[Px*(a + b*x^2 + c*x^4)^p, x], x] /; FreeQ[{a, b, c}, x] && Poly
Q[Px, x] && IGtQ[p, 0]
```

3.62.4 Maple [A] (verified)

Time = 0.13 (sec) , antiderivative size = 141, normalized size of antiderivative = 0.92

method	result
norman	$\frac{c^2 f\, x^{11}}{11}+\frac{c^2 e\, x^{10}}{10}+\left(\frac{2}{9} f b c+\frac{1}{9} c^2 d\right) x^9+\frac{b c e\, x^8}{4}+\left(\frac{2}{7} a c f+\frac{1}{7} b^2 f+\frac{2}{7} b c d\right) x^7+\left(\frac{1}{3} a c e+\frac{1}{6} b^2 e\right) x^6 \cdots$
risch	$\frac{1}{11} c^2 f\, x^{11}+\frac{1}{10} c^2 e\, x^{10}+\frac{2}{9} x^9 f b c+\frac{1}{9} c^2 d\, x^9+\frac{1}{4} b c e\, x^8+\frac{2}{7} x^7 a c f+\frac{1}{7} x^7 b^2 f+\frac{2}{7} x^7 b c d+\frac{1}{3} x^6 a c e \cdots$
parallelrisch	$\frac{1}{11} c^2 f\, x^{11}+\frac{1}{10} c^2 e\, x^{10}+\frac{2}{9} x^9 f b c+\frac{1}{9} c^2 d\, x^9+\frac{1}{4} b c e\, x^8+\frac{2}{7} x^7 a c f+\frac{1}{7} x^7 b^2 f+\frac{2}{7} x^7 b c d+\frac{1}{3} x^6 a c e \cdots$
gosper	$\frac{x(1260 c^2 f\, x^{10}+1386 c^2 e\, x^9+3080 x^8 f b c+1540 x^8 c^2 d+3465 b c e\, x^7+3960 x^6 a c f+1980 x^6 b^2 f+3960 x^6 b c d+4620 x^5 a c e+2310 x^5 b^2 \cdots}{13860}$
default	$\frac{c^2 f\, x^{11}}{11}+\frac{c^2 e\, x^{10}}{10}+\frac{(f b c+c(b f+c d)) x^9}{9}+\frac{b c e\, x^8}{4}+\frac{(a c f+b(b f+c d)+c(a f+b d)) x^7}{7}+\frac{(2 a c e+b^2 e) x^6}{6}+\frac{(a(b f+c d)+b \cdots}{}$

input
```
int((c*x^4+b*x^2+a)*(d*a+a*e*x+(a*f+b*d)*x^2+e*x^3*b+(b*f+c*d)*x^4+c*e*x^5
+c*f*x^6),x,method=_RETURNVERBOSE)
```

output
```
1/11*c^2*f*x^11+1/10*c^2*e*x^10+(2/9*f*b*c+1/9*c^2*d)*x^9+1/4*b*c*e*x^8+(2
/7*a*c*f+1/7*b^2*f+2/7*b*c*d)*x^7+(1/3*a*c*e+1/6*b^2*e)*x^6+(2/5*a*b*f+2/5
*a*c*d+1/5*b^2*d)*x^5+1/2*a*b*e*x^4+(1/3*f*a^2+2/3*d*a*b)*x^3+1/2*a^2*e*x^
2+a^2*d*x
```

3.62.5 Fricas [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 138, normalized size of antiderivative = 0.90

$$\int (a + bx^2 + cx^4)\left(ad + aex + (bd + af)x^2 + bex^3 + (cd + bf)x^4 + cex^5 + cfx^6\right)\,dx$$
$$= \frac{1}{11}\,c^2fx^{11} + \frac{1}{10}\,c^2ex^{10} + \frac{1}{4}\,bcex^8 + \frac{1}{9}\left(c^2d + 2\,bcf\right)x^9 + \frac{1}{6}\left(b^2 + 2\,ac\right)ex^6 + \frac{1}{7}\left(2\,bcd + \left(b^2 + 2\,ac\right)f\right)x^7 + \frac{1}{2}\,abex^4 + \frac{1}{5}\left(2\,abf + \left(b^2 + 2\,ac\right)d\right)x^5 + \frac{1}{2}\,a^2ex^2 + a^2dx + \frac{1}{3}\left(2\,abd + a^2f\right)x^3$$

input
```
integrate((c*x^4+b*x^2+a)*(a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4+c
*e*x^5+c*f*x^6),x, algorithm="fricas")
```

output
```
1/11*c^2*f*x^11 + 1/10*c^2*e*x^10 + 1/4*b*c*e*x^8 + 1/9*(c^2*d + 2*b*c*f)*
x^9 + 1/6*(b^2 + 2*a*c)*e*x^6 + 1/7*(2*b*c*d + (b^2 + 2*a*c)*f)*x^7 + 1/2*
a*b*e*x^4 + 1/5*(2*a*b*f + (b^2 + 2*a*c)*d)*x^5 + 1/2*a^2*e*x^2 + a^2*d*x
+ 1/3*(2*a*b*d + a^2*f)*x^3
```

3.62.6 Sympy [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 165, normalized size of antiderivative = 1.07

$$\int (a + bx^2 + cx^4)\left(ad + aex + (bd + af)x^2 + bex^3 + (cd + bf)x^4 + cex^5 + cfx^6\right)\,dx$$
$$= a^2dx + \frac{a^2ex^2}{2} + \frac{abex^4}{2} + \frac{bcex^8}{4} + \frac{c^2ex^{10}}{10} + \frac{c^2fx^{11}}{11} + x^9\cdot\left(\frac{2bcf}{9} + \frac{c^2d}{9}\right) + x^7\cdot\left(\frac{2acf}{7} + \frac{b^2f}{7} + \frac{2bcd}{7}\right) + x^6\left(\frac{ace}{3} + \frac{b^2e}{6}\right) + x^5\cdot\left(\frac{2abf}{5} + \frac{2acd}{5} + \frac{b^2d}{5}\right) + x^3\left(\frac{a^2f}{3} + \frac{2abd}{3}\right)$$

input
```
integrate((c*x**4+b*x**2+a)*(a*d+a*e*x+(a*f+b*d)*x**2+b*e*x**3+(b*f+c*d)*x
**4+c*e*x**5+c*f*x**6),x)
```

output
```
a**2*d*x + a**2*e*x**2/2 + a*b*e*x**4/2 + b*c*e*x**8/4 + c**2*e*x**10/10 +
 c**2*f*x**11/11 + x**9*(2*b*c*f/9 + c**2*d/9) + x**7*(2*a*c*f/7 + b**2*f/
7 + 2*b*c*d/7) + x**6*(a*c*e/3 + b**2*e/6) + x**5*(2*a*b*f/5 + 2*a*c*d/5 +
 b**2*d/5) + x**3*(a**2*f/3 + 2*a*b*d/3)
```

3.62.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 138, normalized size of antiderivative = 0.90

$$\int (a + bx^2 + cx^4) (ad + aex + (bd + af)x^2 + bex^3 + (cd + bf)x^4 + cex^5 + cfx^6) \, dx$$
$$= \frac{1}{11} c^2 fx^{11} + \frac{1}{10} c^2 ex^{10} + \frac{1}{4} bcex^8 + \frac{1}{9} (c^2 d + 2 bcf)x^9 + \frac{1}{6} (b^2 + 2 ac)ex^6 + \frac{1}{7} (2 bcd + (b^2 + 2 ac)f)x^7 + \frac{1}{2} abex^4 + \frac{1}{5} (2 abf + (b^2 + 2 ac)d)x^5 + \frac{1}{2} a^2 ex^2 + a^2 dx + \frac{1}{3} (2 abd + a^2 f)x^3$$

input
```
integrate((c*x^4+b*x^2+a)*(a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4+c
*e*x^5+c*f*x^6),x, algorithm="maxima")
```

output
```
1/11*c^2*f*x^11 + 1/10*c^2*e*x^10 + 1/4*b*c*e*x^8 + 1/9*(c^2*d + 2*b*c*f)*
x^9 + 1/6*(b^2 + 2*a*c)*e*x^6 + 1/7*(2*b*c*d + (b^2 + 2*a*c)*f)*x^7 + 1/2*
a*b*e*x^4 + 1/5*(2*a*b*f + (b^2 + 2*a*c)*d)*x^5 + 1/2*a^2*e*x^2 + a^2*d*x
+ 1/3*(2*a*b*d + a^2*f)*x^3
```

3.62.8 Giac [A] (verification not implemented)

Time = 0.39 (sec) , antiderivative size = 151, normalized size of antiderivative = 0.98

$$\int (a + bx^2 + cx^4) (ad + aex + (bd + af)x^2 + bex^3 + (cd + bf)x^4 + cex^5 + cfx^6) \, dx = \frac{1}{11} c^2 fx^{11} + \frac{1}{10} c^2 ex^{10} + \frac{1}{9} c^2 dx^9 + \frac{2}{9} bcfx^9 + \frac{1}{4} bcex^8 + \frac{2}{7} bcdx^7 + \frac{1}{7} b^2 fx^7 + \frac{2}{7} acfx^7 + \frac{1}{6} b^2 ex^6 + \frac{1}{3} acex^6 + \frac{1}{5} b^2 dx^5 + \frac{2}{5} acdx^5 + \frac{2}{5} abfx^5 + \frac{1}{2} abex^4 + \frac{2}{3} abdx^3 + \frac{1}{3} a^2 fx^3 + \frac{1}{2} a^2 ex^2 + a^2 dx$$

input
```
integrate((c*x^4+b*x^2+a)*(a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4+c
*e*x^5+c*f*x^6),x, algorithm="giac")
```

output
```
1/11*c^2*f*x^11 + 1/10*c^2*e*x^10 + 1/9*c^2*d*x^9 + 2/9*b*c*f*x^9 + 1/4*b*
c*e*x^8 + 2/7*b*c*d*x^7 + 1/7*b^2*f*x^7 + 2/7*a*c*f*x^7 + 1/6*b^2*e*x^6 +
1/3*a*c*e*x^6 + 1/5*b^2*d*x^5 + 2/5*a*c*d*x^5 + 2/5*a*b*f*x^5 + 1/2*a*b*e*
x^4 + 2/3*a*b*d*x^3 + 1/3*a^2*f*x^3 + 1/2*a^2*e*x^2 + a^2*d*x
```

3.62.9 Mupad [B] (verification not implemented)

Time = 0.07 (sec) , antiderivative size = 138, normalized size of antiderivative = 0.90

$$\int \left(a+bx^2+cx^4\right)\left(ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6\right)\,dx$$
$$=x^5\left(\frac{d\,b^2}{5}+\frac{2\,a\,f\,b}{5}+\frac{2\,a\,c\,d}{5}\right)+x^7\left(\frac{f\,b^2}{7}+\frac{2\,c\,d\,b}{7}+\frac{2\,a\,c\,f}{7}\right)+x^3\left(\frac{f\,a^2}{3}+\frac{2\,b\,d\,a}{3}\right)+x^9\left(\frac{d\,c^2}{9}+\frac{2\,b\,f\,c}{9}\right)+\frac{a^2\,e\,x^2}{2}+\frac{c^2\,e\,x^{10}}{10}+\frac{c^2\,f\,x^{11}}{11}+\frac{e\,x^6\,(b^2+2\,a\,c)}{6}+a^2\,d\,x+\frac{a\,b\,e\,x^4}{2}+\frac{b\,c\,e\,x^8}{4}$$

input
```
int((a + b*x^2 + c*x^4)*(a*d + x^2*(b*d + a*f) + x^4*(c*d + b*f) + a*e*x +
 b*e*x^3 + c*e*x^5 + c*f*x^6),x)
```

output
```
x^5*((b^2*d)/5 + (2*a*c*d)/5 + (2*a*b*f)/5) + x^7*((b^2*f)/7 + (2*b*c*d)/7
 + (2*a*c*f)/7) + x^3*((a^2*f)/3 + (2*a*b*d)/3) + x^9*((c^2*d)/9 + (2*b*c*
f)/9) + (a^2*e*x^2)/2 + (c^2*e*x^10)/10 + (c^2*f*x^11)/11 + (e*x^6*(2*a*c
+ b^2))/6 + a^2*d*x + (a*b*e*x^4)/2 + (b*c*e*x^8)/4
```

3.63 $\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{a+bx^2+cx^4}\,dx$

3.63.1 Optimal result

Integrand size = 63, antiderivative size = 20

$$\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{a+bx^2+cx^4}\,dx = dx+\frac{ex^2}{2}+\frac{fx^3}{3}$$

output
```
d*x+1/2*e*x^2+1/3*f*x^3
```

3.63.2 Mathematica [A] (verified)

Time = 0.00 (sec) , antiderivative size = 20, normalized size of antiderivative = 1.00

$$\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{a+bx^2+cx^4}\,dx = dx+\frac{ex^2}{2}+\frac{fx^3}{3}$$

input
```
Integrate[(a*d + a*e*x + (b*d + a*f)*x^2 + b*e*x^3 + (c*d + b*f)*x^4 + c*e
*x^5 + c*f*x^6)/(a + b*x^2 + c*x^4),x]
```

output
```
d*x + (e*x^2)/2 + (f*x^3)/3
```

3.63.3 Rubi [A] (verified)

Time = 0.20 (sec) , antiderivative size = 20, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.032, Rules used = {2019, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{x^2(af+bd)+ad+aex+x^4(bf+cd)+bex^3+cex^5+cfx^6}{a+bx^2+cx^4}dx$$

$$\Big\downarrow \text{ 2019}$$

$$\int \left(d+ex+fx^2\right)dx$$

$$\Big\downarrow \text{ 2009}$$

$$dx+\frac{ex^2}{2}+\frac{fx^3}{3}$$

input
```
Int[(a*d + a*e*x + (b*d + a*f)*x^2 + b*e*x^3 + (c*d + b*f)*x^4 + c*e*x^5 +
 c*f*x^6)/(a + b*x^2 + c*x^4),x]
```

output
```
d*x + (e*x^2)/2 + (f*x^3)/3
```

3.63.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

3.63.4 Maple [A] (verified)

Time = 0.02 (sec) , antiderivative size = 17, normalized size of antiderivative = 0.85

method	result	size
default	$dx + \frac{1}{2} e\, x^2 + \frac{1}{3} f\, x^3$	17
norman	$dx + \frac{1}{2} e\, x^2 + \frac{1}{3} f\, x^3$	17
risch	$dx + \frac{1}{2} e\, x^2 + \frac{1}{3} f\, x^3$	17
parallelrisch	$dx + \frac{1}{2} e\, x^2 + \frac{1}{3} f\, x^3$	17
parts	$dx + \frac{1}{2} e\, x^2 + \frac{1}{3} f\, x^3$	17
gosper	$\frac{x(2f\, x^2+3ex+6d)}{6}$	18

input
```
int((d*a+a*e*x+(a*f+b*d)*x^2+e*x^3*b+(b*f+c*d)*x^4+c*e*x^5+c*f*x^6)/(c*x^4
+b*x^2+a),x,method=_RETURNVERBOSE)
```

output
```
d*x+1/2*e*x^2+1/3*f*x^3
```

3.63.5 Fricas [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 16, normalized size of antiderivative = 0.80

$$\int \frac{ad + aex + (bd + af)x^2 + bex^3 + (cd + bf)x^4 + cex^5 + cfx^6}{a + bx^2 + cx^4}\, dx = \frac{1}{3} fx^3 + \frac{1}{2} ex^2 + dx$$

input
```
integrate((a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4+c*e*x^5+c*f*x^6)/
(c*x^4+b*x^2+a),x, algorithm="fricas")
```

output
```
1/3*f*x^3 + 1/2*e*x^2 + d*x
```

3.63.6 Sympy [A] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 15, normalized size of antiderivative = 0.75

$$\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{a+bx^2+cx^4}\,dx = dx+\frac{ex^2}{2}+\frac{fx^3}{3}$$

input
```
integrate((a*d+a*e*x+(a*f+b*d)*x**2+b*e*x**3+(b*f+c*d)*x**4+c*e*x**5+c*f*x
**6)/(c*x**4+b*x**2+a),x)
```

output
```
d*x + e*x**2/2 + f*x**3/3
```

3.63.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 16, normalized size of antiderivative = 0.80

$$\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{a+bx^2+cx^4}\,dx = \frac{1}{3}\,fx^3+\frac{1}{2}\,ex^2+dx$$

input
```
integrate((a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4+c*e*x^5+c*f*x^6)/
(c*x^4+b*x^2+a),x, algorithm="maxima")
```

output
```
1/3*f*x^3 + 1/2*e*x^2 + d*x
```

3.63.8 Giac [A] (verification not implemented)

Time = 0.62 (sec) , antiderivative size = 16, normalized size of antiderivative = 0.80

$$\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{a+bx^2+cx^4}\,dx = \frac{1}{3}\,fx^3+\frac{1}{2}\,ex^2+dx$$

input
```
integrate((a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4+c*e*x^5+c*f*x^6)/
(c*x^4+b*x^2+a),x, algorithm="giac")
```

output
```
1/3*f*x^3 + 1/2*e*x^2 + d*x
```

3.63.9 Mupad [B] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 16, normalized size of antiderivative = 0.80

$$\int \frac{ad + aex + (bd + af)x^2 + bex^3 + (cd + bf)x^4 + cex^5 + cfx^6}{a + bx^2 + cx^4}\,dx = \frac{f\,x^3}{3} + \frac{e\,x^2}{2} + d\,x$$

input
```
int((a*d + x^2*(b*d + a*f) + x^4*(c*d + b*f) + a*e*x + b*e*x^3 + c*e*x^5 +
 c*f*x^6)/(a + b*x^2 + c*x^4),x)
```

output
```
d*x + (e*x^2)/2 + (f*x^3)/3
```

3.64 $\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^2}\,dx$

3.64.1 Optimal result

Integrand size = 63, antiderivative size = 211

$$\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^2}\,dx = \frac{\left(f+\frac{2cd-bf}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{\sqrt{2}\sqrt{c}\sqrt{b-\sqrt{b^2-4ac}}} + \frac{\left(f-\frac{2cd-bf}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{\sqrt{2}\sqrt{c}\sqrt{b+\sqrt{b^2-4ac}}} - \frac{e\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\sqrt{b^2-4ac}}$$

output
```
-e*arctanh((2*c*x^2+b)/(-4*a*c+b^2)^(1/2))/(-4*a*c+b^2)^(1/2)+1/2*arctan(x
*2^(1/2)*c^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2))*(f+(-b*f+2*c*d)/(-4*a*c+b^2
)^(1/2))*2^(1/2)/c^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2)+1/2*arctan(x*2^(1/2)
*c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2))*(f+(b*f-2*c*d)/(-4*a*c+b^2)^(1/2))*
2^(1/2)/c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2)
```

3.64.2 Mathematica [A] (verified)

Time = 0.14 (sec) , antiderivative size = 234, normalized size of antiderivative = 1.11

$$\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^2}\,dx = \frac{\frac{\sqrt{2}\left(2cd+\left(-b+\sqrt{b^2-4ac}\right)f\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{\sqrt{c}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{2}\left(-2cd+\left(b+\sqrt{b^2-4ac}\right)f\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{\sqrt{c}\sqrt{b+\sqrt{b^2-4ac}}}+e\log\left(-b+\sqrt{b^2-4ac}\right.}{2\sqrt{b^2-4ac}}$$

input
```
Integrate[(a*d + a*e*x + (b*d + a*f)*x^2 + b*e*x^3 + (c*d + b*f)*x^4 + c*e
*x^5 + c*f*x^6)/(a + b*x^2 + c*x^4)^2,x]
```

output
```
((Sqrt[2]*(2*c*d + (-b + Sqrt[b^2 - 4*a*c])*f)*ArcTan[(Sqrt[2]*Sqrt[c]*x)/
Sqrt[b - Sqrt[b^2 - 4*a*c]]])/(Sqrt[c]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + (Sqr
t[2]*(-2*c*d + (b + Sqrt[b^2 - 4*a*c])*f)*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[
b + Sqrt[b^2 - 4*a*c]]])/(Sqrt[c]*Sqrt[b + Sqrt[b^2 - 4*a*c]]) + e*Log[-b
+ Sqrt[b^2 - 4*a*c] - 2*c*x^2] - e*Log[b + Sqrt[b^2 - 4*a*c] + 2*c*x^2])/(
2*Sqrt[b^2 - 4*a*c])
```

3.64.3 Rubi [A] (verified)

Time = 0.48 (sec) , antiderivative size = 211, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 8, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.127, Rules used = {2019, 2202, 27, 1432, 1083, 219, 1480, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{x^2(af+bd)+ad+aex+x^4(bf+cd)+bex^3+cex^5+cfx^6}{(a+bx^2+cx^4)^2}dx$$

$$\Big\downarrow \text{ 2019}$$

$$\int \frac{d+ex+fx^2}{a+bx^2+cx^4}dx$$

$$\Big\downarrow \text{ 2202}$$

$$\int \frac{fx^2+d}{cx^4+bx^2+a}dx+\int \frac{ex}{cx^4+bx^2+a}dx$$

$$\Big\downarrow \text{ 27}$$

$$\int \frac{fx^2+d}{cx^4+bx^2+a}dx+e\int \frac{x}{cx^4+bx^2+a}dx$$

$$\Big\downarrow \text{1432}$$

$$\int \frac{fx^2+d}{cx^4+bx^2+a}dx+\frac{1}{2}e\int \frac{1}{cx^4+bx^2+a}dx^2$$

$$\Big\downarrow \text{1083}$$

$$\int \frac{fx^2+d}{cx^4+bx^2+a}dx-e\int \frac{1}{-x^4+b^2-4ac}d\left(2cx^2+b\right)$$

$$\Big\downarrow \text{219}$$

$$\int \frac{fx^2+d}{cx^4+bx^2+a}dx-\frac{e\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\sqrt{b^2-4ac}}$$

$$\Big\downarrow \text{1480}$$

$$\frac{1}{2}\left(\frac{2cd-bf}{\sqrt{b^2-4ac}}+f\right)\int \frac{1}{cx^2+\frac{1}{2}\left(b-\sqrt{b^2-4ac}\right)}dx+$$

$$\frac{1}{2}\left(f-\frac{2cd-bf}{\sqrt{b^2-4ac}}\right)\int \frac{1}{cx^2+\frac{1}{2}\left(b+\sqrt{b^2-4ac}\right)}dx-\frac{e\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\sqrt{b^2-4ac}}$$

$$\Big\downarrow \text{218}$$

$$\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{2cd-bf}{\sqrt{b^2-4ac}}+f\right)}{\sqrt{2}\sqrt{c}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(f-\frac{2cd-bf}{\sqrt{b^2-4ac}}\right)}{\sqrt{2}\sqrt{c}\sqrt{\sqrt{b^2-4ac}+b}}-\frac{e\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\sqrt{b^2-4ac}}$$

input
```
Int[(a*d + a*e*x + (b*d + a*f)*x^2 + b*e*x^3 + (c*d + b*f)*x^4 + c*e*x^5 +
 c*f*x^6)/(a + b*x^2 + c*x^4)^2,x]
```

output
```
((f + (2*c*d - b*f)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b -
 Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[c]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + ((f
- (2*c*d - b*f)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b + Sqr
t[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[c]*Sqrt[b + Sqrt[b^2 - 4*a*c]]) - (e*ArcTa
nh[(b + 2*c*x^2)/Sqrt[b^2 - 4*a*c]])/Sqrt[b^2 - 4*a*c]
```

3.64. $\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^2}\,dx$

3.64.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.64. $\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^2}\,dx$

3.64.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.06 (sec) , antiderivative size = 48, normalized size of antiderivative = 0.23

method	result
risch	$\frac{\left(\sum_{\_R=\mathrm{RootOf}\left(c\_Z^4+\_Z^2b+a\right)}\frac{\left(\_R^2f+\_Re+d\right)\ln\left(x-\_R\right)}{2c\_R^3+\_Rb}\right)}{2}$
default	$4c\left(-\frac{\sqrt{-4ac+b^2}\left(-\frac{e\ln\left(2cx^2+\sqrt{-4ac+b^2}+b\right)}{2}+\frac{\left(f\sqrt{-4ac+b^2}+bf-2cd\right)\sqrt{2}\arctan\left(\frac{cx\sqrt{2}}{\sqrt{\left(b+\sqrt{-4ac+b^2}\right)c}}\right)}{2\sqrt{\left(b+\sqrt{-4ac+b^2}\right)c}}\right)}{4c(4ac-b^2)}-\frac{\sqrt{-4ac+b^2}\left(\frac{e\ln\left(-2c\right.}{}\right.}{}\right.$

input
```
int((d*a+a*e*x+(a*f+b*d)*x^2+e*x^3*b+(b*f+c*d)*x^4+c*e*x^5+c*f*x^6)/(c*x^4
+b*x^2+a)^2,x,method=_RETURNVERBOSE)
```

output
```
1/2*sum((_R^2*f+_R*e+d)/(2*_R^3*c+_R*b)*ln(x-_R),_R=RootOf(_Z^4*c+_Z^2*b+a
))
```

3.64.5 Fricas [C] (verification not implemented)

Result contains complex when optimal does not.

Time = 18.53 (sec) , antiderivative size = 723401, normalized size of antiderivative = 3428.44

$$\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^2}\,dx = \text{Too large to display}$$

input
```
integrate((a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4+c*e*x^5+c*f*x^6)/
(c*x^4+b*x^2+a)^2,x, algorithm="fricas")
```

output
```
Too large to include
```

3.64.6 Sympy [F(-1)]

Timed out.

$$\int \frac{ad + aex + (bd + af)x^2 + bex^3 + (cd + bf)x^4 + cex^5 + cfx^6}{(a + bx^2 + cx^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((a*d+a*e*x+(a*f+b*d)*x**2+b*e*x**3+(b*f+c*d)*x**4+c*e*x**5+c*f*x
**6)/(c*x**4+b*x**2+a)**2,x)
```

output
```
Timed out
```

3.64.7 Maxima [F]

$$\int \frac{ad + aex + (bd + af)x^2 + bex^3 + (cd + bf)x^4 + cex^5 + cfx^6}{(a + bx^2 + cx^4)^2}\,dx = \int \frac{cfx^6 + cex^5 + bex^3 + (cd + bf)x^4 + aex + (bd + af)x^2 + ad}{(cx^4 + bx^2 + a)^2}\,dx$$

input
```
integrate((a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4+c*e*x^5+c*f*x^6)/
(c*x^4+b*x^2+a)^2,x, algorithm="maxima")
```

output
```
integrate((c*f*x^6 + c*e*x^5 + b*e*x^3 + (c*d + b*f)*x^4 + a*e*x + (b*d +
a*f)*x^2 + a*d)/(c*x^4 + b*x^2 + a)^2, x)
```

3.64.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 1616 vs. $2(173) = 346$.

Time = 1.23 (sec) , antiderivative size = 1616, normalized size of antiderivative = 7.66

$$\int \frac{ad + aex + (bd + af)x^2 + bex^3 + (cd + bf)x^4 + cex^5 + cfx^6}{(a + bx^2 + cx^4)^2}\,dx = \text{Too large to display}$$

input
```
integrate((a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4+c*e*x^5+c*f*x^6)/
(c*x^4+b*x^2+a)^2,x, algorithm="giac")
```

output
```
-1/2*(b^2*c^2 - 4*a*c^3 - 2*b*c^3 + c^4)*sqrt(b^2 - 4*a*c)*e*log(x^2 + 1/2
*(b + sqrt(b^2 - 4*a*c))/c)/((b^4 - 8*a*b^2*c - 2*b^3*c + 16*a^2*c^2 + 8*a
*b*c^2 + b^2*c^2 - 4*a*c^3)*c^2) - 1/2*(b^2*c^2 - 4*a*c^3 - 2*b*c^3 + c^4)
*sqrt(b^2 - 4*a*c)*e*log(x^2 + 1/2*(b - sqrt(b^2 - 4*a*c))/c)/((b^4 - 8*a*
b^2*c - 2*b^3*c + 16*a^2*c^2 + 8*a*b*c^2 + b^2*c^2 - 4*a*c^3)*c^2) + 1/4*(
(sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^4 - 8*sqrt(2)*sqrt(b*c + sqrt(b
^2 - 4*a*c)*c)*a*b^2*c - 2*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^3*c -
 2*b^4*c + 16*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*c^2 + 8*sqrt(2)*
sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c^2 + sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*
a*c)*c)*b^2*c^2 + 16*a*b^2*c^2 - 2*b^3*c^2 - 4*sqrt(2)*sqrt(b*c + sqrt(b^2
 - 4*a*c)*c)*a*c^3 - 32*a^2*c^3 + 8*a*b*c^3 + sqrt(2)*sqrt(b^2 - 4*a*c)*sq
rt(b*c + sqrt(b^2 - 4*a*c)*c)*b^3 - 4*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c +
 sqrt(b^2 - 4*a*c)*c)*a*b*c - 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(
b^2 - 4*a*c)*c)*b^2*c + sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*
a*c)*c)*b*c^2 + 2*(b^2 - 4*a*c)*b^2*c - 8*(b^2 - 4*a*c)*a*c^2 + 2*(b^2 - 4
*a*c)*b*c^2)*d + 2*(2*a*b^2*c^2 - 8*a^2*c^3 - sqrt(2)*sqrt(b^2 - 4*a*c)*sq
rt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^2 + 4*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c
 + sqrt(b^2 - 4*a*c)*c)*a^2*c + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqr
t(b^2 - 4*a*c)*c)*a*b*c - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 -
4*a*c)*c)*a*c^2 - 2*(b^2 - 4*a*c)*a*c^2)*f)*arctan(2*sqrt(1/2)*x/sqrt((...
```

3.64.9 Mupad [B] (verification not implemented)

Time = 0.64 (sec) , antiderivative size = 3942, normalized size of antiderivative = 18.68

$$\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^2}\,dx = \text{Too large to display}$$

input
```
int((a*d + x^2*(b*d + a*f) + x^4*(c*d + b*f) + a*e*x + b*e*x^3 + c*e*x^5 +
 c*f*x^6)/(a + b*x^2 + c*x^4)^2,x)
```

output
```
symsum(log(c^2*d*e^2 - c^2*d^2*f + c^2*e^3*x - a*c*f^3 - 8*root(16*a*b^4*c
*z^4 - 128*a^2*b^2*c^2*z^4 + 256*a^3*c^3*z^4 - 16*a*b^2*c*d*f*z^2 + 64*a^2
*c^2*d*f*z^2 - 16*a^2*b*c*f^2*z^2 - 8*a*b^2*c*e^2*z^2 - 16*a*b*c^2*d^2*z^2
 + 32*a^2*c^2*e^2*z^2 + 4*b^3*c*d^2*z^2 + 4*a*b^3*f^2*z^2 + 16*a^2*c*e*f^2
*z + 4*b^2*c*d^2*e*z - 4*a*b^2*e*f^2*z - 16*a*c^2*d^2*e*z - 4*a*c*d*e^2*f
+ 2*a*c*d^2*f^2 - 2*b*c*d^3*f - 2*a*b*d*f^3 + b*c*d^2*e^2 + a*b*e^2*f^2 +
a*c*e^4 + b^2*d^2*f^2 + c^2*d^4 + a^2*f^4, z, k)^3*b^3*c^2*x + b*c*d*f^2 -
 16*root(16*a*b^4*c*z^4 - 128*a^2*b^2*c^2*z^4 + 256*a^3*c^3*z^4 - 16*a*b^2
*c*d*f*z^2 + 64*a^2*c^2*d*f*z^2 - 16*a^2*b*c*f^2*z^2 - 8*a*b^2*c*e^2*z^2 -
 16*a*b*c^2*d^2*z^2 + 32*a^2*c^2*e^2*z^2 + 4*b^3*c*d^2*z^2 + 4*a*b^3*f^2*z
^2 + 16*a^2*c*e*f^2*z + 4*b^2*c*d^2*e*z - 4*a*b^2*e*f^2*z - 16*a*c^2*d^2*e
*z - 4*a*c*d*e^2*f + 2*a*c*d^2*f^2 - 2*b*c*d^3*f - 2*a*b*d*f^3 + b*c*d^2*e
^2 + a*b*e^2*f^2 + a*c*e^4 + b^2*d^2*f^2 + c^2*d^4 + a^2*f^4, z, k)^2*a*c^
3*d - 4*root(16*a*b^4*c*z^4 - 128*a^2*b^2*c^2*z^4 + 256*a^3*c^3*z^4 - 16*a
*b^2*c*d*f*z^2 + 64*a^2*c^2*d*f*z^2 - 16*a^2*b*c*f^2*z^2 - 8*a*b^2*c*e^2*z
^2 - 16*a*b*c^2*d^2*z^2 + 32*a^2*c^2*e^2*z^2 + 4*b^3*c*d^2*z^2 + 4*a*b^3*f
^2*z^2 + 16*a^2*c*e*f^2*z + 4*b^2*c*d^2*e*z - 4*a*b^2*e*f^2*z - 16*a*c^2*d
^2*e*z - 4*a*c*d*e^2*f + 2*a*c*d^2*f^2 - 2*b*c*d^3*f - 2*a*b*d*f^3 + b*c*d
^2*e^2 + a*b*e^2*f^2 + a*c*e^4 + b^2*d^2*f^2 + c^2*d^4 + a^2*f^4, z, k)*c^
3*d^2*x + 4*root(16*a*b^4*c*z^4 - 128*a^2*b^2*c^2*z^4 + 256*a^3*c^3*z^4...
```

3.64. $\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^2}\,dx$

3.65 $\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^3}\,dx$

3.65.1 Optimal result

Integrand size = 63, antiderivative size = 368

$$\begin{aligned}&\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^3}\,dx\\&=-\frac{e(b+2cx^2)}{2(b^2-4ac)(a+bx^2+cx^4)}+\frac{x(b^2d-2acd-abf+c(bd-2af)x^2)}{2a(b^2-4ac)(a+bx^2+cx^4)}\\&+\frac{\sqrt{c}\left(bd-2af+\frac{b^2d-12acd+4abf}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{2\sqrt{2}a(b^2-4ac)\sqrt{b-\sqrt{b^2-4ac}}}\\&+\frac{\sqrt{c}\left(bd-2af-\frac{b^2d-12acd+4abf}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{2\sqrt{2}a(b^2-4ac)\sqrt{b+\sqrt{b^2-4ac}}}+\frac{2ce\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}\end{aligned}$$

output
```
-1/2*e*(2*c*x^2+b)/(-4*a*c+b^2)/(c*x^4+b*x^2+a)+1/2*x*(b^2*d-2*a*c*d-a*b*f
+c*(-2*a*f+b*d)*x^2)/a/(-4*a*c+b^2)/(c*x^4+b*x^2+a)+2*c*e*arctanh((2*c*x^2
+b)/(-4*a*c+b^2)^(1/2))/(-4*a*c+b^2)^(3/2)+1/4*arctan(x*2^(1/2)*c^(1/2)/(b
-(-4*a*c+b^2)^(1/2))^(1/2))*c^(1/2)*(b*d-2*a*f+(4*a*b*f-12*a*c*d+b^2*d)/(-
4*a*c+b^2)^(1/2))/a/(-4*a*c+b^2)*2^(1/2)/(b-(-4*a*c+b^2)^(1/2))^(1/2)+1/4*
arctan(x*2^(1/2)*c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2))*c^(1/2)*(b*d-2*a*f+
(-4*a*b*f+12*a*c*d-b^2*d)/(-4*a*c+b^2)^(1/2))/a/(-4*a*c+b^2)*2^(1/2)/(b+(-
4*a*c+b^2)^(1/2))^(1/2)
```

3.65.2 Mathematica [A] (verified)

Time = 0.74 (sec) , antiderivative size = 398, normalized size of antiderivative = 1.08

$$\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^3}\,dx$$
$$=\frac{1}{4}\left(\frac{2ab(e+fx)-2bdx(b+cx^2)+4acx(d+x(e+fx))}{a(-b^2+4ac)(a+bx^2+cx^4)}\right.$$
$$+\frac{\sqrt{2}\sqrt{c}\left(b^2d+b\left(\sqrt{b^2-4ac}d+4af\right)-2a\left(6cd+\sqrt{b^2-4ac}f\right)\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)}{a(b^2-4ac)^{3/2}\sqrt{b-\sqrt{b^2-4ac}}}$$
$$+\frac{\sqrt{2}\sqrt{c}\left(-b^2d+12acd+b\sqrt{b^2-4ac}d-4abf-2a\sqrt{b^2-4ac}f\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{a(b^2-4ac)^{3/2}\sqrt{b+\sqrt{b^2-4ac}}}$$
$$\left.-\frac{4ce\log\left(-b+\sqrt{b^2-4ac}-2cx^2\right)}{(b^2-4ac)^{3/2}}+\frac{4ce\log\left(b+\sqrt{b^2-4ac}+2cx^2\right)}{(b^2-4ac)^{3/2}}\right)$$

input
```
Integrate[(a*d + a*e*x + (b*d + a*f)*x^2 + b*e*x^3 + (c*d + b*f)*x^4 + c*e
*x^5 + c*f*x^6)/(a + b*x^2 + c*x^4)^3,x]
```

output
```
((2*a*b*(e + f*x) - 2*b*d*x*(b + c*x^2) + 4*a*c*x*(d + x*(e + f*x)))/(a*(-
b^2 + 4*a*c)*(a + b*x^2 + c*x^4)) + (Sqrt[2]*Sqrt[c]*(b^2*d + b*(Sqrt[b^2
- 4*a*c]*d + 4*a*f) - 2*a*(6*c*d + Sqrt[b^2 - 4*a*c]*f))*ArcTan[(Sqrt[2]*S
qrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]])/(a*(b^2 - 4*a*c)^(3/2)*Sqrt[b - Sq
rt[b^2 - 4*a*c]]) + (Sqrt[2]*Sqrt[c]*(-(b^2*d) + 12*a*c*d + b*Sqrt[b^2 - 4
*a*c]*d - 4*a*b*f - 2*a*Sqrt[b^2 - 4*a*c]*f)*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sq
rt[b + Sqrt[b^2 - 4*a*c]]])/(a*(b^2 - 4*a*c)^(3/2)*Sqrt[b + Sqrt[b^2 - 4*a
*c]]) - (4*c*e*Log[-b + Sqrt[b^2 - 4*a*c] - 2*c*x^2])/(b^2 - 4*a*c)^(3/2)
+ (4*c*e*Log[b + Sqrt[b^2 - 4*a*c] + 2*c*x^2])/(b^2 - 4*a*c)^(3/2))/4
```

3.65.3 Rubi [A] (verified)

Time = 0.71 (sec) , antiderivative size = 356, normalized size of antiderivative = 0.97, number of steps used = 12, number of rules used = 11, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.175, Rules used = {2019, 2202, 27, 1432, 1086, 1083, 219, 1492, 25, 1480, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

3.65. $\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^3}\,dx$

$$\int \frac{x^2(af+bd)+ad+aex+x^4(bf+cd)+bex^3+cex^5+cfx^6}{(a+bx^2+cx^4)^3}dx$$

$$\Big\downarrow \text{2019}$$

$$\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^2}dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^2}dx+\int \frac{ex}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \text{27}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^2}dx+e\int \frac{x}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \text{1432}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^2}dx+\frac{1}{2}e\int \frac{1}{(cx^4+bx^2+a)^2}dx^2$$

$$\Big\downarrow \text{1086}$$

$$\frac{1}{2}e\left(-\frac{2c\int \frac{1}{cx^4+bx^2+a}dx^2}{b^2-4ac}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)+\int \frac{fx^2+d}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \text{1083}$$

$$\frac{1}{2}e\left(\frac{4c\int \frac{1}{-x^4+b^2-4ac}d(2cx^2+b)}{b^2-4ac}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)+\int \frac{fx^2+d}{(cx^4+bx^2+a)^2}dx$$

$$\Big\downarrow \text{219}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^2}dx+\frac{1}{2}e\left(\frac{4c\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)$$

$$\Big\downarrow \text{1492}$$

$$-\frac{\int -\frac{db^2+afb+c(bd-2af)x^2-6acd}{cx^4+bx^2+a}dx}{2a(b^2-4ac)}+\frac{1}{2}e\left(\frac{4c\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)+\frac{x(cx^2(bd-2af)-abf-2acd+b^2d)}{2a(b^2-4ac)(a+bx^2+cx^4)}$$

$$\Big\downarrow \text{25}$$

3.65. $\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^3}dx$

$$\frac{\int \frac{db^2+afb+c(bd-2af)x^2-6acd}{cx^4+bx^2+a}dx}{2a\left(b^2-4ac\right)}+\frac{1}{2}e\left(\frac{4c\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{b+2cx^2}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

$$\Big\downarrow \text{1480}$$

$$\frac{\frac{1}{2}c\left(\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)\int \frac{1}{cx^2+\frac{1}{2}\left(b-\sqrt{b^2-4ac}\right)}dx+\frac{1}{2}c\left(-\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)\int \frac{1}{cx^2+\frac{1}{2}\left(b+\sqrt{b^2-4ac}\right)}dx}{2a\left(b^2-4ac\right)}+\frac{1}{2}e\left(\frac{4c\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{b+2cx^2}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

$$\Big\downarrow \text{218}$$

$$\frac{\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{4abf-12acd+b^2d}{\sqrt{b^2-4ac}}-2af+bd\right)}{\sqrt{2}\sqrt{\sqrt{b^2-4ac}+b}}}{2a\left(b^2-4ac\right)}+\frac{1}{2}e\left(\frac{4c\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{b+2cx^2}{\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}\right)+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{2a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)}$$

input
```
Int[(a*d + a*e*x + (b*d + a*f)*x^2 + b*e*x^3 + (c*d + b*f)*x^4 + c*e*x^5 +
 c*f*x^6)/(a + b*x^2 + c*x^4)^3,x]
```

output
```
(x*(b^2*d - 2*a*c*d - a*b*f + c*(b*d - 2*a*f)*x^2))/(2*a*(b^2 - 4*a*c)*(a
+ b*x^2 + c*x^4)) + ((Sqrt[c]*(b*d - 2*a*f + (b^2*d - 12*a*c*d + 4*a*b*f)/
Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[b^2 - 4*a*c]]]
)/(Sqrt[2]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + (Sqrt[c]*(b*d - 2*a*f - (b^2*d -
 12*a*c*d + 4*a*b*f)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b
+ Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[b + Sqrt[b^2 - 4*a*c]]))/(2*a*(b^2 -
4*a*c)) + (e*(-((b + 2*c*x^2)/((b^2 - 4*a*c)*(a + b*x^2 + c*x^4))) + (4*c*
ArcTan h[(b + 2*c*x^2)/Sqrt[b^2 - 4*a*c]])/(b^2 - 4*a*c)^(3/2)))/2
```

3.65. $\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{\left(a+bx^2+cx^4\right)^3}\,dx$

3.65.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1086
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] - Simp[2*c*((2*p +
 3)/((p + 1)*(b^2 - 4*a*c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; Fre
eQ[{a, b, c}, x] && ILtQ[p, -1]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

3.65. $\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^3}\,dx$

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.65.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.24 (sec) , antiderivative size = 232, normalized size of antiderivative = 0.63

method	result
risch	$\frac{\frac{c(2af-bd)x^3}{2a(4ac-b^2)}+\frac{c x^2 e}{4ac-b^2}+\frac{(abf+2acd-b^2d)x}{2a(4ac-b^2)}+\frac{be}{8ac-2b^2}}{c x^4+b x^2+a}+\frac{\left(\sum_{\_R=\mathrm{RootOf}(c\_Z^4+\_Z^2b+a)}\frac{\left(\frac{c(2af-bd)\_R^2}{a(4ac-b^2)}+\frac{4ce\_R}{4ac-b^2}-\frac{abf-6acd+b^2d}{a(4ac-b^2)}\right.}{2c\_R^3+\_Rb}\right.}{4}$
default	$16c^2\left(-\frac{-\frac{\frac{(-4acd\sqrt{-4ac+b^2}+b^2d\sqrt{-4ac+b^2}+8a^2cf-2ab^2f-4abcd+b^3d)x}{16ac}-\frac{e(4ac-b^2)}{8c}}{x^2+\frac{b}{2c}-\frac{\sqrt{-4ac+b^2}}{2c}}+\frac{2ae\sqrt{-4ac+b^2}\ln(-2cx^2+\sqrt{-4ac+b^2}-b)+\frac{(4\sqrt{}}{}}{}}{4c(4ac-b^2)^2}\right.$

3.65. $\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^3}\,dx$

input
```
int((d*a+a*e*x+(a*f+b*d)*x^2+e*x^3*b+(b*f+c*d)*x^4+c*e*x^5+c*f*x^6)/(c*x^4
+b*x^2+a)^3,x,method=_RETURNVERBOSE)
```

output
```
(1/2*c*(2*a*f-b*d)/a/(4*a*c-b^2)*x^3+c/(4*a*c-b^2)*x^2*e+1/2*(a*b*f+2*a*c*
d-b^2*d)/a/(4*a*c-b^2)*x+1/2/(4*a*c-b^2)*b*e)/(c*x^4+b*x^2+a)+1/4*sum((c*(
2*a*f-b*d)/a/(4*a*c-b^2)*_R^2+4*c/(4*a*c-b^2)*e*_R-(a*b*f-6*a*c*d+b^2*d)/a
/(4*a*c-b^2))/(2*_R^3*c+_R*b)*ln(x-_R),_R=RootOf(_Z^4*c+_Z^2*b+a))
```

3.65.5 Fricas [F(-1)]

Timed out.

$$\int \frac{ad + aex + (bd + af)x^2 + bex^3 + (cd + bf)x^4 + cex^5 + cfx^6}{(a + bx^2 + cx^4)^3} \, dx = \text{Timed out}$$

input
```
integrate((a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4+c*e*x^5+c*f*x^6)/
(c*x^4+b*x^2+a)^3,x, algorithm="fricas")
```

output
```
Timed out
```

3.65.6 Sympy [F(-1)]

Timed out.

$$\int \frac{ad + aex + (bd + af)x^2 + bex^3 + (cd + bf)x^4 + cex^5 + cfx^6}{(a + bx^2 + cx^4)^3} \, dx = \text{Timed out}$$

input
```
integrate((a*d+a*e*x+(a*f+b*d)*x**2+b*e*x**3+(b*f+c*d)*x**4+c*e*x**5+c*f*x
**6)/(c*x**4+b*x**2+a)**3,x)
```

output
```
Timed out
```

3.65.7 Maxima [F]

$$\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^3}\,dx = \int \frac{cfx^6+cex^5+bex^3+(cd+bf)x^4+aex+(bd+af)x^2+ad}{(cx^4+bx^2+a)^3}\,dx$$

input
```
integrate((a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4+c*e*x^5+c*f*x^6)/
(c*x^4+b*x^2+a)^3,x, algorithm="maxima")
```

output
```
-1/2*(2*a*c*e*x^2 - (b*c*d - 2*a*c*f)*x^3 + a*b*e + (a*b*f - (b^2 - 2*a*c)
*d)*x)/((a*b^2*c - 4*a^2*c^2)*x^4 + a^2*b^2 - 4*a^3*c + (a*b^3 - 4*a^2*b*c
)*x^2) - 1/2*integrate((4*a*c*e*x - a*b*f - (b*c*d - 2*a*c*f)*x^2 - (b^2 -
 6*a*c)*d)/(c*x^4 + b*x^2 + a), x)/(a*b^2 - 4*a^2*c)
```

3.65.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 5159 vs. $2(320) = 640$.

Time = 2.82 (sec) , antiderivative size = 5159, normalized size of antiderivative = 14.02

$$\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^3}\,dx = \text{Too large to display}$$

input
```
integrate((a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4+c*e*x^5+c*f*x^6)/
(c*x^4+b*x^2+a)^3,x, algorithm="giac")
```

output
```
1/2*(b*c*d*x^3 - 2*a*c*f*x^3 - 2*a*c*e*x^2 + b^2*d*x - 2*a*c*d*x - a*b*f*x
 - a*b*e)/((c*x^4 + b*x^2 + a)*(a*b^2 - 4*a^2*c)) - 1/16*((2*b^3*c^2 - 8*a
*b*c^3 - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^3 + 4
*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c + 2*sqrt(
2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^2*c - sqrt(2)*sqrt(
b^2 - 4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b*c^2 - 2*(b^2 - 4*a*c)*b*c^2
)*(a*b^2 - 4*a^2*c)^2*d - 2*(2*a*b^2*c^2 - 8*a^2*c^3 - sqrt(2)*sqrt(b^2 -
4*a*c)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^2 + 4*sqrt(2)*sqrt(b^2 - 4*a*c)
*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*c + 2*sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(
b*c + sqrt(b^2 - 4*a*c)*c)*a*b*c - sqrt(2)*sqrt(b^2 - 4*a*c)*sqrt(b*c + sq
rt(b^2 - 4*a*c)*c)*a*c^2 - 2*(b^2 - 4*a*c)*a*c^2)*(a*b^2 - 4*a^2*c)^2*f -
2*(sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^6 - 14*sqrt(2)*sqrt(b*c + s
qrt(b^2 - 4*a*c)*c)*a^2*b^4*c - 2*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*
a*b^5*c - 2*a*b^6*c + 64*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*b^2*c
^2 + 20*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^3*c^2 + sqrt(2)*sqrt
(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^4*c^2 + 28*a^2*b^4*c^2 - 96*sqrt(2)*sqrt(b
*c + sqrt(b^2 - 4*a*c)*c)*a^4*c^3 - 48*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c
)*c)*a^3*b*c^3 - 10*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^2*c^3 -
128*a^3*b^2*c^3 + 24*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^3*c^4 + 192
*a^4*c^4 + 2*(b^2 - 4*a*c)*a*b^4*c - 20*(b^2 - 4*a*c)*a^2*b^2*c^2 + 48*...
```

3.65.9 Mupad [B] (verification not implemented)

Time = 8.62 (sec) , antiderivative size = 4707, normalized size of antiderivative = 12.79

$$\int \frac{ad + aex + (bd + af)x^2 + bex^3 + (cd + bf)x^4 + cex^5 + cfx^6}{(a + bx^2 + cx^4)^3} \, dx = \text{Too large to display}$$

input
```
int((a*d + x^2*(b*d + a*f) + x^4*(c*d + b*f) + a*e*x + b*e*x^3 + c*e*x^5 +
 c*f*x^6)/(a + b*x^2 + c*x^4)^3,x)
```

output

```
symsum(log((5*b^3*c^4*d^3 + 8*a^3*c^4*f^3 - 96*a^2*c^5*d*e^2 + 72*a^2*c^5*
d^2*f - 3*b^4*c^3*d^2*f + 6*a^2*b^2*c^3*f^3 - 36*a*b*c^5*d^3 + 16*a*b^2*c^
4*d*e^2 + 18*a*b^2*c^4*d^2*f + 3*a*b^3*c^3*d*f^2 - 60*a^2*b*c^4*d*f^2 + 16
*a^2*b*c^4*e^2*f)/(8*(a^2*b^6 - 64*a^5*c^3 - 12*a^3*b^4*c + 48*a^4*b^2*c^2
)) - root(1572864*a^8*b^2*c^5*z^4 - 983040*a^7*b^4*c^4*z^4 + 327680*a^6*b^
6*c^3*z^4 - 61440*a^5*b^8*c^2*z^4 + 6144*a^4*b^10*c*z^4 - 1048576*a^9*c^6*
z^4 - 256*a^3*b^12*z^4 + 576*a^2*b^8*c*d*f*z^2 + 24576*a^5*b^2*c^4*d*f*z^2
 - 3072*a^3*b^6*c^2*d*f*z^2 + 2048*a^4*b^4*c^3*d*f*z^2 + 12288*a^6*b*c^4*f
^2*z^2 + 61440*a^5*b*c^5*d^2*z^2 - 49152*a^6*c^5*d*f*z^2 + 432*a*b^9*c*d^2
*z^2 - 8192*a^5*b^3*c^3*f^2*z^2 + 1536*a^4*b^5*c^2*f^2*z^2 + 24576*a^5*b^2
*c^4*e^2*z^2 - 6144*a^4*b^4*c^3*e^2*z^2 + 512*a^3*b^6*c^2*e^2*z^2 - 61440*
a^4*b^3*c^4*d^2*z^2 + 24064*a^3*b^5*c^3*d^2*z^2 - 4608*a^2*b^7*c^2*d^2*z^2
 - 32*a*b^10*d*f*z^2 - 32768*a^6*c^5*e^2*z^2 - 16*a^2*b^9*f^2*z^2 - 16*b^1
1*d^2*z^2 - 4096*a^4*b*c^4*d*e*f*z + 64*a*b^7*c*d*e*f*z + 3072*a^3*b^3*c^3
*d*e*f*z - 768*a^2*b^5*c^2*d*e*f*z + 32*a^2*b^6*c*e*f^2*z - 672*a*b^6*c^2*
d^2*e*z + 1536*a^4*b^2*c^3*e*f^2*z - 384*a^3*b^4*c^2*e*f^2*z - 15872*a^3*b
^2*c^4*d^2*e*z + 4992*a^2*b^4*c^3*d^2*e*z - 2048*a^5*c^4*e*f^2*z + 18432*a
^4*c^5*d^2*e*z + 32*b^8*c*d^2*e*z - 32*a*b^4*c^2*d*e^2*f + 192*a^2*b^2*c^3
*d*e^2*f - 192*a^3*b*c^3*e^2*f^2 + 198*a*b^4*c^2*d^2*f^2 + 144*a^2*b^3*c^2
*d*f^3 - 960*a^2*b*c^4*d^2*e^2 + 240*a*b^3*c^3*d^2*e^2 + 768*a^3*c^4*d*...
```

3.65. $\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^3}\,dx$

3.66 $\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^4}\,dx$

3.66.1 Optimal result

Integrand size = 63, antiderivative size = 621

$$\begin{aligned}
&\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^4}\,dx \\
&= -\frac{e(b+2cx^2)}{4(b^2-4ac)(a+bx^2+cx^4)^2} + \frac{x(b^2d-2acd-abf+c(bd-2af)x^2)}{4a(b^2-4ac)(a+bx^2+cx^4)^2} \\
&\quad + \frac{3ce(b+2cx^2)}{2(b^2-4ac)^2(a+bx^2+cx^4)} \\
&\quad + \frac{x(3b^4d-25ab^2cd+28a^2c^2d+ab^3f+8a^2bcf+c(3b^3d-24abcd+ab^2f+20a^2cf)x^2)}{8a^2(b^2-4ac)^2(a+bx^2+cx^4)} \\
&\quad + \frac{\sqrt{c}(3b^4d+b^3(3\sqrt{b^2-4ac}d+af)-4abc(6\sqrt{b^2-4ac}d+13af)-ab^2(30cd-\sqrt{b^2-4ac}f)+4a^2c(42}{8\sqrt{2}a^2(b^2-4ac)^{5/2}\sqrt{b-\sqrt{b^2-4ac}}} \\
&\quad + \frac{\sqrt{c}\left(3b^3d-24abcd+ab^2f+20a^2cf-\frac{3b^4d-30ab^2cd+168a^2c^2d+ab^3f-52a^2bcf}{\sqrt{b^2-4ac}}\right)\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b+\sqrt{b^2-4ac}}}\right)}{8\sqrt{2}a^2(b^2-4ac)^2\sqrt{b+\sqrt{b^2-4ac}}} \\
&\quad - \frac{6c^2e\,\mathrm{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{5/2}}
\end{aligned}$$

output
```
-1/4*e*(2*c*x^2+b)/(-4*a*c+b^2)/(c*x^4+b*x^2+a)^2+1/4*x*(b^2*d-2*a*c*d-a*b
*f+c*(-2*a*f+b*d)*x^2)/a/(-4*a*c+b^2)/(c*x^4+b*x^2+a)^2+3/2*c*e*(2*c*x^2+b
)/(-4*a*c+b^2)^2/(c*x^4+b*x^2+a)+1/8*x*(3*b^4*d-25*a*b^2*c*d+28*a^2*c^2*d+
a*b^3*f+8*a^2*b*c*f+c*(20*a^2*c*f+a*b^2*f-24*a*b*c*d+3*b^3*d)*x^2)/a^2/(-4
*a*c+b^2)^2/(c*x^4+b*x^2+a)-6*c^2*e*arctanh((2*c*x^2+b)/(-4*a*c+b^2)^(1/2)
)/(-4*a*c+b^2)^(5/2)+1/16*arctan(x*2^(1/2)*c^(1/2)/(b-(-4*a*c+b^2)^(1/2))^
(1/2))*c^(1/2)*(3*b^4*d+b^3*(a*f+3*d*(-4*a*c+b^2)^(1/2))-4*a*b*c*(13*a*f+6
*d*(-4*a*c+b^2)^(1/2))-a*b^2*(30*c*d-f*(-4*a*c+b^2)^(1/2))+4*a^2*c*(42*c*d
+5*f*(-4*a*c+b^2)^(1/2)))/a^2/(-4*a*c+b^2)^(5/2)*2^(1/2)/(b-(-4*a*c+b^2)^(
1/2))^(1/2)+1/16*arctan(x*2^(1/2)*c^(1/2)/(b+(-4*a*c+b^2)^(1/2))^(1/2))*c^
(1/2)*(3*b^3*d-24*a*b*c*d+a*b^2*f+20*a^2*c*f+(52*a^2*b*c*f-168*a^2*c^2*d-a
*b^3*f+30*a*b^2*c*d-3*b^4*d)/(-4*a*c+b^2)^(1/2))/a^2/(-4*a*c+b^2)^2*2^(1/2
)/(b+(-4*a*c+b^2)^(1/2))^(1/2)
```

3.66.2 Mathematica [A] (verified)

Time = 2.03 (sec) , antiderivative size = 625, normalized size of antiderivative = 1.01

$$\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^4}\,dx$$

$$= \frac{1}{16}\left(\frac{4ab(e+fx)-4bdx(b+cx^2)+8acx(d+x(e+fx))}{a(-b^2+4ac)(a+bx^2+cx^4)^2}\right.$$

$$+\frac{6b^3dx(b+cx^2)+2abx(-25bcd+b^2f-24c^2dx^2+bcfx^2)+8a^2c(b(3e+2fx)+cx(7d+6ex+5fx^2))}{a^2(b^2-4ac)^2(a+bx^2+cx^4)}$$

$$+\frac{\sqrt{2}\sqrt{c}\left(3b^4d+b^3\left(3\sqrt{b^2-4ac}d+af\right)-4abc\left(6\sqrt{b^2-4ac}d+13af\right)+ab^2\left(-30cd+\sqrt{b^2-4ac}f\right)+4a}{a^2(b^2-4ac)^{5/2}\sqrt{b-\sqrt{b^2-4ac}}}$$

$$+\frac{\sqrt{2}\sqrt{c}\left(-3b^4d+b^3\left(3\sqrt{b^2-4ac}d-af\right)+4abc\left(-6\sqrt{b^2-4ac}d+13af\right)+ab^2\left(30cd+\sqrt{b^2-4ac}f\right)+4}{a^2(b^2-4ac)^{5/2}\sqrt{b+\sqrt{b^2-4ac}}}$$

$$\left.+\frac{48c^2e\log\left(-b+\sqrt{b^2-4ac}-2cx^2\right)}{(b^2-4ac)^{5/2}}-\frac{48c^2e\log\left(b+\sqrt{b^2-4ac}+2cx^2\right)}{(b^2-4ac)^{5/2}}\right)$$

input
```
Integrate[(a*d + a*e*x + (b*d + a*f)*x^2 + b*e*x^3 + (c*d + b*f)*x^4 + c*e
*x^5 + c*f*x^6)/(a + b*x^2 + c*x^4)^4,x]
```

output
```
((4*a*b*(e + f*x) - 4*b*d*x*(b + c*x^2) + 8*a*c*x*(d + x*(e + f*x)))/(a*(-
b^2 + 4*a*c)*(a + b*x^2 + c*x^4)^2) + (6*b^3*d*x*(b + c*x^2) + 2*a*b*x*(-2
5*b*c*d + b^2*f - 24*c^2*d*x^2 + b*c*f*x^2) + 8*a^2*c*(b*(3*e + 2*f*x) + c
*x*(7*d + 6*e*x + 5*f*x^2)))/(a^2*(b^2 - 4*a*c)^2*(a + b*x^2 + c*x^4)) + (
Sqrt[2]*Sqrt[c]*(3*b^4*d + b^3*(3*Sqrt[b^2 - 4*a*c]*d + a*f) - 4*a*b*c*(6*
Sqrt[b^2 - 4*a*c]*d + 13*a*f) + a*b^2*(-30*c*d + Sqrt[b^2 - 4*a*c]*f) + 4*
a^2*c*(42*c*d + 5*Sqrt[b^2 - 4*a*c]*f))*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b
- Sqrt[b^2 - 4*a*c]]])/(a^2*(b^2 - 4*a*c)^(5/2)*Sqrt[b - Sqrt[b^2 - 4*a*c]
]) + (Sqrt[2]*Sqrt[c]*(-3*b^4*d + b^3*(3*Sqrt[b^2 - 4*a*c]*d - a*f) + 4*a*
b*c*(-6*Sqrt[b^2 - 4*a*c]*d + 13*a*f) + a*b^2*(30*c*d + Sqrt[b^2 - 4*a*c]*
f) + 4*a^2*c*(-42*c*d + 5*Sqrt[b^2 - 4*a*c]*f))*ArcTan[(Sqrt[2]*Sqrt[c]*x)
/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(a^2*(b^2 - 4*a*c)^(5/2)*Sqrt[b + Sqrt[b^2
- 4*a*c]]) + (48*c^2*e*Log[-b + Sqrt[b^2 - 4*a*c] - 2*c*x^2])/(b^2 - 4*a*c
)^(5/2) - (48*c^2*e*Log[b + Sqrt[b^2 - 4*a*c] + 2*c*x^2])/(b^2 - 4*a*c)^(5
/2))/16
```

3.66.3 Rubi [A] (verified)

Time = 1.11 (sec) , antiderivative size = 605, normalized size of antiderivative = 0.97, number of steps used = 15, number of rules used = 14, $\frac{\text{number of rules}}{\text{integrand size}} = 0.222$, Rules used = {2019, 2202, 27, 1432, 1086, 1086, 1083, 219, 1492, 25, 1492, 25, 1480, 218}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{x^2(af+bd)+ad+aex+x^4(bf+cd)+bex^3+cex^5+cfx^6}{(a+bx^2+cx^4)^4}\,dx$$

$$\Big\downarrow \text{ 2019}$$

$$\int \frac{d+ex+fx^2}{(a+bx^2+cx^4)^3}dx$$

$$\Big\downarrow \text{ 2202}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^3}dx+\int \frac{ex}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{ 27}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^3}dx+e\int \frac{x}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{ 1432}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^3}dx+\frac{1}{2}e\int \frac{1}{(cx^4+bx^2+a)^3}dx^2$$

$$\Big\downarrow \text{1086}$$

$$\frac{1}{2}e\left(-\frac{3c\int \frac{1}{(cx^4+bx^2+a)^2}dx^2}{b^2-4ac}-\frac{b+2cx^2}{2(b^2-4ac)(a+bx^2+cx^4)^2}\right)+\int \frac{fx^2+d}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{1086}$$

$$\frac{1}{2}e\left(-\frac{3c\left(-\frac{2c\int \frac{1}{cx^4+bx^2+a}dx^2}{b^2-4ac}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2(b^2-4ac)(a+bx^2+cx^4)^2}\right)+\int \frac{fx^2+d}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{1083}$$

$$\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\int \frac{1}{-x^4+b^2-4ac}d(2cx^2+b)}{b^2-4ac}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2(b^2-4ac)(a+bx^2+cx^4)^2}\right)+\int \frac{fx^2+d}{(cx^4+bx^2+a)^3}dx$$

$$\Big\downarrow \text{219}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^3}dx+\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2(b^2-4ac)(a+bx^2+cx^4)^2}\right)$$

$$\Big\downarrow \text{1492}$$

3.66. $\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^4}\,dx$

$$-\frac{\int -\frac{3db^2+afb+5c(bd-2af)x^2-14acd}{(cx^4+bx^2+a)^2}dx}{4a\left(b^2-4ac\right)}+\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$$\Big\downarrow \; 25$$

$$\frac{\int \frac{3db^2+afb+5c(bd-2af)x^2-14acd}{(cx^4+bx^2+a)^2}dx}{4a\left(b^2-4ac\right)}+\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$$\Big\downarrow \; 1492$$

$$\frac{\frac{x\left(cx^2\left(20a^2cf+ab^2f-24abcd+3b^3d\right)+8a^2bcf+28a^2c^2d+ab^3f-25ab^2cd+3b^4d\right)}{2a(b^2-4ac)(a+bx^2+cx^4)}-\frac{\int -\frac{3db^4+afb^3-27acdb^2-16a^2cfb+c\left(3db^3+afb^2-24acdb+20a^2cf\right)}{cx^4+bx^2+a}}{2a(b^2-4ac)}}{4a\left(b^2-4ac\right)}+\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\operatorname{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$$\Big\downarrow \; 25$$

3.66. $\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^4}\,dx$

$$\frac{\frac{\int \frac{3db^4+afb^3-27acdb^2-16a^2cfb+c\left(3db^3+afb^2-24acdb+20a^2cf\right)x^2+84a^2c^2d}{cx^4+bx^2+a}dx}{2a(b^2-4ac)}+\frac{x\left(cx^2\left(20a^2cf+ab^2f-24abcd+3b^3d\right)+8a^2bcf+28a^2c^2d+ab^3f-25}{2a(b^2-4ac)(a+bx^2+cx^4)}}{4a\left(b^2-4ac\right)}$$

$$\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)+$$

$$\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$\downarrow$ 1480

$$\frac{\frac{\frac{1}{2}c\left(\frac{-52a^2bcf+168a^2c^2d+ab^3f-30ab^2cd+3b^4d}{\sqrt{b^2-4ac}}+20a^2cf+ab^2f-24abcd+3b^3d\right)\int\frac{1}{cx^2+\frac{1}{2}\left(b-\sqrt{b^2-4ac}\right)}dx+\frac{1}{2}c\left(-\frac{-52a^2bcf+168a^2c^2d+ab^3f-30ab^2cd+}{\sqrt{b^2-4ac}}\right.}{2a(b^2-4ac)}}{4a\left(b^2\right.}$$

$$\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)+$$

$$\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

$\downarrow$ 218

$$\frac{\frac{\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{b-\sqrt{b^2-4ac}}}\right)\left(\frac{-52a^2bcf+168a^2c^2d+ab^3f-30ab^2cd+3b^4d}{\sqrt{b^2-4ac}}+20a^2cf+ab^2f-24abcd+3b^3d\right)}{\sqrt{2}\sqrt{b-\sqrt{b^2-4ac}}}+\frac{\sqrt{c}\arctan\left(\frac{\sqrt{2}\sqrt{c}x}{\sqrt{\sqrt{b^2-4ac}+b}}\right)\left(-\frac{-52a^2bcf+168a^2c^2}{\sqrt{b}}\right.}{\sqrt{2}\sqrt{}}}{2a(b^2-4ac)}}{4a\left(b^2-4a\right.}$$

$$\frac{1}{2}e\left(-\frac{3c\left(\frac{4c\,\text{arctanh}\left(\frac{b+2cx^2}{\sqrt{b^2-4ac}}\right)}{\left(b^2-4ac\right)^{3/2}}-\frac{b+2cx^2}{(b^2-4ac)(a+bx^2+cx^4)}\right)}{b^2-4ac}-\frac{b+2cx^2}{2\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}\right)+$$

$$\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{4a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^2}$$

3.66. $\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{\left(a+bx^2+cx^4\right)^4}\,dx$

input
```
Int[(a*d + a*e*x + (b*d + a*f)*x^2 + b*e*x^3 + (c*d + b*f)*x^4 + c*e*x^5 +
 c*f*x^6)/(a + b*x^2 + c*x^4)^4,x]
```

output
```
(x*(b^2*d - 2*a*c*d - a*b*f + c*(b*d - 2*a*f)*x^2))/(4*a*(b^2 - 4*a*c)*(a
+ b*x^2 + c*x^4)^2) + ((x*(3*b^4*d - 25*a*b^2*c*d + 28*a^2*c^2*d + a*b^3*f
 + 8*a^2*b*c*f + c*(3*b^3*d - 24*a*b*c*d + a*b^2*f + 20*a^2*c*f)*x^2))/(2*
a*(b^2 - 4*a*c)*(a + b*x^2 + c*x^4)) + ((Sqrt[c]*(3*b^3*d - 24*a*b*c*d + a
*b^2*f + 20*a^2*c*f + (3*b^4*d - 30*a*b^2*c*d + 168*a^2*c^2*d + a*b^3*f -
52*a^2*b*c*f)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x)/Sqrt[b - Sqrt[
b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[b - Sqrt[b^2 - 4*a*c]]) + (Sqrt[c]*(3*b^3*d
- 24*a*b*c*d + a*b^2*f + 20*a^2*c*f - (3*b^4*d - 30*a*b^2*c*d + 168*a^2*c^
2*d + a*b^3*f - 52*a^2*b*c*f)/Sqrt[b^2 - 4*a*c])*ArcTan[(Sqrt[2]*Sqrt[c]*x
)/Sqrt[b + Sqrt[b^2 - 4*a*c]]])/(Sqrt[2]*Sqrt[b + Sqrt[b^2 - 4*a*c]]))/(2*
a*(b^2 - 4*a*c)))/(4*a*(b^2 - 4*a*c)) + (e*(-1/2*(b + 2*c*x^2)/((b^2 - 4*a
*c)*(a + b*x^2 + c*x^4)^2) - (3*c*(-((b + 2*c*x^2)/((b^2 - 4*a*c)*(a + b*x
^2 + c*x^4))) + (4*c*ArcTanh[(b + 2*c*x^2)/Sqrt[b^2 - 4*a*c]])/(b^2 - 4*a*
c)^(3/2)))/(b^2 - 4*a*c)))/2
```

3.66.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 218
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1083
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Simp[-2   Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x]
```

rule 1086
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] - Simp[2*c*((2*p +
 3)/((p + 1)*(b^2 - 4*a*c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; Fre
eQ[{a, b, c}, x] && ILtQ[p, -1]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 1480
```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[(e/2 + (2*c*d - b*e)/(2*q))   Int[1/(
b/2 - q/2 + c*x^2), x], x] + Simp[(e/2 - (2*c*d - b*e)/(2*q))   Int[1/(b/2
+ q/2 + c*x^2), x], x]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
 && NeQ[c*d^2 - a*e^2, 0] && PosQ[b^2 - 4*a*c]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.66. $\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^4}\,dx$

3.66.4 Maple [C] (verified)

Result contains higher order function than in optimal. Order 9 vs. order 3.

Time = 0.56 (sec) , antiderivative size = 607, normalized size of antiderivative = 0.98

method	result
risch	$\frac{\frac{c^{2}\left(20a^{2}cf+a\,b^{2}f-24abcd+3b^{3}d\right)x^{7}}{8a^{2}\left(16a^{2}c^{2}-8a\,b^{2}c+b^{4}\right)}+\frac{3c^{3}e\,x^{6}}{16a^{2}c^{2}-8a\,b^{2}c+b^{4}}+\frac{c\left(28a^{2}bcf+28a^{2}c^{2}d+2a\,b^{3}f-49a\,b^{2}cd+6d\,b^{4}\right)x^{5}}{8a^{2}\left(16a^{2}c^{2}-8a\,b^{2}c+b^{4}\right)}+\frac{9b\,c^{2}e\,x^{4}}{2\left(16a^{2}c^{2}-8a\,b^{2}c+b^{4}\right)}+\frac{\left(36a^{3}c^{2}\ldots\right.}{\ldots}}{\left(c\,x^{4}+b\,x\ldots\right.}$
default	Expression too large to display

input

```
int((d*a+a*e*x+(a*f+b*d)*x^2+e*x^3*b+(b*f+c*d)*x^4+c*e*x^5+c*f*x^6)/(c*x^4
+b*x^2+a)^4,x,method=_RETURNVERBOSE)
```

output

```
(1/8*c^2*(20*a^2*c*f+a*b^2*f-24*a*b*c*d+3*b^3*d)/a^2/(16*a^2*c^2-8*a*b^2*c
+b^4)*x^7+3*c^3*e/(16*a^2*c^2-8*a*b^2*c+b^4)*x^6+1/8/a^2*c*(28*a^2*b*c*f+2
8*a^2*c^2*d+2*a*b^3*f-49*a*b^2*c*d+6*b^4*d)/(16*a^2*c^2-8*a*b^2*c+b^4)*x^5
+9/2*b*c^2*e/(16*a^2*c^2-8*a*b^2*c+b^4)*x^4+1/8*(36*a^3*c^2*f+5*a^2*b^2*c*
f-4*a^2*b*c^2*d+a*b^4*f-20*a*b^3*c*d+3*b^5*d)/a^2/(16*a^2*c^2-8*a*b^2*c+b^
4)*x^3+(5*a*c+b^2)*c*e/(16*a^2*c^2-8*a*b^2*c+b^4)*x^2+1/8*(16*a^2*b*c*f+44
*a^2*c^2*d-a*b^3*f-37*a*b^2*c*d+5*b^4*d)/(16*a^2*c^2-8*a*b^2*c+b^4)/a*x+1/
4*b*(10*a*c-b^2)*e/(16*a^2*c^2-8*a*b^2*c+b^4))/(c*x^4+b*x^2+a)^2+1/16*sum(
(c*(20*a^2*c*f+a*b^2*f-24*a*b*c*d+3*b^3*d)/a^2/(16*a^2*c^2-8*a*b^2*c+b^4)*
_R^2+48*c^2*e/(16*a^2*c^2-8*a*b^2*c+b^4)*_R-(16*a^2*b*c*f-84*a^2*c^2*d-a*b
^3*f+27*a*b^2*c*d-3*b^4*d)/a^2/(16*a^2*c^2-8*a*b^2*c+b^4))/(2*_R^3*c+_R*b)
*ln(x-_R),_R=RootOf(_Z^4*c+_Z^2*b+a))
```

3.66.5 Fricas [F(-1)]

Timed out.

$$\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^4}\,dx=\text{Timed out}$$

input

```
integrate((a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4+c*e*x^5+c*f*x^6)/
(c*x^4+b*x^2+a)^4,x, algorithm="fricas")
```

output

```
Timed out
```

3.66.6 Sympy [F(-1)]

Timed out.

$$\int \frac{ad + aex + (bd + af)x^2 + bex^3 + (cd + bf)x^4 + cex^5 + cfx^6}{(a + bx^2 + cx^4)^4} dx = \text{Timed out}$$

input
```
integrate((a*d+a*e*x+(a*f+b*d)*x**2+b*e*x**3+(b*f+c*d)*x**4+c*e*x**5+c*f*x
**6)/(c*x**4+b*x**2+a)**4,x)
```

output
```
Timed out
```

3.66.7 Maxima [F]

$$\int \frac{ad + aex + (bd + af)x^2 + bex^3 + (cd + bf)x^4 + cex^5 + cfx^6}{(a + bx^2 + cx^4)^4} dx$$
$$= \int \frac{cfx^6 + cex^5 + bex^3 + (cd + bf)x^4 + aex + (bd + af)x^2 + ad}{(cx^4 + bx^2 + a)^4} dx$$

input
```
integrate((a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4+c*e*x^5+c*f*x^6)/
(c*x^4+b*x^2+a)^4,x, algorithm="maxima")
```

output
```
1/8*(24*a^2*c^3*e*x^6 + 36*a^2*b*c^2*e*x^4 + (3*(b^3*c^2 - 8*a*b*c^3)*d +
(a*b^2*c^2 + 20*a^2*c^3)*f)*x^7 + ((6*b^4*c - 49*a*b^2*c^2 + 28*a^2*c^3)*d
 + 2*(a*b^3*c + 14*a^2*b*c^2)*f)*x^5 + 8*(a^2*b^2*c + 5*a^3*c^2)*e*x^2 + (
(3*b^5 - 20*a*b^3*c - 4*a^2*b*c^2)*d + (a*b^4 + 5*a^2*b^2*c + 36*a^3*c^2)*
f)*x^3 - 2*(a^2*b^3 - 10*a^3*b*c)*e + ((5*a*b^4 - 37*a^2*b^2*c + 44*a^3*c^
2)*d - (a^2*b^3 - 16*a^3*b*c)*f)*x)/((a^2*b^4*c^2 - 8*a^3*b^2*c^3 + 16*a^4
*c^4)*x^8 + a^4*b^4 - 8*a^5*b^2*c + 16*a^6*c^2 + 2*(a^2*b^5*c - 8*a^3*b^3*
c^2 + 16*a^4*b*c^3)*x^6 + (a^2*b^6 - 6*a^3*b^4*c + 32*a^5*c^3)*x^4 + 2*(a^
3*b^5 - 8*a^4*b^3*c + 16*a^5*b*c^2)*x^2) + 1/8*integrate((48*a^2*c^2*e*x +
 (3*(b^3*c - 8*a*b*c^2)*d + (a*b^2*c + 20*a^2*c^2)*f)*x^2 + 3*(b^4 - 9*a*b
^2*c + 28*a^2*c^2)*d + (a*b^3 - 16*a^2*b*c)*f)/(c*x^4 + b*x^2 + a), x)/(a^
2*b^4 - 8*a^3*b^2*c + 16*a^4*c^2)
```

3.66.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 5280 vs. $2(561) = 1122$.

Time = 1.60 (sec) , antiderivative size = 5280, normalized size of antiderivative = 8.50

$$\int \frac{ad + aex + (bd + af)x^2 + bex^3 + (cd + bf)x^4 + cex^5 + cfx^6}{(a + bx^2 + cx^4)^4} \, dx = \text{Too large to display}$$

input
```
integrate((a*d+a*e*x+(a*f+b*d)*x^2+b*e*x^3+(b*f+c*d)*x^4+c*e*x^5+c*f*x^6)/
(c*x^4+b*x^2+a)^4,x, algorithm="giac")
```

output
```
-3*(b^2*c^4 - 4*a*c^5 - 2*b*c^5 + c^6)*sqrt(b^2 - 4*a*c)*e*log(x^2 + 1/2*(
a^2*b^5 - 8*a^3*b^3*c + 16*a^4*b*c^2 + sqrt((a^2*b^5 - 8*a^3*b^3*c + 16*a^
4*b*c^2)^2 - 4*(a^3*b^4 - 8*a^4*b^2*c + 16*a^5*c^2)*(a^2*b^4*c - 8*a^3*b^2
*c^2 + 16*a^4*c^3)))/(a^2*b^4*c - 8*a^3*b^2*c^2 + 16*a^4*c^3))/((b^8 - 16*
a*b^6*c - 2*b^7*c + 96*a^2*b^4*c^2 + 24*a*b^5*c^2 + b^6*c^2 - 256*a^3*b^2*
c^3 - 96*a^2*b^3*c^3 - 12*a*b^4*c^3 + 256*a^4*c^4 + 128*a^3*b*c^4 + 48*a^2
*b^2*c^4 - 64*a^3*c^5)*c^2) - 3*(b^2*c^4 - 4*a*c^5 - 2*b*c^5 + c^6)*sqrt(b
^2 - 4*a*c)*e*log(x^2 + 1/2*(a^2*b^5 - 8*a^3*b^3*c + 16*a^4*b*c^2 - sqrt((
a^2*b^5 - 8*a^3*b^3*c + 16*a^4*b*c^2)^2 - 4*(a^3*b^4 - 8*a^4*b^2*c + 16*a^
5*c^2)*(a^2*b^4*c - 8*a^3*b^2*c^2 + 16*a^4*c^3)))/(a^2*b^4*c - 8*a^3*b^2*c
^2 + 16*a^4*c^3))/((b^8 - 16*a*b^6*c - 2*b^7*c + 96*a^2*b^4*c^2 + 24*a*b^5
*c^2 + b^6*c^2 - 256*a^3*b^2*c^3 - 96*a^2*b^3*c^3 - 12*a*b^4*c^3 + 256*a^4
*c^4 + 128*a^3*b*c^4 + 48*a^2*b^2*c^4 - 64*a^3*c^5)*c^2) + 1/32*(3*(sqrt(2
)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^8 - 17*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4
*a*c)*c)*a*b^6*c - 2*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*b^7*c - 2*b^8
*c + 116*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a^2*b^4*c^2 + 26*sqrt(2)*
sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^5*c^2 + sqrt(2)*sqrt(b*c + sqrt(b^2 -
4*a*c)*c)*b^6*c^2 + 34*a*b^6*c^2 + 2*b^7*c^2 - 368*sqrt(2)*sqrt(b*c + sqrt
(b^2 - 4*a*c)*c)*a^3*b^2*c^3 - 128*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)
*a^2*b^3*c^3 - 13*sqrt(2)*sqrt(b*c + sqrt(b^2 - 4*a*c)*c)*a*b^4*c^3 - 2...
```

3.66.9 Mupad [B] (verification not implemented)

Time = 9.34 (sec) , antiderivative size = 8689, normalized size of antiderivative = 13.99

$$\int \frac{ad + aex + (bd + af)x^2 + bex^3 + (cd + bf)x^4 + cex^5 + cfx^6}{(a + bx^2 + cx^4)^4} \, dx = \text{Too large to display}$$

input
```
int((a*d + x^2*(b*d + a*f) + x^4*(c*d + b*f) + a*e*x + b*e*x^3 + c*e*x^5 +
 c*f*x^6)/(a + b*x^2 + c*x^4)^4,x)
```

output
```
((x^2*(5*a*c^2*e + b^2*c*e))/(b^4 + 16*a^2*c^2 - 8*a*b^2*c) - (b^3*e - 10*
a*b*c*e)/(4*(b^4 + 16*a^2*c^2 - 8*a*b^2*c)) + (x^5*(28*a^2*c^3*d + 6*b^4*c
*d + 2*a*b^3*c*f - 49*a*b^2*c^2*d + 28*a^2*b*c^2*f))/(8*a^2*(b^4 + 16*a^2*
c^2 - 8*a*b^2*c)) + (x*(5*b^4*d + 44*a^2*c^2*d - a*b^3*f - 37*a*b^2*c*d +
16*a^2*b*c*f))/(8*a*(b^4 + 16*a^2*c^2 - 8*a*b^2*c)) + (3*c^3*e*x^6)/(b^4 +
 16*a^2*c^2 - 8*a*b^2*c) + (x^3*(3*b^5*d + 36*a^3*c^2*f + a*b^4*f - 20*a*b
^3*c*d - 4*a^2*b*c^2*d + 5*a^2*b^2*c*f))/(8*a^2*(b^4 + 16*a^2*c^2 - 8*a*b^
2*c)) + (9*b*c^2*e*x^4)/(2*(b^4 + 16*a^2*c^2 - 8*a*b^2*c)) + (c*x^7*(20*a^
2*c^2*f + 3*b^3*c*d - 24*a*b*c^2*d + a*b^2*c*f))/(8*a^2*(b^4 + 16*a^2*c^2
- 8*a*b^2*c)))/(x^4*(2*a*c + b^2) + a^2 + c^2*x^8 + 2*a*b*x^2 + 2*b*c*x^6)
 + symsum(log(root(56371445760*a^11*b^8*c^6*z^4 - 503316480*a^8*b^14*c^3*z
^4 + 47185920*a^7*b^16*c^2*z^4 - 171798691840*a^14*b^2*c^9*z^4 + 193273528
320*a^13*b^4*c^8*z^4 - 128849018880*a^12*b^6*c^7*z^4 - 16911433728*a^10*b^
10*c^5*z^4 + 3523215360*a^9*b^12*c^4*z^4 - 2621440*a^6*b^18*c*z^4 + 687194
76736*a^15*c^10*z^4 + 65536*a^5*b^20*z^4 - 73728*a^2*b^16*c*d*f*z^2 - 1321
205760*a^9*b^2*c^8*d*f*z^2 + 732168192*a^7*b^6*c^6*d*f*z^2 - 366280704*a^6
*b^8*c^5*d*f*z^2 - 330301440*a^8*b^4*c^7*d*f*z^2 + 96583680*a^5*b^10*c^4*d
*f*z^2 - 15175680*a^4*b^12*c^3*d*f*z^2 + 1428480*a^3*b^14*c^2*d*f*z^2 - 44
0401920*a^10*b*c^8*f^2*z^2 + 1761607680*a^10*c^9*d*f*z^2 - 14080*a^3*b^15*
c*f^2*z^2 + 6936330240*a^8*b^3*c^8*d^2*z^2 + 2464874496*a^6*b^7*c^6*d^2...
```

3.66. $\int \frac{ad+aex+(bd+af)x^2+bex^3+(cd+bf)x^4+cex^5+cfx^6}{(a+bx^2+cx^4)^4}\,dx$

3.67 $\int \frac{2-x-2x^2+x^3}{4-5x^2+x^4}\,dx$

3.67.1 Optimal result

Integrand size = 26, antiderivative size = 4

$$\int \frac{2-x-2x^2+x^3}{4-5x^2+x^4}\,dx = \log(2+x)$$

output
```
ln(2+x)
```

3.67.2 Mathematica [A] (verified)

Time = 0.00 (sec) , antiderivative size = 4, normalized size of antiderivative = 1.00

$$\int \frac{2-x-2x^2+x^3}{4-5x^2+x^4}\,dx = \log(2+x)$$

input
```
Integrate[(2 - x - 2*x^2 + x^3)/(4 - 5*x^2 + x^4),x]
```

output
```
Log[2 + x]
```

3.67.3 Rubi [A] (verified)

Time = 0.15 (sec) , antiderivative size = 4, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.077, Rules used = {2019, 16}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{x^3-2x^2-x+2}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{ 2019}$$

$$\int \frac{1}{x+2}dx$$

$$\Big\downarrow \text{ 16}$$

$$\log(x+2)$$

input
```
Int[(2 - x - 2*x^2 + x^3)/(4 - 5*x^2 + x^4),x]
```

output
```
Log[2 + x]
```

3.67.3.1 Defintions of rubi rules used

rule 16
```
Int[(c_.)/((a_.) + (b_.)*(x_)), x_Symbol] :> Simp[c*(Log[RemoveContent[a +
b*x, x]]/b), x] /; FreeQ[{a, b, c}, x]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

3.67.4 Maple [A] (verified)

Time = 0.02 (sec) , antiderivative size = 5, normalized size of antiderivative = 1.25

method	result	size
default	$\ln(x+2)$	5
norman	$\ln(x+2)$	5
risch	$\ln(x+2)$	5
parallelrisch	$\ln(x+2)$	5

input
```
int((x^3-2*x^2-x+2)/(x^4-5*x^2+4),x,method=_RETURNVERBOSE)
```

output
```
ln(x+2)
```

3.67.5 Fricas [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 4, normalized size of antiderivative = 1.00

$$\int \frac{2-x-2x^2+x^3}{4-5x^2+x^4}\,dx = \log(x+2)$$

input
```
integrate((x^3-2*x^2-x+2)/(x^4-5*x^2+4),x, algorithm="fricas")
```

output
```
log(x + 2)
```

3.67.6 Sympy [A] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 3, normalized size of antiderivative = 0.75

$$\int \frac{2-x-2x^2+x^3}{4-5x^2+x^4}\,dx = \log(x+2)$$

input
```
integrate((x**3-2*x**2-x+2)/(x**4-5*x**2+4),x)
```

output
```
log(x + 2)
```

3.67.7 Maxima [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 4, normalized size of antiderivative = 1.00

$$\int \frac{2-x-2x^2+x^3}{4-5x^2+x^4}\,dx = \log{(x+2)}$$

input
```
integrate((x^3-2*x^2-x+2)/(x^4-5*x^2+4),x, algorithm="maxima")
```

output
```
log(x + 2)
```

3.67.8 Giac [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 5, normalized size of antiderivative = 1.25

$$\int \frac{2-x-2x^2+x^3}{4-5x^2+x^4}\,dx = \log{(|x+2|)}$$

input
```
integrate((x^3-2*x^2-x+2)/(x^4-5*x^2+4),x, algorithm="giac")
```

output
```
log(abs(x + 2))
```

3.67.9 Mupad [B] (verification not implemented)

Time = 0.01 (sec) , antiderivative size = 4, normalized size of antiderivative = 1.00

$$\int \frac{2-x-2x^2+x^3}{4-5x^2+x^4}\,dx = \ln{(x+2)}$$

input
```
int(-(x + 2*x^2 - x^3 - 2)/(x^4 - 5*x^2 + 4),x)
```

output
```
log(x + 2)
```

3.68 $\int \frac{(d+ex)(2-x-2x^2+x^3)}{4-5x^2+x^4}\,dx$

3.68.1 Optimal result

Integrand size = 31, antiderivative size = 14

$$\int \frac{(d+ex)(2-x-2x^2+x^3)}{4-5x^2+x^4}\,dx = ex+(d-2e)\log(2+x)$$

output
```
e*x+(d-2*e)*ln(2+x)
```

3.68.2 Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 16, normalized size of antiderivative = 1.14

$$\int \frac{(d+ex)(2-x-2x^2+x^3)}{4-5x^2+x^4}\,dx = e(2+x)+(d-2e)\log(2+x)$$

input
```
Integrate[((d + e*x)*(2 - x - 2*x^2 + x^3))/(4 - 5*x^2 + x^4),x]
```

output
```
e*(2 + x) + (d - 2*e)*Log[2 + x]
```

3.68.3 Rubi [A] (verified)

Time = 0.18 (sec) , antiderivative size = 14, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.097, Rules used = {2019, 49, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^3-2x^2-x+2)(d+ex)}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{2019}$$

$$\int \frac{d+ex}{x+2}dx$$

$$\Big\downarrow \text{49}$$

$$\int \left(\frac{d-2e}{x+2}+e\right)dx$$

$$\Big\downarrow \text{2009}$$

$$(d-2e)\log(x+2)+ex$$

input
```
Int[((d + e*x)*(2 - x - 2*x^2 + x^3))/(4 - 5*x^2 + x^4),x]
```

output
```
e*x + (d - 2*e)*Log[2 + x]
```

3.68.3.1 Defintions of rubi rules used

rule 49
```
Int[((a_.) + (b_.)*(x_))^(m_.)*((c_.) + (d_.)*(x_))^(n_.), x_Symbol] :> Int
[ExpandIntegrand[(a + b*x)^m*(c + d*x)^n, x], x] /; FreeQ[{a, b, c, d}, x]
&& IGtQ[m, 0] && IGtQ[m + n + 2, 0]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

3.68.4 Maple [A] (verified)

Time = 0.03 (sec) , antiderivative size = 15, normalized size of antiderivative = 1.07

method	result	size
default	$ex + (d - 2e)\ln(x + 2)$	15
norman	$ex + (d - 2e)\ln(x + 2)$	15
risch	$ex + \ln(x + 2)\, d - 2\ln(x + 2)\, e$	18
parallelrisch	$ex + \ln(x + 2)\, d - 2\ln(x + 2)\, e$	18

input
```
int((e*x+d)*(x^3-2*x^2-x+2)/(x^4-5*x^2+4),x,method=_RETURNVERBOSE)
```

output
```
e*x+(d-2*e)*ln(x+2)
```

3.68.5 Fricas [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 14, normalized size of antiderivative = 1.00

$$\int \frac{(d + ex)(2 - x - 2x^2 + x^3)}{4 - 5x^2 + x^4}\, dx = ex + (d - 2\,e)\log(x + 2)$$

input
```
integrate((e*x+d)*(x^3-2*x^2-x+2)/(x^4-5*x^2+4),x, algorithm="fricas")
```

output
```
e*x + (d - 2*e)*log(x + 2)
```

3.68.6 Sympy [A] (verification not implemented)

Time = 0.06 (sec) , antiderivative size = 12, normalized size of antiderivative = 0.86

$$\int \frac{(d + ex)(2 - x - 2x^2 + x^3)}{4 - 5x^2 + x^4}\, dx = ex + (d - 2e)\log(x + 2)$$

input
```
integrate((e*x+d)*(x**3-2*x**2-x+2)/(x**4-5*x**2+4),x)
```

output
```
e*x + (d - 2*e)*log(x + 2)
```

3.68.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 14, normalized size of antiderivative = 1.00

$$\int \frac{(d+ex)(2-x-2x^2+x^3)}{4-5x^2+x^4}\,dx = ex+(d-2\,e)\log{(x+2)}$$

input
```
integrate((e*x+d)*(x^3-2*x^2-x+2)/(x^4-5*x^2+4),x, algorithm="maxima")
```

output
```
e*x + (d - 2*e)*log(x + 2)
```

3.68.8 Giac [A] (verification not implemented)

Time = 0.31 (sec) , antiderivative size = 15, normalized size of antiderivative = 1.07

$$\int \frac{(d+ex)(2-x-2x^2+x^3)}{4-5x^2+x^4}\,dx = ex+(d-2\,e)\log{(|x+2|)}$$

input
```
integrate((e*x+d)*(x^3-2*x^2-x+2)/(x^4-5*x^2+4),x, algorithm="giac")
```

output
```
e*x + (d - 2*e)*log(abs(x + 2))
```

3.68.9 Mupad [B] (verification not implemented)

Time = 7.86 (sec) , antiderivative size = 14, normalized size of antiderivative = 1.00

$$\int \frac{(d+ex)(2-x-2x^2+x^3)}{4-5x^2+x^4}\,dx = \ln{(x+2)}\ (d-2\,e)+e\,x$$

input
```
int(-((d + e*x)*(x + 2*x^2 - x^3 - 2))/(x^4 - 5*x^2 + 4),x)
```

output
```
log(x + 2)*(d - 2*e) + e*x
```

3.69 $\int \frac{(d+ex+fx^2)(2-x-2x^2+x^3)}{4-5x^2+x^4}\,dx$

3.69.1 Optimal result

Integrand size = 36, antiderivative size = 31

$$\int \frac{(d+ex+fx^2)(2-x-2x^2+x^3)}{4-5x^2+x^4}\,dx=(e-4f)x+\frac{1}{2}f(2+x)^2+(d-2e+4f)\log(2+x)$$

output
```
(e-4*f)*x+1/2*f*(2+x)^2+(d-2*e+4*f)*ln(2+x)
```

3.69.2 Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 30, normalized size of antiderivative = 0.97

$$\int \frac{(d+ex+fx^2)(2-x-2x^2+x^3)}{4-5x^2+x^4}\,dx=\frac{1}{2}(2e+f(-6+x))(2+x)+(d-2e+4f)\log(2+x)$$

input
```
Integrate[((d + e*x + f*x^2)*(2 - x - 2*x^2 + x^3))/(4 - 5*x^2 + x^4),x]
```

output
```
((2*e + f*(-6 + x))*(2 + x))/2 + (d - 2*e + 4*f)*Log[2 + x]
```

3.69.3 Rubi [A] (verified)

Time = 0.22 (sec) , antiderivative size = 31, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.083, Rules used = {2019, 1140, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^3-2x^2-x+2)(d+ex+fx^2)}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{2019}$$

$$\int \frac{d+ex+fx^2}{x+2}dx$$

$$\Big\downarrow \text{1140}$$

$$\int \left(\frac{d-2e+4f}{x+2}+e+f(x+2)-4f\right)dx$$

$$\Big\downarrow \text{2009}$$

$$\log(x+2)(d-2e+4f)+x(e-4f)+\frac{1}{2}f(x+2)^2$$

input
```
Int[((d + e*x + f*x^2)*(2 - x - 2*x^2 + x^3))/(4 - 5*x^2 + x^4),x]
```

output
```
(e - 4*f)*x + (f*(2 + x)^2)/2 + (d - 2*e + 4*f)*Log[2 + x]
```

3.69.3.1 Defintions of rubi rules used

rule 1140
```
Int[((d_.) + (e_.)*(x_))^(m_.)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_.), x
_Symbol] :> Int[ExpandIntegrand[(d + e*x)^m*(a + b*x + c*x^2)^p, x], x] /;
FreeQ[{a, b, c, d, e, m}, x] && IGtQ[p, 0]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

3.69.4 Maple [A] (verified)

Time = 0.03 (sec) , antiderivative size = 28, normalized size of antiderivative = 0.90

method	result	size
default	$\frac{f\,x^{2}}{2}+ex-2fx+\left(d-2e+4f\right)\ln\left(x+2\right)$	28
norman	$\left(e-2f\right)x+\frac{f\,x^{2}}{2}+\left(d-2e+4f\right)\ln\left(x+2\right)$	28
risch	$\frac{f\,x^{2}}{2}+ex-2fx+\ln\left(x+2\right)d-2\ln\left(x+2\right)e+4\ln\left(x+2\right)f$	35
parallelrisch	$\frac{f\,x^{2}}{2}+ex-2fx+\ln\left(x+2\right)d-2\ln\left(x+2\right)e+4\ln\left(x+2\right)f$	35

input
```
int((f*x^2+e*x+d)*(x^3-2*x^2-x+2)/(x^4-5*x^2+4),x,method=_RETURNVERBOSE)
```

output
```
1/2*f*x^2+e*x-2*f*x+(d-2*e+4*f)*ln(x+2)
```

3.69.5 Fricas [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 27, normalized size of antiderivative = 0.87

$$\int \frac{(d+ex+fx^2)\left(2-x-2x^2+x^3\right)}{4-5x^2+x^4}\,dx=\frac{1}{2}\,fx^2+(e-2\,f)x+(d-2\,e+4\,f)\log\left(x+2\right)$$

input
```
integrate((f*x^2+e*x+d)*(x^3-2*x^2-x+2)/(x^4-5*x^2+4),x, algorithm="fricas
")
```

output
```
1/2*f*x^2 + (e - 2*f)*x + (d - 2*e + 4*f)*log(x + 2)
```

3.69.6 Sympy [A] (verification not implemented)

Time = 0.08 (sec) , antiderivative size = 26, normalized size of antiderivative = 0.84

$$\int \frac{(d+ex+fx^2)\left(2-x-2x^2+x^3\right)}{4-5x^2+x^4}\,dx=\frac{fx^2}{2}+x(e-2f)+(d-2e+4f)\log\left(x+2\right)$$

input
```
integrate((f*x**2+e*x+d)*(x**3-2*x**2-x+2)/(x**4-5*x**2+4),x)
```

output
```
f*x**2/2 + x*(e - 2*f) + (d - 2*e + 4*f)*log(x + 2)
```

3.69.7 Maxima [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 27, normalized size of antiderivative = 0.87

$$\int \frac{(d+ex+fx^2)(2-x-2x^2+x^3)}{4-5x^2+x^4}\,dx = \frac{1}{2}\,fx^2+(e-2\,f)x+(d-2\,e+4\,f)\log{(x+2)}$$

input
```
integrate((f*x^2+e*x+d)*(x^3-2*x^2-x+2)/(x^4-5*x^2+4),x, algorithm="maxima
")
```

output
```
1/2*f*x^2 + (e - 2*f)*x + (d - 2*e + 4*f)*log(x + 2)
```

3.69.8 Giac [A] (verification not implemented)

Time = 0.31 (sec) , antiderivative size = 28, normalized size of antiderivative = 0.90

$$\int \frac{(d+ex+fx^2)(2-x-2x^2+x^3)}{4-5x^2+x^4}\,dx = \frac{1}{2}\,fx^2+ex-2\,fx+(d-2\,e+4\,f)\log{(|x+2|)}$$

input
```
integrate((f*x^2+e*x+d)*(x^3-2*x^2-x+2)/(x^4-5*x^2+4),x, algorithm="giac")
```

output
```
1/2*f*x^2 + e*x - 2*f*x + (d - 2*e + 4*f)*log(abs(x + 2))
```

3.69.9 Mupad [B] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 27, normalized size of antiderivative = 0.87

$$\int \frac{(d+ex+fx^2)(2-x-2x^2+x^3)}{4-5x^2+x^4}\,dx = x\,(e-2\,f)+\frac{f\,x^2}{2}+\ln{(x+2)}\,(d-2\,e+4\,f)$$

input
```
int(-((d + e*x + f*x^2)*(x + 2*x^2 - x^3 - 2))/(x^4 - 5*x^2 + 4),x)
```

output
```
x*(e - 2*f) + (f*x^2)/2 + log(x + 2)*(d - 2*e + 4*f)
```

3.70 $\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3)}{4-5x^2+x^4}\,dx$

3.70.1 Optimal result

Integrand size = 41, antiderivative size = 51

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3)}{4-5x^2+x^4}\,dx$$
$$= (e-4f+12g)x+\frac{1}{2}(f-6g)(2+x)^2+\frac{1}{3}g(2+x)^3+(d-2e+4f-8g)\log(2+x)$$

output
```
(e-4*f+12*g)*x+1/2*(f-6*g)*(2+x)^2+1/3*g*(2+x)^3+(d-2*e+4*f-8*g)*ln(2+x)
```

3.70.2 Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 45, normalized size of antiderivative = 0.88

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3)}{4-5x^2+x^4}\,dx$$
$$= \frac{1}{6}(2+x)\left(6e+3f(-6+x)+2g\left(22-5x+x^2\right)\right)+(d-2e+4f-8g)\log(2+x)$$

input
```
Integrate[((2 - x - 2*x^2 + x^3)*(d + e*x + f*x^2 + g*x^3))/(4 - 5*x^2 + x
^4),x]
```

output
```
((2 + x)*(6*e + 3*f*(-6 + x) + 2*g*(22 - 5*x + x^2)))/6 + (d - 2*e + 4*f -
 8*g)*Log[2 + x]
```

3.70.3 Rubi [A] (verified)

Time = 0.28 (sec) , antiderivative size = 51, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.073$, Rules used = {2019, 2389, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^3 - 2x^2 - x + 2)(d + ex + fx^2 + gx^3)}{x^4 - 5x^2 + 4} \, dx$$

$$\Big\downarrow \; 2019$$

$$\int \frac{d + ex + fx^2 + gx^3}{x + 2} dx$$

$$\Big\downarrow \; 2389$$

$$\int \left(\frac{d - 2e + 4f - 8g}{x + 2} + e + (x + 2)(f - 6g) - 4f + g(x + 2)^2 + 12g\right) dx$$

$$\Big\downarrow \; 2009$$

$$\log(x + 2)(d - 2e + 4f - 8g) + x(e - 4f + 12g) + \frac{1}{2}(x + 2)^2(f - 6g) + \frac{1}{3}g(x + 2)^3$$

input
```
Int[((2 - x - 2*x^2 + x^3)*(d + e*x + f*x^2 + g*x^3))/(4 - 5*x^2 + x^4),x]
```

output
```
(e - 4*f + 12*g)*x + ((f - 6*g)*(2 + x)^2)/2 + (g*(2 + x)^3)/3 + (d - 2*e
+ 4*f - 8*g)*Log[2 + x]
```

3.70.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2389
```
Int[(Pq_)*((a_) + (b_.)*(x_)^(n_.))^(p_.), x_Symbol] :> Int[ExpandIntegrand
[Pq*(a + b*x^n)^p, x], x] /; FreeQ[{a, b, n}, x] && PolyQ[Pq, x] && (IGtQ[p
, 0] || EqQ[n, 1])
```

3.70.4 Maple [A] (verified)

Time = 0.04 (sec) , antiderivative size = 45, normalized size of antiderivative = 0.88

method	result
norman	$\left(\frac{f}{2}-g\right)x^2+(e-2f+4g)\,x+\frac{g\,x^3}{3}+(d-2e+4f-8g)\ln(x+2)$
default	$\frac{g\,x^3}{3}+\frac{f\,x^2}{2}-g\,x^2+ex-2fx+4gx+(d-2e+4f-8g)\ln(x+2)$
risch	$\frac{g\,x^3}{3}+\frac{f\,x^2}{2}-g\,x^2+ex-2fx+4gx+\ln(x+2)\,d-2\ln(x+2)\,e+4\ln(x+2)\,f-8\ln(x$
parallelrisch	$\frac{g\,x^3}{3}+\frac{f\,x^2}{2}-g\,x^2+ex-2fx+4gx+\ln(x+2)\,d-2\ln(x+2)\,e+4\ln(x+2)\,f-8\ln(x$

input
```
int((x^3-2*x^2-x+2)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x,method=_RETURNVERB
OSE)
```

output
```
(1/2*f-g)*x^2+(e-2*f+4*g)*x+1/3*g*x^3+(d-2*e+4*f-8*g)*ln(x+2)
```

3.70.5 Fricas [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 43, normalized size of antiderivative = 0.84

$$\int \frac{(2-x-2x^2+x^3)\,(d+ex+fx^2+gx^3)}{4-5x^2+x^4}\,dx$$
$$=\frac{1}{3}\,gx^3+\frac{1}{2}\,(f-2\,g)x^2+(e-2\,f+4\,g)x+(d-2\,e+4\,f-8\,g)\log(x+2)$$

input
```
integrate((x^3-2*x^2-x+2)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="
fricas")
```

output
```
1/3*g*x^3 + 1/2*(f - 2*g)*x^2 + (e - 2*f + 4*g)*x + (d - 2*e + 4*f - 8*g)*
log(x + 2)
```

3.70.6 Sympy [A] (verification not implemented)

Time = 0.08 (sec) , antiderivative size = 41, normalized size of antiderivative = 0.80

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3)}{4-5x^2+x^4}\,dx$$
$$=\frac{gx^3}{3}+x^2\left(\frac{f}{2}-g\right)+x(e-2f+4g)+(d-2e+4f-8g)\log(x+2)$$

input
```
integrate((x**3-2*x**2-x+2)*(g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4),x)
```

output
```
g*x**3/3 + x**2*(f/2 - g) + x*(e - 2*f + 4*g) + (d - 2*e + 4*f - 8*g)*log(
x + 2)
```

3.70.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 43, normalized size of antiderivative = 0.84

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3)}{4-5x^2+x^4}\,dx$$
$$=\frac{1}{3}\,gx^3+\frac{1}{2}\,(f-2\,g)x^2+(e-2\,f+4\,g)x+(d-2\,e+4\,f-8\,g)\log(x+2)$$

input
```
integrate((x^3-2*x^2-x+2)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="
maxima")
```

output
```
1/3*g*x^3 + 1/2*(f - 2*g)*x^2 + (e - 2*f + 4*g)*x + (d - 2*e + 4*f - 8*g)*
log(x + 2)
```

3.70.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 47, normalized size of antiderivative = 0.92

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3)}{4-5x^2+x^4}\,dx$$
$$=\frac{1}{3}\,gx^3+\frac{1}{2}\,fx^2-gx^2+ex-2\,fx+4\,gx+(d-2\,e+4\,f-8\,g)\log(|x+2|)$$

input
```
integrate((x^3-2*x^2-x+2)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="
giac")
```

output
```
1/3*g*x^3 + 1/2*f*x^2 - g*x^2 + e*x - 2*f*x + 4*g*x + (d - 2*e + 4*f - 8*g
)*log(abs(x + 2))
```

3.70.9 Mupad [B] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.86

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3)}{4-5x^2+x^4}\,dx = x^2\left(\frac{f}{2}-g\right)+x\,(e-2\,f+4\,g)+\frac{g\,x^3}{3}+\ln(x+2)\,(d-2\,e+4\,f-8\,g)$$

input
```
int(-((d + e*x + f*x^2 + g*x^3)*(x + 2*x^2 - x^3 - 2))/(x^4 - 5*x^2 + 4),x
)
```

output
```
x^2*(f/2 - g) + x*(e - 2*f + 4*g) + (g*x^3)/3 + log(x + 2)*(d - 2*e + 4*f
- 8*g)
```

3.71 $\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx$

3.71.1 Optimal result

Integrand size = 46, antiderivative size = 68

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx = (e-2f+4g-8h)x+\frac{1}{2}(f-2g+4h)x^2+\frac{1}{3}(g-2h)x^3 + \frac{hx^4}{4}+(d-2e+4f-8g+16h)\log(2+x)$$

output
```
(e-2*f+4*g-8*h)*x+1/2*(f-2*g+4*h)*x^2+1/3*(g-2*h)*x^3+1/4*h*x^4+(d-2*e+4*f
-8*g+16*h)*ln(2+x)
```

3.71.2 Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 68, normalized size of antiderivative = 1.00

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx = (e-2f+4g-8h)x+\frac{1}{2}(f-2g+4h)x^2+\frac{1}{3}(g-2h)x^3 + \frac{hx^4}{4}+(d-2e+4f-8g+16h)\log(2+x)$$

input
```
Integrate[((2 - x - 2*x^2 + x^3)*(d + e*x + f*x^2 + g*x^3 + h*x^4))/(4 - 5
*x^2 + x^4),x]
```

output
```
(e - 2*f + 4*g - 8*h)*x + ((f - 2*g + 4*h)*x^2)/2 + ((g - 2*h)*x^3)/3 + (h
*x^4)/4 + (d - 2*e + 4*f - 8*g + 16*h)*Log[2 + x]
```

3.71.3 Rubi [A] (verified)

Time = 0.32 (sec) , antiderivative size = 68, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.065$, Rules used = {2019, 2389, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^3-2x^2-x+2)(d+ex+fx^2+gx^3+hx^4)}{x^4-5x^2+4}\,dx$$

$$\Big\downarrow \text{2019}$$

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{x+2}\,dx$$

$$\Big\downarrow \text{2389}$$

$$\int \left(\frac{d-2e+4f-8g+16h}{x+2}+e\left(1-\frac{2(f-2g+4h)}{e}\right)+x(f-2g+4h)+x^2(g-2h)+hx^3\right)dx$$

$$\Big\downarrow \text{2009}$$

$$\log(x+2)(d-2e+4f-8g+16h)+x(e-2f+4g-8h)+\frac{1}{2}x^2(f-2g+4h)+\frac{1}{3}x^3(g-2h)+\frac{hx^4}{4}$$

input
```
Int[((2 - x - 2*x^2 + x^3)*(d + e*x + f*x^2 + g*x^3 + h*x^4))/(4 - 5*x^2 +
 x^4),x]
```

output
```
(e - 2*f + 4*g - 8*h)*x + ((f - 2*g + 4*h)*x^2)/2 + ((g - 2*h)*x^3)/3 + (h
*x^4)/4 + (d - 2*e + 4*f - 8*g + 16*h)*Log[2 + x]
```

3.71.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2389
```
Int[(Pq_)*((a_) + (b_.)*(x_)^(n_.))^(p_.), x_Symbol] :> Int[ExpandIntegrand
[Pq*(a + b*x^n)^p, x], x] /; FreeQ[{a, b, n}, x] && PolyQ[Pq, x] && (IGtQ[p
, 0] || EqQ[n, 1])
```

3.71.4 Maple [A] (verified)

Time = 0.04 (sec) , antiderivative size = 65, normalized size of antiderivative = 0.96

method	result
norman	$\left(\frac{g}{3}-\frac{2h}{3}\right)x^3+\left(\frac{f}{2}-g+2h\right)x^2+\left(e-2f+4g-8h\right)x+\frac{h\,x^4}{4}+\left(d-2e+4f-8g+16h\right)\ln$
default	$\frac{h\,x^4}{4}+\frac{g\,x^3}{3}-\frac{2h\,x^3}{3}+\frac{f\,x^2}{2}-g\,x^2+2h\,x^2+ex-2fx+4gx-8hx+\left(d-2e+4f-8g+16\right.$
risch	$\frac{h\,x^4}{4}+\frac{g\,x^3}{3}-\frac{2h\,x^3}{3}+\frac{f\,x^2}{2}-g\,x^2+2h\,x^2+ex-2fx+4gx-8hx+\ln\left(x+2\right)d-2\ln\left(x+\right.$
parallelrisch	$\frac{h\,x^4}{4}+\frac{g\,x^3}{3}-\frac{2h\,x^3}{3}+\frac{f\,x^2}{2}-g\,x^2+2h\,x^2+ex-2fx+4gx-8hx+\ln\left(x+2\right)d-2\ln\left(x+\right.$

input
```
int((x^3-2*x^2-x+2)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x,method=_RETU
RNVERBOSE)
```

output
```
(1/3*g-2/3*h)*x^3+(1/2*f-g+2*h)*x^2+(e-2*f+4*g-8*h)*x+1/4*h*x^4+(d-2*e+4*f
-8*g+16*h)*ln(x+2)
```

3.71.5 Fricas [A] (verification not implemented)

Time = 0.24 (sec) , antiderivative size = 62, normalized size of antiderivative = 0.91

$$\int \frac{(2-x-2x^2+x^3)\,(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx = \frac{1}{4}\,hx^4+\frac{1}{3}\,(g-2\,h)x^3+\frac{1}{2}\,(f-2\,g+4\,h)x^2 + (e-2\,f+4\,g-8\,h)x+(d-2\,e+4\,f-8\,g+16\,h)\log{(x+2)}$$

input
```
integrate((x^3-2*x^2-x+2)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algor
ithm="fricas")
```

output
```
1/4*h*x^4 + 1/3*(g - 2*h)*x^3 + 1/2*(f - 2*g + 4*h)*x^2 + (e - 2*f + 4*g -
 8*h)*x + (d - 2*e + 4*f - 8*g + 16*h)*log(x + 2)
```

3.71.6 Sympy [A] (verification not implemented)

Time = 0.10 (sec) , antiderivative size = 63, normalized size of antiderivative = 0.93

$$\int \frac{(2-x-2x^2+x^3)\,(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx = \frac{hx^4}{4}+x^3\left(\frac{g}{3}-\frac{2h}{3}\right)+x^2\left(\frac{f}{2}-g+2h\right) + x(e-2f+4g-8h)+(d-2e+4f-8g+16h)\log{(x+2)}$$

input
```
integrate((x**3-2*x**2-x+2)*(h*x**4+g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4),x
)
```

output
```
h*x**4/4 + x**3*(g/3 - 2*h/3) + x**2*(f/2 - g + 2*h) + x*(e - 2*f + 4*g -
8*h) + (d - 2*e + 4*f - 8*g + 16*h)*log(x + 2)
```

3.71.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 62, normalized size of antiderivative = 0.91

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx$$
$$=\frac{1}{4}hx^4+\frac{1}{3}(g-2h)x^3+\frac{1}{2}(f-2g+4h)x^2+(e-2f+4g-8h)x+(d-2e+4f-8g+16h)\log(x+2)$$

input
```
integrate((x^3-2*x^2-x+2)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algor
ithm="maxima")
```

output
```
1/4*h*x^4 + 1/3*(g - 2*h)*x^3 + 1/2*(f - 2*g + 4*h)*x^2 + (e - 2*f + 4*g -
 8*h)*x + (d - 2*e + 4*f - 8*g + 16*h)*log(x + 2)
```

3.71.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 72, normalized size of antiderivative = 1.06

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx$$
$$=\frac{1}{4}hx^4+\frac{1}{3}gx^3-\frac{2}{3}hx^3+\frac{1}{2}fx^2-gx^2+2hx^2+ex-2fx+4gx-8hx+(d-2e+4f-8g+16h)\log(|x+2|)$$

input
```
integrate((x^3-2*x^2-x+2)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algor
ithm="giac")
```

output
```
1/4*h*x^4 + 1/3*g*x^3 - 2/3*h*x^3 + 1/2*f*x^2 - g*x^2 + 2*h*x^2 + e*x - 2*
f*x + 4*g*x - 8*h*x + (d - 2*e + 4*f - 8*g + 16*h)*log(abs(x + 2))
```

3.71.9 Mupad [B] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 64, normalized size of antiderivative = 0.94

$$\int \frac{(2-x-2x^2+x^3)\,(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx$$
$$= x^3\left(\frac{g}{3}-\frac{2\,h}{3}\right)+\ln\left(x+2\right)\,\left(d-2\,e+4\,f-8\,g+16\,h\right)$$
$$+\frac{h\,x^4}{4}+x^2\left(\frac{f}{2}-g+2\,h\right)+x\,\left(e-2\,f+4\,g-8\,h\right)$$

input
```
int(-((x + 2*x^2 - x^3 - 2)*(d + e*x + f*x^2 + g*x^3 + h*x^4))/(x^4 - 5*x^
2 + 4),x)
```

output
```
x^3*(g/3 - (2*h)/3) + log(x + 2)*(d - 2*e + 4*f - 8*g + 16*h) + (h*x^4)/4
+ x^2*(f/2 - g + 2*h) + x*(e - 2*f + 4*g - 8*h)
```

3.72 $\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4+ix^5)}{4-5x^2+x^4}\,dx$

3.72.1 Optimal result

Integrand size = 51, antiderivative size = 92

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4+ix^5)}{4-5x^2+x^4}\,dx = (e-2f+4g-8h+16i)x+\frac{1}{2}(f-2g+4h-8i)x^2+\frac{1}{3}(g-2h+4i)x^3 + \frac{1}{4}(h-2i)x^4+\frac{ix^5}{5}+(d-2e+4f-8g+16h-32i)\log(2+x)$$

output
```
(e-2*f+4*g-8*h+16*i)*x+1/2*(f-2*g+4*h-8*i)*x^2+1/3*(g-2*h+4*i)*x^3+1/4*(h-
2*i)*x^4+1/5*i*x^5+(d-2*e+4*f-8*g+16*h-32*i)*ln(2+x)
```

3.72.2 Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 92, normalized size of antiderivative = 1.00

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4+ix^5)}{4-5x^2+x^4}\,dx = (e-2f+4g-8h+16i)x+\frac{1}{2}(f-2g+4h-8i)x^2+\frac{1}{3}(g-2h+4i)x^3 + \frac{1}{4}(h-2i)x^4+\frac{ix^5}{5}+(d-2e+4f-8g+16h-32i)\log(2+x)$$

input
```
Integrate[((2 - x - 2*x^2 + x^3)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)
)/(4 - 5*x^2 + x^4),x]
```

output
```
(e - 2*f + 4*g - 8*h + 16*i)*x + ((f - 2*g + 4*h - 8*i)*x^2)/2 + ((g - 2*h
 + 4*i)*x^3)/3 + ((h - 2*i)*x^4)/4 + (i*x^5)/5 + (d - 2*e + 4*f - 8*g + 16
*h - 32*i)*Log[2 + x]
```

3.72.3 Rubi [A] (verified)

Time = 0.35 (sec) , antiderivative size = 92, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.059, Rules used = {2019, 2389, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^3-2x^2-x+2)(d+ex+fx^2+gx^3+hx^4+ix^5)}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{2019}$$

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{x+2}dx$$

$$\Big\downarrow \text{2389}$$

$$\int \left(\frac{d-2e+4f-8g+16h-32i}{x+2}+e\left(1-\frac{2(f-2g+4h-8i)}{e}\right)+x(f-2g+4h-8i)+x^2(g-2h+4i)+x^3$$

$$\Big\downarrow \text{2009}$$

$$\log(x+2)(d-2e+4f-8g+16h-32i)+x(e-2f+4g-8h+16i)+\frac{1}{2}x^2(f-2g+4h-8i)+\frac{1}{3}x^3(g-2h+4i)+\frac{1}{4}x^4(h-2i)+\frac{ix^5}{5}$$

input
```
Int[((2 - x - 2*x^2 + x^3)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5))/(4 -
 5*x^2 + x^4),x]
```

output
```
(e - 2*f + 4*g - 8*h + 16*i)*x + ((f - 2*g + 4*h - 8*i)*x^2)/2 + ((g - 2*h
 + 4*i)*x^3)/3 + ((h - 2*i)*x^4)/4 + (i*x^5)/5 + (d - 2*e + 4*f - 8*g + 16
*h - 32*i)*Log[2 + x]
```

3.72.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2389
```
Int[(Pq_)*((a_) + (b_.)*(x_)^(n_.))^(p_.), x_Symbol] :> Int[ExpandIntegrand
[Pq*(a + b*x^n)^p, x], x] /; FreeQ[{a, b, n}, x] && PolyQ[Pq, x] && (IGtQ[p
, 0] || EqQ[n, 1])
```

3.72.4 Maple [A] (verified)

Time = 0.04 (sec) , antiderivative size = 88, normalized size of antiderivative = 0.96

method	result
norman	$\left(\frac{h}{4}-\frac{i}{2}\right)x^4+\left(\frac{g}{3}-\frac{2h}{3}+\frac{4i}{3}\right)x^3+\left(\frac{f}{2}-g+2h-4i\right)x^2+(e-2f+4g-8h+16i)x+\frac{i\,x^5}{5}+$
default	$\frac{i\,x^5}{5}+\frac{h\,x^4}{4}-\frac{i\,x^4}{2}+\frac{g\,x^3}{3}-\frac{2h\,x^3}{3}+\frac{4i\,x^3}{3}+\frac{f\,x^2}{2}-g\,x^2+2h\,x^2-4i\,x^2+ex-2fx+4gx-8hx$
risch	$\frac{i\,x^5}{5}+\frac{h\,x^4}{4}-\frac{i\,x^4}{2}+\frac{g\,x^3}{3}-\frac{2h\,x^3}{3}+\frac{4i\,x^3}{3}+\frac{f\,x^2}{2}-g\,x^2+2h\,x^2-4i\,x^2+ex-2fx+4gx-8hx$
parallelrisch	$\frac{i\,x^5}{5}+\frac{h\,x^4}{4}-\frac{i\,x^4}{2}+\frac{g\,x^3}{3}-\frac{2h\,x^3}{3}+\frac{4i\,x^3}{3}+\frac{f\,x^2}{2}-g\,x^2+2h\,x^2-4i\,x^2+ex-2fx+4gx-8hx$

input
```
int((x^3-2*x^2-x+2)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x,method
=_RETURNVERBOSE)
```

output
```
(1/4*h-1/2*i)*x^4+(1/3*g-2/3*h+4/3*i)*x^3+(1/2*f-g+2*h-4*i)*x^2+(e-2*f+4*g
-8*h+16*i)*x+1/5*i*x^5+(d-2*e+4*f-8*g+16*h-32*i)*ln(x+2)
```

3.72.5 Fricas [A] (verification not implemented)

Time = 0.25 (sec) , antiderivative size = 84, normalized size of antiderivative = 0.91

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4+ix^5)}{4-5x^2+x^4}\,dx$$
$$=\frac{1}{5}\,ix^5+\frac{1}{4}\,(h-2\,i)x^4+\frac{1}{3}\,(g-2\,h+4\,i)x^3+\frac{1}{2}\,(f-2\,g+4\,h-8\,i)x^2+(e-2\,f+4\,g-8\,h+16\,i)x+(d-2\,e+4\,f-8\,g+16\,h-32\,i)\log(x+2)$$

input
```
integrate((x^3-2*x^2-x+2)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x,
 algorithm="fricas")
```

output
```
1/5*i*x^5 + 1/4*(h - 2*i)*x^4 + 1/3*(g - 2*h + 4*i)*x^3 + 1/2*(f - 2*g + 4
*h - 8*i)*x^2 + (e - 2*f + 4*g - 8*h + 16*i)*x + (d - 2*e + 4*f - 8*g + 16
*h - 32*i)*log(x + 2)
```

3.72.6 Sympy [A] (verification not implemented)

Time = 0.13 (sec) , antiderivative size = 88, normalized size of antiderivative = 0.96

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4+ix^5)}{4-5x^2+x^4}\,dx$$
$$=\frac{ix^5}{5}+x^4\left(\frac{h}{4}-\frac{i}{2}\right)+x^3\left(\frac{g}{3}-\frac{2h}{3}+\frac{4i}{3}\right)+x^2\left(\frac{f}{2}-g+2h-4i\right)+x(e-2f+4g-8h+16i)+(d-2e+4f-8g+16h-32i)\log(x+2)$$

input
```
integrate((x**3-2*x**2-x+2)*(i*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(x**4-5*x*
*2+4),x)
```

output
```
i*x**5/5 + x**4*(h/4 - i/2) + x**3*(g/3 - 2*h/3 + 4*i/3) + x**2*(f/2 - g +
 2*h - 4*i) + x*(e - 2*f + 4*g - 8*h + 16*i) + (d - 2*e + 4*f - 8*g + 16*h
 - 32*i)*log(x + 2)
```

3.72.7 Maxima [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 84, normalized size of antiderivative = 0.91

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4+ix^5)}{4-5x^2+x^4}\,dx$$
$$= \frac{1}{5}\,ix^5 + \frac{1}{4}\,(h-2\,i)x^4 + \frac{1}{3}\,(g-2\,h+4\,i)x^3 + \frac{1}{2}\,(f-2\,g+4\,h-8\,i)x^2 + (e-2\,f+4\,g-8\,h+16\,i)x + (d-2\,e+4\,f-8\,g+16\,h-32\,i)\log(x+2)$$

input
```
integrate((x^3-2*x^2-x+2)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x,
 algorithm="maxima")
```

output
```
1/5*i*x^5 + 1/4*(h - 2*i)*x^4 + 1/3*(g - 2*h + 4*i)*x^3 + 1/2*(f - 2*g + 4
*h - 8*i)*x^2 + (e - 2*f + 4*g - 8*h + 16*i)*x + (d - 2*e + 4*f - 8*g + 16
*h - 32*i)*log(x + 2)
```

3.72.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 103, normalized size of antiderivative = 1.12

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4+ix^5)}{4-5x^2+x^4}\,dx$$
$$= \frac{1}{5}\,ix^5 + \frac{1}{4}\,hx^4 - \frac{1}{2}\,ix^4 + \frac{1}{3}\,gx^3 - \frac{2}{3}\,hx^3 + \frac{4}{3}\,ix^3 + \frac{1}{2}\,fx^2 - gx^2 + 2\,hx^2 - 4\,ix^2 + ex - 2\,fx + 4\,gx - 8\,hx + 16\,ix + (d-2\,e+4\,f-8\,g+16\,h-32\,i)\log(|x+2|)$$

input
```
integrate((x^3-2*x^2-x+2)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x,
 algorithm="giac")
```

output
```
1/5*i*x^5 + 1/4*h*x^4 - 1/2*i*x^4 + 1/3*g*x^3 - 2/3*h*x^3 + 4/3*i*x^3 + 1/
2*f*x^2 - g*x^2 + 2*h*x^2 - 4*i*x^2 + e*x - 2*f*x + 4*g*x - 8*h*x + 16*i*x
 + (d - 2*e + 4*f - 8*g + 16*h - 32*i)*log(abs(x + 2))
```

3.72.9 Mupad [B] (verification not implemented)

Time = 0.02 (sec) , antiderivative size = 87, normalized size of antiderivative = 0.95

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4+ix^5)}{4-5x^2+x^4}\,dx$$
$$= x^4\left(\frac{h}{4}-\frac{i}{2}\right)+\ln(x+2)\,(d-2\,e+4\,f-8\,g+16\,h-32\,i)+\frac{i\,x^5}{5}$$
$$+x^2\left(\frac{f}{2}-g+2\,h-4\,i\right)+x\,(e-2\,f+4\,g-8\,h+16\,i)+x^3\left(\frac{g}{3}-\frac{2\,h}{3}+\frac{4\,i}{3}\right)$$

input
```
int(-((x + 2*x^2 - x^3 - 2)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5))/(x^
4 - 5*x^2 + 4),x)
```

output
```
x^4*(h/4 - i/2) + log(x + 2)*(d - 2*e + 4*f - 8*g + 16*h - 32*i) + (i*x^5)
/5 + x^2*(f/2 - g + 2*h - 4*i) + x*(e - 2*f + 4*g - 8*h + 16*i) + x^3*(g/3
 - (2*h)/3 + (4*i)/3)
```

3.73 $\int \frac{2-3x+x^2}{4-5x^2+x^4} dx$

3.73.1 Optimal result

Integrand size = 21, antiderivative size = 11

$$\int \frac{2-3x+x^2}{4-5x^2+x^4} \, dx = \log(1+x) - \log(2+x)$$

output
```
ln(1+x)-ln(2+x)
```

3.73.2 Mathematica [A] (verified)

Time = 0.05 (sec) , antiderivative size = 11, normalized size of antiderivative = 1.00

$$\int \frac{2-3x+x^2}{4-5x^2+x^4} \, dx = \log(1+x) - \log(2+x)$$

input
```
Integrate[(2 - 3*x + x^2)/(4 - 5*x^2 + x^4),x]
```

output
```
Log[1 + x] - Log[2 + x]
```

3.73.3 Rubi [A] (verified)

Time = 0.18 (sec) , antiderivative size = 11, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.143, Rules used = {2019, 1081, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{x^2-3x+2}{x^4-5x^2+4}dx$$
$$\Big\downarrow \text{2019}$$
$$\int \frac{1}{x^2+3x+2}dx$$
$$\Big\downarrow \text{1081}$$
$$\int \left(\frac{1}{x+1}+\frac{1}{-x-2}\right)dx$$
$$\Big\downarrow \text{2009}$$
$$\log(x+1)-\log(x+2)$$

input
```
Int[(2 - 3*x + x^2)/(4 - 5*x^2 + x^4),x]
```

output
```
Log[1 + x] - Log[2 + x]
```

3.73.3.1 Defintions of rubi rules used

rule 1081
```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> With[{q = Rt[b^2
- 4*a*c, 2]}, Simp[c   Int[ExpandIntegrand[1/((b/2 - q/2 + c*x)*(b/2 + q/2
+ c*x)), x], x], x]] /; FreeQ[{a, b, c}, x] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

3.73.4 Maple [A] (verified)

Time = 0.03 (sec) , antiderivative size = 12, normalized size of antiderivative = 1.09

method	result	size
default	$\ln(x+1)-\ln(x+2)$	12
norman	$\ln(x+1)-\ln(x+2)$	12
risch	$\ln(x+1)-\ln(x+2)$	12
parallelrisch	$\ln(x+1)-\ln(x+2)$	12

input
```
int((x^2-3*x+2)/(x^4-5*x^2+4),x,method=_RETURNVERBOSE)
```

output
```
ln(x+1)-ln(x+2)
```

3.73.5 Fricas [A] (verification not implemented)

Time = 0.25 (sec) , antiderivative size = 11, normalized size of antiderivative = 1.00

$$\int \frac{2-3x+x^2}{4-5x^2+x^4}\,dx = -\log(x+2)+\log(x+1)$$

input
```
integrate((x^2-3*x+2)/(x^4-5*x^2+4),x, algorithm="fricas")
```

output
```
-log(x + 2) + log(x + 1)
```

3.73.6 Sympy [A] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 8, normalized size of antiderivative = 0.73

$$\int \frac{2-3x+x^2}{4-5x^2+x^4}\,dx = \log(x+1)-\log(x+2)$$

input
```
integrate((x**2-3*x+2)/(x**4-5*x**2+4),x)
```

output
```
log(x + 1) - log(x + 2)
```

3.73.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 11, normalized size of antiderivative = 1.00

$$\int \frac{2-3x+x^2}{4-5x^2+x^4}\,dx = -\log{(x+2)} + \log{(x+1)}$$

input
```
integrate((x^2-3*x+2)/(x^4-5*x^2+4),x, algorithm="maxima")
```

output
```
-log(x + 2) + log(x + 1)
```

3.73.8 Giac [A] (verification not implemented)

Time = 0.31 (sec) , antiderivative size = 13, normalized size of antiderivative = 1.18

$$\int \frac{2-3x+x^2}{4-5x^2+x^4}\,dx = -\log{(|x+2|)} + \log{(|x+1|)}$$

input
```
integrate((x^2-3*x+2)/(x^4-5*x^2+4),x, algorithm="giac")
```

output
```
-log(abs(x + 2)) + log(abs(x + 1))
```

3.73.9 Mupad [B] (verification not implemented)

Time = 0.05 (sec) , antiderivative size = 8, normalized size of antiderivative = 0.73

$$\int \frac{2-3x+x^2}{4-5x^2+x^4}\,dx = -2\,\mathrm{atanh}(2\,x+3)$$

input
```
int((x^2 - 3*x + 2)/(x^4 - 5*x^2 + 4),x)
```

output
```
-2*atanh(2*x + 3)
```

3.74 $\int \frac{(d+ex)(2-3x+x^2)}{4-5x^2+x^4}\,dx$

3.74.1 Optimal result

Integrand size = 26, antiderivative size = 22

$$\int \frac{(d+ex)\left(2-3x+x^2\right)}{4-5x^2+x^4}\,dx=(d-e)\log(1+x)-(d-2e)\log(2+x)$$

output
```
(d-e)*ln(1+x)-(d-2*e)*ln(2+x)
```

3.74.2 Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 23, normalized size of antiderivative = 1.05

$$\int \frac{(d+ex)\left(2-3x+x^2\right)}{4-5x^2+x^4}\,dx=(d-e)\log(1+x)+(-d+2e)\log(2+x)$$

input
```
Integrate[((d + e*x)*(2 - 3*x + x^2))/(4 - 5*x^2 + x^4),x]
```

output
```
(d - e)*Log[1 + x] + (-d + 2*e)*Log[2 + x]
```

3.74.3 Rubi [A] (verified)

Time = 0.20 (sec) , antiderivative size = 22, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.115, Rules used = {2019, 1141, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^2-3x+2)(d+ex)}{x^4-5x^2+4}dx$$
$$\Big\downarrow \text{2019}$$
$$\int \frac{d+ex}{x^2+3x+2}dx$$
$$\Big\downarrow \text{1141}$$
$$\int \left(\frac{d-e}{x+1}-\frac{d-2e}{x+2}\right)dx$$
$$\Big\downarrow \text{2009}$$
$$(d-e)\log(x+1)-(d-2e)\log(x+2)$$

input
```
Int[((d + e*x)*(2 - 3*x + x^2))/(4 - 5*x^2 + x^4),x]
```

output
```
(d - e)*Log[1 + x] - (d - 2*e)*Log[2 + x]
```

3.74.3.1 Defintions of rubi rules used

rule 1141
```
Int[((d_.) + (e_.)*(x_))^(m_.)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_
Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[1/c^p   Int[ExpandIntegrand[
(d + e*x)^m*(b/2 - q/2 + c*x)^p*(b/2 + q/2 + c*x)^p, x], x], x] /; EqQ[p, -
1] ||  !FractionalPowerFactorQ[q]] /; FreeQ[{a, b, c, d, e}, x] && ILtQ[p,
0] && IntegerQ[m] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

3.74.4 Maple [A] (verified)

Time = 0.04 (sec) , antiderivative size = 24, normalized size of antiderivative = 1.09

method	result	size
default	$(-d+2e)\ln(x+2)+(d-e)\ln(x+1)$	24
norman	$(-d+2e)\ln(x+2)+(d-e)\ln(x+1)$	24
parallelrisch	$\ln(x+1)\,d-\ln(x+1)\,e-\ln(x+2)\,d+2\ln(x+2)\,e$	29
risch	$-\ln(x+2)\,d+2\ln(x+2)\,e+\ln(-x-1)\,d-\ln(-x-1)\,e$	33

input
```
int((e*x+d)*(x^2-3*x+2)/(x^4-5*x^2+4),x,method=_RETURNVERBOSE)
```

output
```
(-d+2*e)*ln(x+2)+(d-e)*ln(x+1)
```

3.74.5 Fricas [A] (verification not implemented)

Time = 0.25 (sec) , antiderivative size = 22, normalized size of antiderivative = 1.00

$$\int \frac{(d+ex)(2-3x+x^2)}{4-5x^2+x^4}\,dx = -(d-2\,e)\log(x+2)+(d-e)\log(x+1)$$

input
```
integrate((e*x+d)*(x^2-3*x+2)/(x^4-5*x^2+4),x, algorithm="fricas")
```

output
```
-(d - 2*e)*log(x + 2) + (d - e)*log(x + 1)
```

3.74.6 Sympy [A] (verification not implemented)

Time = 0.17 (sec) , antiderivative size = 29, normalized size of antiderivative = 1.32

$$\int \frac{(d+ex)\left(2-3x+x^2\right)}{4-5x^2+x^4}\,dx=(-d+2e)\log\left(x+\frac{4d-6e}{2d-3e}\right)+(d-e)\log\left(x+1\right)$$

input
```
integrate((e*x+d)*(x**2-3*x+2)/(x**4-5*x**2+4),x)
```

output
```
(-d + 2*e)*log(x + (4*d - 6*e)/(2*d - 3*e)) + (d - e)*log(x + 1)
```

3.74.7 Maxima [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 22, normalized size of antiderivative = 1.00

$$\int \frac{(d+ex)\left(2-3x+x^2\right)}{4-5x^2+x^4}\,dx=-(d-2\,e)\log\left(x+2\right)+(d-e)\log\left(x+1\right)$$

input
```
integrate((e*x+d)*(x^2-3*x+2)/(x^4-5*x^2+4),x, algorithm="maxima")
```

output
```
-(d - 2*e)*log(x + 2) + (d - e)*log(x + 1)
```

3.74.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 24, normalized size of antiderivative = 1.09

$$\int \frac{(d+ex)\left(2-3x+x^2\right)}{4-5x^2+x^4}\,dx=-(d-2\,e)\log\left(|x+2|\right)+(d-e)\log\left(|x+1|\right)$$

input
```
integrate((e*x+d)*(x^2-3*x+2)/(x^4-5*x^2+4),x, algorithm="giac")
```

output
```
-(d - 2*e)*log(abs(x + 2)) + (d - e)*log(abs(x + 1))
```

3.74.9 Mupad [B] (verification not implemented)

Time = 7.89 (sec) , antiderivative size = 22, normalized size of antiderivative = 1.00

$$\int \frac{(d+ex)\,(2-3x+x^2)}{4-5x^2+x^4}\,dx = \ln(x+1)\,(d-e) - \ln(x+2)\,(d-2\,e)$$

input
```
int(((d + e*x)*(x^2 - 3*x + 2))/(x^4 - 5*x^2 + 4),x)
```

output
```
log(x + 1)*(d - e) - log(x + 2)*(d - 2*e)
```

3.75 $\int \frac{(2-3x+x^2)(d+ex+fx^2)}{4-5x^2+x^4}\,dx$

3.75.1 Optimal result

Integrand size = 31, antiderivative size = 29

$$\int \frac{(2-3x+x^2)(d+ex+fx^2)}{4-5x^2+x^4}\,dx = fx+(d-e+f)\log(1+x)-(d-2e+4f)\log(2+x)$$

output
```
f*x+(d-e+f)*ln(1+x)-(d-2*e+4*f)*ln(2+x)
```

3.75.2 Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 30, normalized size of antiderivative = 1.03

$$\int \frac{(2-3x+x^2)(d+ex+fx^2)}{4-5x^2+x^4}\,dx = fx+(d-e+f)\log(1+x)+(-d+2e-4f)\log(2+x)$$

input
```
Integrate[((2 - 3*x + x^2)*(d + e*x + f*x^2))/(4 - 5*x^2 + x^4),x]
```

output
```
f*x + (d - e + f)*Log[1 + x] + (-d + 2*e - 4*f)*Log[2 + x]
```

3.75.3 Rubi [A] (verified)

Time = 0.24 (sec) , antiderivative size = 29, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.097, Rules used = {2019, 2188, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^2-3x+2)(d+ex+fx^2)}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{2019}$$

$$\int \frac{d+ex+fx^2}{x^2+3x+2}dx$$

$$\Big\downarrow \text{2188}$$

$$\int \left(\frac{d+x(e-3f)-2f}{x^2+3x+2}+f\right)dx$$

$$\Big\downarrow \text{2009}$$

$$\log(x+1)(d-e+f)-\log(x+2)(d-2e+4f)+fx$$

input
```
Int[((2 - 3*x + x^2)*(d + e*x + f*x^2))/(4 - 5*x^2 + x^4),x]
```

output
```
f*x + (d - e + f)*Log[1 + x] - (d - 2*e + 4*f)*Log[2 + x]
```

3.75.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2188
```
Int[(Pq_)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_.), x_Symbol] :> Int[Expand
Integrand[Pq*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c}, x] && PolyQ[Pq
, x] && IGtQ[p, -2]
```

3.75.4 Maple [A] (verified)

Time = 0.04 (sec) , antiderivative size = 31, normalized size of antiderivative = 1.07

method	result
default	$fx+(-d+2e-4f)\ln(x+2)+(d-e+f)\ln(x+1)$
norman	$fx+(-d+2e-4f)\ln(x+2)+(d-e+f)\ln(x+1)$
parallelrisch	$\ln(x+1)d-\ln(x+1)e+\ln(x+1)f-\ln(x+2)d+2\ln(x+2)e-4\ln(x+2)f+fx$
risch	$fx+\ln(-x-1)d-\ln(-x-1)e+\ln(-x-1)f-\ln(x+2)d+2\ln(x+2)e-4\ln(x$

input
```
int((x^2-3*x+2)*(f*x^2+e*x+d)/(x^4-5*x^2+4),x,method=_RETURNVERBOSE)
```

output
```
f*x+(-d+2*e-4*f)*ln(x+2)+(d-e+f)*ln(x+1)
```

3.75.5 Fricas [A] (verification not implemented)

Time = 0.24 (sec) , antiderivative size = 29, normalized size of antiderivative = 1.00

$$\int \frac{(2-3x+x^2)(d+ex+fx^2)}{4-5x^2+x^4}\,dx = fx-(d-2\,e+4\,f)\log(x+2)+(d-e+f)\log(x+1)$$

input
```
integrate((x^2-3*x+2)*(f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="fricas")
```

output
```
f*x - (d - 2*e + 4*f)*log(x + 2) + (d - e + f)*log(x + 1)
```

3.75.6 Sympy [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 44, normalized size of antiderivative = 1.52

$$\int \frac{(2-3x+x^2)(d+ex+fx^2)}{4-5x^2+x^4}\,dx = fx+(-d+2e-4f)\log\left(x+\frac{4d-6e+10f}{2d-3e+5f}\right) + (d-e+f)\log(x+1)$$

input
```
integrate((x**2-3*x+2)*(f*x**2+e*x+d)/(x**4-5*x**2+4),x)
```

output
```
f*x + (-d + 2*e - 4*f)*log(x + (4*d - 6*e + 10*f)/(2*d - 3*e + 5*f)) + (d
- e + f)*log(x + 1)
```

3.75.7 Maxima [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 29, normalized size of antiderivative = 1.00

$$\int \frac{(2-3x+x^2)(d+ex+fx^2)}{4-5x^2+x^4}\,dx = fx-(d-2e+4f)\log(x+2)+(d-e+f)\log(x+1)$$

input
```
integrate((x^2-3*x+2)*(f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="maxima")
```

output
```
f*x - (d - 2*e + 4*f)*log(x + 2) + (d - e + f)*log(x + 1)
```

3.75.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 31, normalized size of antiderivative = 1.07

$$\int \frac{(2-3x+x^2)(d+ex+fx^2)}{4-5x^2+x^4}\,dx = fx-(d-2e+4f)\log(|x+2|) + (d-e+f)\log(|x+1|)$$

input
```
integrate((x^2-3*x+2)*(f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="giac")
```

output
```
f*x - (d - 2*e + 4*f)*log(abs(x + 2)) + (d - e + f)*log(abs(x + 1))
```

3.75.9 Mupad [B] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 29, normalized size of antiderivative = 1.00

$$\int \frac{(2-3x+x^2)(d+ex+fx^2)}{4-5x^2+x^4}\,dx = f\,x+\ln(x+1)\,(d-e+f)-\ln(x+2)\,(d-2e+4f)$$

input
```
int(((x^2 - 3*x + 2)*(d + e*x + f*x^2))/(x^4 - 5*x^2 + 4),x)
```

output
```
f*x + log(x + 1)*(d - e + f) - log(x + 2)*(d - 2*e + 4*f)
```

3.76 $\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3)}{4-5x^2+x^4}\,dx$

3.76.1 Optimal result

Integrand size = 36, antiderivative size = 47

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3)}{4-5x^2+x^4}\,dx = (f-3g)x+\frac{gx^2}{2}+(d-e+f-g)\log(1+x) - (d-2e+4f-8g)\log(2+x)$$

output
```
(f-3*g)*x+1/2*g*x^2+(d-e+f-g)*ln(1+x)-(d-2*e+4*f-8*g)*ln(2+x)
```

3.76.2 Mathematica [A] (verified)

Time = 0.07 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.94

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3)}{4-5x^2+x^4}\,dx = fx+\frac{1}{2}g(-6+x)x+(d-e+f-g)\log(1+x) - (d-2e+4f-8g)\log(2+x)$$

input
```
Integrate[((2 - 3*x + x^2)*(d + e*x + f*x^2 + g*x^3))/(4 - 5*x^2 + x^4),x]
```

output
```
f*x + (g*(-6 + x)*x)/2 + (d - e + f - g)*Log[1 + x] - (d - 2*e + 4*f - 8*g
)*Log[2 + x]
```

3.76.3 Rubi [A] (verified)

Time = 0.27 (sec) , antiderivative size = 47, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.083, Rules used = {2019, 2188, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^2-3x+2)(d+ex+fx^2+gx^3)}{x^4-5x^2+4}dx$$
$$\Big\downarrow \text{2019}$$
$$\int \frac{d+ex+fx^2+gx^3}{x^2+3x+2}dx$$
$$\Big\downarrow \text{2188}$$
$$\int \left(\frac{d+x(e-3f+7g)-2f+6g}{x^2+3x+2}+f+gx-3g\right)dx$$
$$\Big\downarrow \text{2009}$$
$$\log(x+1)(d-e+f-g)-\log(x+2)(d-2e+4f-8g)+x(f-3g)+\frac{gx^2}{2}$$

input
```
Int[((2 - 3*x + x^2)*(d + e*x + f*x^2 + g*x^3))/(4 - 5*x^2 + x^4),x]
```

output
```
(f - 3*g)*x + (g*x^2)/2 + (d - e + f - g)*Log[1 + x] - (d - 2*e + 4*f - 8*
g)*Log[2 + x]
```

3.76.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2188
```
Int[(Pq_)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_.), x_Symbol] :> Int[Expand
Integrand[Pq*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c}, x] && PolyQ[Pq
, x] && IGtQ[p, -2]
```

3.76.4 Maple [A] (verified)

Time = 0.05 (sec) , antiderivative size = 47, normalized size of antiderivative = 1.00

method	result
default	$\frac{g\,x^{2}}{2}+fx-3gx+\left(-d+2e-4f+8g\right)\ln\left(x+2\right)+\left(d-e+f-g\right)\ln\left(x+1\right)$
norman	$\left(f-3g\right)x+\frac{g\,x^{2}}{2}+\left(-d+2e-4f+8g\right)\ln\left(x+2\right)+\left(d-e+f-g\right)\ln\left(x+1\right)$
parallelrisch	$\frac{g\,x^{2}}{2}+fx-3gx+\ln\left(x+1\right)d-\ln\left(x+1\right)e+\ln\left(x+1\right)f-\ln\left(x+1\right)g-\ln\left(x+2\right)d+2$
risch	$\frac{g\,x^{2}}{2}+fx-3gx+\ln\left(-x-1\right)d-\ln\left(-x-1\right)e+\ln\left(-x-1\right)f-\ln\left(-x-1\right)g-\ln\left(x+2\right.$

input
```
int((x^2-3*x+2)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x,method=_RETURNVERBOSE)
```

output
```
1/2*g*x^2+f*x-3*g*x+(-d+2*e-4*f+8*g)*ln(x+2)+(d-e+f-g)*ln(x+1)
```

3.76.5 Fricas [A] (verification not implemented)

Time = 0.25 (sec) , antiderivative size = 45, normalized size of antiderivative = 0.96

$$\int \frac{(2-3x+x^2)\left(d+ex+fx^2+gx^3\right)}{4-5x^2+x^4}\,dx = \frac{1}{2}\,gx^2+(f-3\,g)x - (d-2\,e+4\,f-8\,g)\log(x+2) + (d-e+f-g)\log(x+1)$$

input
```
integrate((x^2-3*x+2)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="fric
as")
```

output
```
1/2*g*x^2 + (f - 3*g)*x - (d - 2*e + 4*f - 8*g)*log(x + 2) + (d - e + f -
g)*log(x + 1)
```

3.76.6 Sympy [A] (verification not implemented)

Time = 0.48 (sec) , antiderivative size = 66, normalized size of antiderivative = 1.40

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3)}{4-5x^2+x^4}\,dx = \frac{gx^2}{2} + x(f-3g) + (-d+2e-4f+8g)\log\left(x+\frac{4d-6e+10f-18g}{2d-3e+5f-9g}\right) + (d-e+f-g)\log(x+1)$$

input
```
integrate((x**2-3*x+2)*(g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4),x)
```

output
```
g*x**2/2 + x*(f - 3*g) + (-d + 2*e - 4*f + 8*g)*log(x + (4*d - 6*e + 10*f
- 18*g)/(2*d - 3*e + 5*f - 9*g)) + (d - e + f - g)*log(x + 1)
```

3.76.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 45, normalized size of antiderivative = 0.96

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3)}{4-5x^2+x^4}\,dx = \frac{1}{2}\,gx^2 + (f-3\,g)x - (d-2\,e+4\,f-8\,g)\log(x+2) + (d-e+f-g)\log(x+1)$$

input
```
integrate((x^2-3*x+2)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="maxi
ma")
```

output
```
1/2*g*x^2 + (f - 3*g)*x - (d - 2*e + 4*f - 8*g)*log(x + 2) + (d - e + f -
g)*log(x + 1)
```

3.76.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 47, normalized size of antiderivative = 1.00

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3)}{4-5x^2+x^4}\,dx = \frac{1}{2}\,gx^2 + fx - 3\,gx - (d-2\,e+4\,f-8\,g)\log(|x+2|) + (d-e+f-g)\log(|x+1|)$$

input
```
integrate((x^2-3*x+2)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="giac
")
```

output
```
1/2*g*x^2 + f*x - 3*g*x - (d - 2*e + 4*f - 8*g)*log(abs(x + 2)) + (d - e +
 f - g)*log(abs(x + 1))
```

3.76.9 Mupad [B] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 45, normalized size of antiderivative = 0.96

$$\int \frac{(2-3x+x^2)\,(d+ex+fx^2+gx^3)}{4-5x^2+x^4}\,dx = \ln(x+1)\,(d-e+f-g)+x\,(f-3\,g) + \frac{g\,x^2}{2} - \ln(x+2)\,(d-2\,e+4\,f-8\,g)$$

input
```
int(((x^2 - 3*x + 2)*(d + e*x + f*x^2 + g*x^3))/(x^4 - 5*x^2 + 4),x)
```

output
```
log(x + 1)*(d - e + f - g) + x*(f - 3*g) + (g*x^2)/2 - log(x + 2)*(d - 2*e
 + 4*f - 8*g)
```

3.77 $\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx$

3.77.1 Optimal result

Integrand size = 41, antiderivative size = 66

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx = (f-3g+7h)x+\frac{1}{2}(g-3h)x^2+\frac{hx^3}{3}+(d-e+f-g+h)\log(1+x) - (d-2e+4f-8g+16h)\log(2+x)$$

output
```
(f-3*g+7*h)*x+1/2*(g-3*h)*x^2+1/3*h*x^3+(d-e+f-g+h)*ln(1+x)-(d-2*e+4*f-8*g
+16*h)*ln(2+x)
```

3.77.2 Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 67, normalized size of antiderivative = 1.02

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx = (f-3g+7h)x+\frac{1}{2}(g-3h)x^2+\frac{hx^3}{3}+(d-e+f-g+h)\log(1+x) + (-d+2e-4f+8g-16h)\log(2+x)$$

input
```
Integrate[((2 - 3*x + x^2)*(d + e*x + f*x^2 + g*x^3 + h*x^4))/(4 - 5*x^2 +
 x^4),x]
```

output
```
(f - 3*g + 7*h)*x + ((g - 3*h)*x^2)/2 + (h*x^3)/3 + (d - e + f - g + h)*Lo
g[1 + x] + (-d + 2*e - 4*f + 8*g - 16*h)*Log[2 + x]
```

3.77.3 Rubi [A] (verified)

Time = 0.30 (sec) , antiderivative size = 66, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.073$, Rules used = {2019, 2188, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^2-3x+2)(d+ex+fx^2+gx^3+hx^4)}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{2019}$$

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{x^2+3x+2}dx$$

$$\Big\downarrow \text{2188}$$

$$\int \left(\frac{d+x(e-3f+7g-15h)-2f+6g-14h}{x^2+3x+2}+f+x(g-3h)-3g+hx^2+7h\right)dx$$

$$\Big\downarrow \text{2009}$$

$$\log(x+1)(d-e+f-g+h)-\log(x+2)(d-2e+4f-8g+16h)+x(f-3g+7h)+\frac{1}{2}x^2(g-3h)+\frac{hx^3}{3}$$

input
```
Int[((2 - 3*x + x^2)*(d + e*x + f*x^2 + g*x^3 + h*x^4))/(4 - 5*x^2 + x^4),
x]
```

output
```
(f - 3*g + 7*h)*x + ((g - 3*h)*x^2)/2 + (h*x^3)/3 + (d - e + f - g + h)*Lo
g[1 + x] - (d - 2*e + 4*f - 8*g + 16*h)*Log[2 + x]
```

3.77.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2188
```
Int[(Pq_)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_.), x_Symbol] :> Int[Expand
Integrand[Pq*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c}, x] && PolyQ[Pq
, x] && IGtQ[p, -2]
```

3.77.4 Maple [A] (verified)

Time = 0.06 (sec) , antiderivative size = 65, normalized size of antiderivative = 0.98

method	result
norman	$\left(\frac{g}{2}-\frac{3h}{2}\right)x^2+(f-3g+7h)\,x+\frac{h\,x^3}{3}+(-d+2e-4f+8g-16h)\ln(x+2)+(d-e+f$
default	$\frac{h\,x^3}{3}+\frac{g\,x^2}{2}-\frac{3h\,x^2}{2}+fx-3gx+7hx+(-d+2e-4f+8g-16h)\ln(x+2)+(d-e+f$
parallelrisch	$\frac{h\,x^3}{3}+\frac{g\,x^2}{2}-\frac{3h\,x^2}{2}+fx-3gx+7hx+\ln(x+1)\,d-\ln(x+1)\,e+\ln(x+1)\,f-\ln(x+1)$
risch	$\frac{h\,x^3}{3}+\frac{g\,x^2}{2}-\frac{3h\,x^2}{2}+fx-3gx+7hx+\ln(-x-1)\,d-\ln(-x-1)\,e+\ln(-x-1)\,f-\ln($

input
```
int((x^2-3*x+2)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x,method=_RETURNVE
RBOSE)
```

output
```
(1/2*g-3/2*h)*x^2+(f-3*g+7*h)*x+1/3*h*x^3+(-d+2*e-4*f+8*g-16*h)*ln(x+2)+(d
-e+f-g+h)*ln(x+1)
```

3.77.5 Fricas [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 62, normalized size of antiderivative = 0.94

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx$$
$$=\frac{1}{3}\,hx^3+\frac{1}{2}\,(g-3\,h)x^2+(f-3\,g+7\,h)x$$
$$-(d-2\,e+4\,f-8\,g+16\,h)\log(x+2)+(d-e+f-g+h)\log(x+1)$$

input
```
integrate((x^2-3*x+2)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm
="fricas")
```

output
```
1/3*h*x^3 + 1/2*(g - 3*h)*x^2 + (f - 3*g + 7*h)*x - (d - 2*e + 4*f - 8*g +
 16*h)*log(x + 2) + (d - e + f - g + h)*log(x + 1)
```

3.77.6 Sympy [A] (verification not implemented)

Time = 0.85 (sec) , antiderivative size = 94, normalized size of antiderivative = 1.42

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx$$
$$=\frac{hx^3}{3}+x^2\left(\frac{g}{2}-\frac{3h}{2}\right)+x(f-3g+7h)$$
$$+(-d+2e-4f+8g-16h)\log\left(x+\frac{4d-6e+10f-18g+34h}{2d-3e+5f-9g+17h}\right)$$
$$+(d-e+f-g+h)\log(x+1)$$

input
```
integrate((x**2-3*x+2)*(h*x**4+g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4),x)
```

output
```
h*x**3/3 + x**2*(g/2 - 3*h/2) + x*(f - 3*g + 7*h) + (-d + 2*e - 4*f + 8*g
- 16*h)*log(x + (4*d - 6*e + 10*f - 18*g + 34*h)/(2*d - 3*e + 5*f - 9*g +
17*h)) + (d - e + f - g + h)*log(x + 1)
```

3.77.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 62, normalized size of antiderivative = 0.94

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx = \frac{1}{3}\,hx^3+\frac{1}{2}\,(g-3\,h)x^2+(f-3\,g+7\,h)x - (d-2\,e+4\,f-8\,g+16\,h)\log(x+2)+(d-e+f-g+h)\log(x+1)$$

input
```
integrate((x^2-3*x+2)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm
="maxima")
```

output
```
1/3*h*x^3 + 1/2*(g - 3*h)*x^2 + (f - 3*g + 7*h)*x - (d - 2*e + 4*f - 8*g +
 16*h)*log(x + 2) + (d - e + f - g + h)*log(x + 1)
```

3.77.8 Giac [A] (verification not implemented)

Time = 0.31 (sec) , antiderivative size = 67, normalized size of antiderivative = 1.02

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx = \frac{1}{3}\,hx^3+\frac{1}{2}\,gx^2-\frac{3}{2}\,hx^2+fx-3\,gx+7\,hx - (d-2\,e+4\,f-8\,g+16\,h)\log(|x+2|)+(d-e+f-g+h)\log(|x+1|)$$

input
```
integrate((x^2-3*x+2)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm
="giac")
```

output
```
1/3*h*x^3 + 1/2*g*x^2 - 3/2*h*x^2 + f*x - 3*g*x + 7*h*x - (d - 2*e + 4*f -
 8*g + 16*h)*log(abs(x + 2)) + (d - e + f - g + h)*log(abs(x + 1))
```

3.77.9 Mupad [B] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 63, normalized size of antiderivative = 0.95

$$\int \frac{(2-3x+x^2)\,(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx$$
$$= x^2\left(\frac{g}{2}-\frac{3\,h}{2}\right)+x\,(f-3\,g+7\,h)-\ln(x+2)\,(d-2\,e+4\,f-8\,g+16\,h) + \frac{h\,x^3}{3}+\ln(x+1)\,(d-e+f-g+h)$$

input
```
int(((x^2 - 3*x + 2)*(d + e*x + f*x^2 + g*x^3 + h*x^4))/(x^4 - 5*x^2 + 4),
x)
```

output
```
x^2*(g/2 - (3*h)/2) + x*(f - 3*g + 7*h) - log(x + 2)*(d - 2*e + 4*f - 8*g
+ 16*h) + (h*x^3)/3 + log(x + 1)*(d - e + f - g + h)
```

3.78 $\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4+ix^5)}{4-5x^2+x^4}\,dx$

3.78.1 Optimal result

Integrand size = 46, antiderivative size = 90

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4+ix^5)}{4-5x^2+x^4}\,dx = (f-3g+7h-15i)x+\frac{1}{2}(g-3h+7i)x^2+\frac{1}{3}(h-3i)x^3+\frac{ix^4}{4} + (d-e+f-g+h-i)\log(1+x)-(d-2e+4f-8g+16h-32i)\log(2+x)$$

output
```
(f-3*g+7*h-15*i)*x+1/2*(g-3*h+7*i)*x^2+1/3*(h-3*i)*x^3+1/4*i*x^4+(d-e+f-g+
h-i)*ln(1+x)-(d-2*e+4*f-8*g+16*h-32*i)*ln(2+x)
```

3.78.2 Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 91, normalized size of antiderivative = 1.01

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4+ix^5)}{4-5x^2+x^4}\,dx = (f-3g+7h-15i)x+\frac{1}{2}(g-3h+7i)x^2+\frac{1}{3}(h-3i)x^3+\frac{ix^4}{4} + (d-e+f-g+h-i)\log(1+x)+(-d+2e-4f+8g-16h+32i)\log(2+x)$$

input
```
Integrate[((2 - 3*x + x^2)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5))/(4 -
 5*x^2 + x^4),x]
```

output
```
(f - 3*g + 7*h - 15*i)*x + ((g - 3*h + 7*i)*x^2)/2 + ((h - 3*i)*x^3)/3 + (
i*x^4)/4 + (d - e + f - g + h - i)*Log[1 + x] + (-d + 2*e - 4*f + 8*g - 16
*h + 32*i)*Log[2 + x]
```

3.78.3 Rubi [A] (verified)

Time = 0.33 (sec) , antiderivative size = 90, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.065, Rules used = {2019, 2188, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^2-3x+2)(d+ex+fx^2+gx^3+hx^4+ix^5)}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{2019}$$

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{x^2+3x+2}dx$$

$$\Big\downarrow \text{2188}$$

$$\int \left(\frac{d+x(e-3f+7g-15h+31i)-2f+6g-14h+30i}{x^2+3x+2}+f+x(g-3h+7i)-3g+x^2(h-3i)+7h+ix^3-\right.$$

$$\Big\downarrow \text{2009}$$

$$\log(x+1)(d-e+f-g+h-i)-\log(x+2)(d-2e+4f-8g+16h-32i)+x(f-3g+7h-15i)+\frac{1}{2}x^2(g-3h+7i)+\frac{1}{3}x^3(h-3i)+\frac{ix^4}{4}$$

input
```
Int[((2 - 3*x + x^2)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5))/(4 - 5*x^2
 + x^4),x]
```

output
```
(f - 3*g + 7*h - 15*i)*x + ((g - 3*h + 7*i)*x^2)/2 + ((h - 3*i)*x^3)/3 + (
i*x^4)/4 + (d - e + f - g + h - i)*Log[1 + x] - (d - 2*e + 4*f - 8*g + 16*
h - 32*i)*Log[2 + x]
```

3.78.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2188
```
Int[(Pq_)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_.), x_Symbol] :> Int[Expand
Integrand[Pq*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c}, x] && PolyQ[Pq
, x] && IGtQ[p, -2]
```

3.78.4 Maple [A] (verified)

Time = 0.06 (sec) , antiderivative size = 88, normalized size of antiderivative = 0.98

method	result
norman	$\left(\frac{h}{3}-i\right)x^3+\left(\frac{g}{2}-\frac{3h}{2}+\frac{7i}{2}\right)x^2+\left(f-3g+7h-15i\right)x+\frac{i\,x^4}{4}+\left(-d+2e-4f+8g-16h-\right.$
default	$\frac{i\,x^4}{4}+\frac{h\,x^3}{3}-i\,x^3+\frac{g\,x^2}{2}-\frac{3h\,x^2}{2}+\frac{7i\,x^2}{2}+fx-3gx+7hx-15ix+\left(-d+2e-4f+8g-16\right.$
parallelrisch	$\frac{i\,x^4}{4}+\frac{h\,x^3}{3}-i\,x^3+\frac{g\,x^2}{2}-\frac{3h\,x^2}{2}+\frac{7i\,x^2}{2}+fx-3gx+7hx-15ix+\ln\left(x+1\right)d-\ln\left(x+1\right)e$
risch	$\frac{i\,x^4}{4}+\frac{h\,x^3}{3}-i\,x^3+\frac{g\,x^2}{2}-\frac{3h\,x^2}{2}+\frac{7i\,x^2}{2}+fx-3gx+7hx-15ix+\ln\left(-x-1\right)d-\ln\left(-x-\right.$

input
```
int((x^2-3*x+2)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x,method=_RE
TURNVERBOSE)
```

output
```
(1/3*h-i)*x^3+(1/2*g-3/2*h+7/2*i)*x^2+(f-3*g+7*h-15*i)*x+1/4*i*x^4+(-d+2*e
-4*f+8*g-16*h+32*i)*ln(x+2)+(d-e+f-g+h-i)*ln(x+1)
```

3.78.5 Fricas [A] (verification not implemented)

Time = 0.25 (sec) , antiderivative size = 84, normalized size of antiderivative = 0.93

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4+ix^5)}{4-5x^2+x^4}\,dx$$
$$= \frac{1}{4}ix^4 + \frac{1}{3}(h-3i)x^3 + \frac{1}{2}(g-3h+7i)x^2 + (f-3g+7h-15i)x - (d-2e+4f-8g+16h-32i)\log(x+2) + (d-e+f-g+h-i)\log(x+1)$$

input
```
integrate((x^2-3*x+2)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, alg
orithm="fricas")
```

output
```
1/4*i*x^4 + 1/3*(h - 3*i)*x^3 + 1/2*(g - 3*h + 7*i)*x^2 + (f - 3*g + 7*h -
 15*i)*x - (d - 2*e + 4*f - 8*g + 16*h - 32*i)*log(x + 2) + (d - e + f - g
 + h - i)*log(x + 1)
```

3.78.6 Sympy [A] (verification not implemented)

Time = 1.39 (sec) , antiderivative size = 122, normalized size of antiderivative = 1.36

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4+ix^5)}{4-5x^2+x^4}\,dx$$
$$= \frac{ix^4}{4} + x^3\left(\frac{h}{3}-i\right) + x^2\left(\frac{g}{2}-\frac{3h}{2}+\frac{7i}{2}\right) + x(f-3g+7h-15i) + (-d+2e-4f+8g-16h+32i)\log\left(x+\frac{4d-6e+10f-18g+34h-66i}{2d-3e+5f-9g+17h-33i}\right) + (d-e+f-g+h-i)\log(x+1)$$

input
```
integrate((x**2-3*x+2)*(i*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4)
,x)
```

output
```
i*x**4/4 + x**3*(h/3 - i) + x**2*(g/2 - 3*h/2 + 7*i/2) + x*(f - 3*g + 7*h
- 15*i) + (-d + 2*e - 4*f + 8*g - 16*h + 32*i)*log(x + (4*d - 6*e + 10*f -
 18*g + 34*h - 66*i)/(2*d - 3*e + 5*f - 9*g + 17*h - 33*i)) + (d - e + f -
 g + h - i)*log(x + 1)
```

3.78.7 Maxima [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 84, normalized size of antiderivative = 0.93

$$\int \frac{(2-3x+x^2)\,(d+ex+fx^2+gx^3+hx^4+ix^5)}{4-5x^2+x^4}\,dx$$
$$= \frac{1}{4}\,ix^4 + \frac{1}{3}\,(h-3\,i)x^3 + \frac{1}{2}\,(g-3\,h+7\,i)x^2 + (f-3\,g+7\,h-15\,i)x - (d-2\,e+4\,f-8\,g+16\,h-32\,i)\log{(x+2)} + (d-e+f-g+h-i)\log{(x+1)}$$

input
```
integrate((x^2-3*x+2)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, alg
orithm="maxima")
```

output
```
1/4*i*x^4 + 1/3*(h - 3*i)*x^3 + 1/2*(g - 3*h + 7*i)*x^2 + (f - 3*g + 7*h -
 15*i)*x - (d - 2*e + 4*f - 8*g + 16*h - 32*i)*log(x + 2) + (d - e + f - g
 + h - i)*log(x + 1)
```

3.78.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 95, normalized size of antiderivative = 1.06

$$\int \frac{(2-3x+x^2)\,(d+ex+fx^2+gx^3+hx^4+ix^5)}{4-5x^2+x^4}\,dx$$
$$= \frac{1}{4}\,ix^4 + \frac{1}{3}\,hx^3 - ix^3 + \frac{1}{2}\,gx^2 - \frac{3}{2}\,hx^2 + \frac{7}{2}\,ix^2 + fx - 3\,gx + 7\,hx - 15\,ix - (d-2\,e+4\,f-8\,g+16\,h-32\,i)\log{(|x+2|)} + (d-e+f-g+h-i)\log{(|x+1|)}$$

input
```
integrate((x^2-3*x+2)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, alg
orithm="giac")
```

output
```
1/4*i*x^4 + 1/3*h*x^3 - i*x^3 + 1/2*g*x^2 - 3/2*h*x^2 + 7/2*i*x^2 + f*x -
3*g*x + 7*h*x - 15*i*x - (d - 2*e + 4*f - 8*g + 16*h - 32*i)*log(abs(x + 2
)) + (d - e + f - g + h - i)*log(abs(x + 1))
```

3.78.9 Mupad [B] (verification not implemented)

Time = 0.05 (sec) , antiderivative size = 86, normalized size of antiderivative = 0.96

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4+ix^5)}{4-5x^2+x^4}\,dx$$
$$= x^3\left(\frac{h}{3}-i\right)-\ln(x+2)\,(d-2\,e+4\,f-8\,g+16\,h-32\,i) + \ln(x+1)\,(d-e+f-g+h-i)+\frac{i\,x^4}{4}+x^2\left(\frac{g}{2}-\frac{3\,h}{2}+\frac{7\,i}{2}\right)+x\,(f-3\,g+7\,h-15\,i)$$

input
```
int(((x^2 - 3*x + 2)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5))/(x^4 - 5*x
^2 + 4),x)
```

output
```
x^3*(h/3 - i) - log(x + 2)*(d - 2*e + 4*f - 8*g + 16*h - 32*i) + log(x + 1
)*(d - e + f - g + h - i) + (i*x^4)/4 + x^2*(g/2 - (3*h)/2 + (7*i)/2) + x*
(f - 3*g + 7*h - 15*i)
```

3.79 $\int \frac{2+x}{4-5x^2+x^4} \, dx$

3.79.1 Optimal result

Integrand size = 16, antiderivative size = 29

$$\int \frac{2+x}{4-5x^2+x^4} \, dx = -\frac{1}{2}\log(1-x)+\frac{1}{3}\log(2-x)+\frac{1}{6}\log(1+x)$$

output
```
-1/2*ln(1-x)+1/3*ln(2-x)+1/6*ln(1+x)
```

3.79.2 Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 29, normalized size of antiderivative = 1.00

$$\int \frac{2+x}{4-5x^2+x^4} \, dx = -\frac{1}{2}\log(1-x)+\frac{1}{3}\log(2-x)+\frac{1}{6}\log(1+x)$$

input
```
Integrate[(2 + x)/(4 - 5*x^2 + x^4),x]
```

output
```
-1/2*Log[1 - x] + Log[2 - x]/3 + Log[1 + x]/6
```

3.79.3 Rubi [A] (verified)

Time = 0.20 (sec) , antiderivative size = 29, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.188, Rules used = {2019, 2462, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{x+2}{x^4-5x^2+4}\,dx$$

$$\Big\downarrow \ 2019$$

$$\int \frac{1}{x^3-2x^2-x+2}dx$$

$$\Big\downarrow \ 2462$$

$$\int \left(-\frac{1}{2(x-1)}+\frac{1}{6(x+1)}+\frac{1}{3(x-2)}\right)dx$$

$$\Big\downarrow \ 2009$$

$$-\frac{1}{2}\log(1-x)+\frac{1}{3}\log(2-x)+\frac{1}{6}\log(x+1)$$

input
```
Int[(2 + x)/(4 - 5*x^2 + x^4),x]
```

output
```
-1/2*Log[1 - x] + Log[2 - x]/3 + Log[1 + x]/6
```

3.79.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2462
```
Int[(u_.)*(Px_)^(p_), x_Symbol] :> With[{Qx = Factor[Px]}, Int[ExpandIntegr
and[u*Qx^p, x], x] /;  !SumQ[NonfreeFactors[Qx, x]]] /; PolyQ[Px, x] && GtQ
[Expon[Px, x], 2] &&  !BinomialQ[Px, x] &&  !TrinomialQ[Px, x] && ILtQ[p, 0
] && RationalFunctionQ[u, x]
```

3.79.4 Maple [A] (verified)

Time = 0.04 (sec) , antiderivative size = 20, normalized size of antiderivative = 0.69

method	result	size
default	$\frac{\ln(x+1)}{6}-\frac{\ln(x-1)}{2}+\frac{\ln(x-2)}{3}$	20
norman	$\frac{\ln(x+1)}{6}-\frac{\ln(x-1)}{2}+\frac{\ln(x-2)}{3}$	20
risch	$\frac{\ln(x+1)}{6}-\frac{\ln(x-1)}{2}+\frac{\ln(x-2)}{3}$	20
parallelrisch	$\frac{\ln(x+1)}{6}-\frac{\ln(x-1)}{2}+\frac{\ln(x-2)}{3}$	20

input
```
int((x+2)/(x^4-5*x^2+4),x,method=_RETURNVERBOSE)
```

output
```
1/6*ln(x+1)-1/2*ln(x-1)+1/3*ln(x-2)
```

3.79.5 Fricas [A] (verification not implemented)

Time = 0.25 (sec) , antiderivative size = 19, normalized size of antiderivative = 0.66

$$\int \frac{2+x}{4-5x^2+x^4}\,dx = \frac{1}{6}\log\left(x+1\right)-\frac{1}{2}\log\left(x-1\right)+\frac{1}{3}\log\left(x-2\right)$$

input
```
integrate((2+x)/(x^4-5*x^2+4),x, algorithm="fricas")
```

output
```
1/6*log(x + 1) - 1/2*log(x - 1) + 1/3*log(x - 2)
```

3.79.6 Sympy [A] (verification not implemented)

Time = 0.06 (sec) , antiderivative size = 19, normalized size of antiderivative = 0.66

$$\int \frac{2+x}{4-5x^2+x^4}\,dx = \frac{\log\left(x-2\right)}{3}-\frac{\log\left(x-1\right)}{2}+\frac{\log\left(x+1\right)}{6}$$

input
```
integrate((2+x)/(x**4-5*x**2+4),x)
```

output
```
log(x - 2)/3 - log(x - 1)/2 + log(x + 1)/6
```

3.79.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 19, normalized size of antiderivative = 0.66

$$\int \frac{2+x}{4-5x^2+x^4}\,dx = \frac{1}{6}\,\log\left(x+1\right) - \frac{1}{2}\,\log\left(x-1\right) + \frac{1}{3}\,\log\left(x-2\right)$$

input
```
integrate((2+x)/(x^4-5*x^2+4),x, algorithm="maxima")
```

output
```
1/6*log(x + 1) - 1/2*log(x - 1) + 1/3*log(x - 2)
```

3.79.8 Giac [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 22, normalized size of antiderivative = 0.76

$$\int \frac{2+x}{4-5x^2+x^4}\,dx = \frac{1}{6}\,\log\left(|x+1|\right) - \frac{1}{2}\,\log\left(|x-1|\right) + \frac{1}{3}\,\log\left(|x-2|\right)$$

input
```
integrate((2+x)/(x^4-5*x^2+4),x, algorithm="giac")
```

output
```
1/6*log(abs(x + 1)) - 1/2*log(abs(x - 1)) + 1/3*log(abs(x - 2))
```

3.79.9 Mupad [B] (verification not implemented)

Time = 0.04 (sec) , antiderivative size = 19, normalized size of antiderivative = 0.66

$$\int \frac{2+x}{4-5x^2+x^4}\,dx = \frac{\ln\left(x+1\right)}{6} - \frac{\ln\left(x-1\right)}{2} + \frac{\ln\left(x-2\right)}{3}$$

input
```
int((x + 2)/(x^4 - 5*x^2 + 4),x)
```

output
```
log(x + 1)/6 - log(x - 1)/2 + log(x - 2)/3
```

3.80 $\int \frac{(2+x)(d+ex)}{4-5x^2+x^4}\,dx$

3.80.1 Optimal result

Integrand size = 21, antiderivative size = 42

$$\int \frac{(2+x)(d+ex)}{4-5x^2+x^4}\,dx = -\frac{1}{2}(d+e)\log(1-x)+\frac{1}{3}(d+2e)\log(2-x)+\frac{1}{6}(d-e)\log(1+x)$$

output
```
-1/2*(d+e)*ln(1-x)+1/3*(d+2*e)*ln(2-x)+1/6*(d-e)*ln(1+x)
```

3.80.2 Mathematica [A] (verified)

Time = 0.01 (sec) , antiderivative size = 39, normalized size of antiderivative = 0.93

$$\int \frac{(2+x)(d+ex)}{4-5x^2+x^4}\,dx = \frac{1}{6}(-3(d+e)\log(1-x)+2(d+2e)\log(2-x)+(d-e)\log(1+x))$$

input
```
Integrate[((2 + x)*(d + e*x))/(4 - 5*x^2 + x^4),x]
```

output
```
(-3*(d + e)*Log[1 - x] + 2*(d + 2*e)*Log[2 - x] + (d - e)*Log[1 + x])/6
```

3.80.3 Rubi [A] (verified)

Time = 0.23 (sec) , antiderivative size = 42, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.143, Rules used = {2019, 2462, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x+2)(d+ex)}{x^4-5x^2+4} \, dx$$

$$\Big\downarrow \text{ 2019}$$

$$\int \frac{d+ex}{x^3-2x^2-x+2}dx$$

$$\Big\downarrow \text{ 2462}$$

$$\int \left(\frac{-d-e}{2(x-1)}+\frac{d+2e}{3(x-2)}+\frac{d-e}{6(x+1)}\right)dx$$

$$\Big\downarrow \text{ 2009}$$

$$-\frac{1}{2}(d+e)\log(1-x)+\frac{1}{3}(d+2e)\log(2-x)+\frac{1}{6}(d-e)\log(x+1)$$

input
```
Int[((2 + x)*(d + e*x))/(4 - 5*x^2 + x^4),x]
```

output
```
-1/2*((d + e)*Log[1 - x]) + ((d + 2*e)*Log[2 - x])/3 + ((d - e)*Log[1 + x]
)/6
```

3.80.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2462
```
Int[(u_.)*(Px_)^(p_), x_Symbol] :> With[{Qx = Factor[Px]}, Int[ExpandIntegr
and[u*Qx^p, x], x] /;  !SumQ[NonfreeFactors[Qx, x]]] /; PolyQ[Px, x] && GtQ
[Expon[Px, x], 2] &&  !BinomialQ[Px, x] &&  !TrinomialQ[Px, x] && ILtQ[p, 0
] && RationalFunctionQ[u, x]
```

3.80.4 Maple [A] (verified)

Time = 0.05 (sec) , antiderivative size = 38, normalized size of antiderivative = 0.90

method	result	size
default	$\left(\frac{d}{6}-\frac{e}{6}\right)\ln\left(x+1\right)+\left(-\frac{d}{2}-\frac{e}{2}\right)\ln\left(x-1\right)+\left(\frac{d}{3}+\frac{2e}{3}\right)\ln\left(x-2\right)$	38
norman	$\left(\frac{d}{6}-\frac{e}{6}\right)\ln\left(x+1\right)+\left(-\frac{d}{2}-\frac{e}{2}\right)\ln\left(x-1\right)+\left(\frac{d}{3}+\frac{2e}{3}\right)\ln\left(x-2\right)$	38
parallelrisch	$\frac{\ln(x-2)d}{3}+\frac{2\ln(x-2)e}{3}-\frac{\ln(x-1)d}{2}-\frac{\ln(x-1)e}{2}+\frac{\ln(x+1)d}{6}-\frac{\ln(x+1)e}{6}$	44
risch	$\frac{\ln(x+1)d}{6}-\frac{\ln(x+1)e}{6}-\frac{\ln(1-x)d}{2}-\frac{\ln(1-x)e}{2}+\frac{\ln(2-x)d}{3}+\frac{2\ln(2-x)e}{3}$	52

input
```
int((x+2)*(e*x+d)/(x^4-5*x^2+4),x,method=_RETURNVERBOSE)
```

output
```
(1/6*d-1/6*e)*ln(x+1)+(-1/2*d-1/2*e)*ln(x-1)+(1/3*d+2/3*e)*ln(x-2)
```

3.80.5 Fricas [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 32, normalized size of antiderivative = 0.76

$$\int \frac{(2+x)(d+ex)}{4-5x^2+x^4}\,dx=\frac{1}{6}\,(d-e)\log\left(x+1\right)-\frac{1}{2}\,(d+e)\log\left(x-1\right)+\frac{1}{3}\,(d+2\,e)\log\left(x-2\right)$$

input
```
integrate((2+x)*(e*x+d)/(x^4-5*x^2+4),x, algorithm="fricas")
```

output
```
1/6*(d - e)*log(x + 1) - 1/2*(d + e)*log(x - 1) + 1/3*(d + 2*e)*log(x - 2)
```

3.80.6 Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 304 vs. $2(37) = 74$.

Time = 1.11 (sec) , antiderivative size = 304, normalized size of antiderivative = 7.24

$$\int \frac{(2+x)(d+ex)}{4-5x^2+x^4}\,dx = \frac{(d-e)\log\left(x+\frac{26d^3+66d^2e-9d^2(d-e)+78de^2-12de(d-e)-7d(d-e)^2+46e^3+3e^2(d-e)-8e(d-e)^2}{10d^3+69d^2e+102de^2+35e^3}\right)}{6} - \frac{(d+e)\log\left(x+\frac{26d^3+66d^2e+27d^2(d+e)+78de^2+36de(d+e)-63d(d+e)^2+46e^3-9e^2(d+e)-72e(d+e)^2}{10d^3+69d^2e+102de^2+35e^3}\right)}{2} + \frac{(d+2e)\log\left(x+\frac{26d^3+66d^2e-18d^2(d+2e)+78de^2-24de(d+2e)-28d(d+2e)^2+46e^3+6e^2(d+2e)-32e(d+2e)^2}{10d^3+69d^2e+102de^2+35e^3}\right)}{3}$$

input
```
integrate((2+x)*(e*x+d)/(x**4-5*x**2+4),x)
```

output
```
(d - e)*log(x + (26*d**3 + 66*d**2*e - 9*d**2*(d - e) + 78*d*e**2 - 12*d*e
*(d - e) - 7*d*(d - e)**2 + 46*e**3 + 3*e**2*(d - e) - 8*e*(d - e)**2)/(10
*d**3 + 69*d**2*e + 102*d*e**2 + 35*e**3))/6 - (d + e)*log(x + (26*d**3 +
66*d**2*e + 27*d**2*(d + e) + 78*d*e**2 + 36*d*e*(d + e) - 63*d*(d + e)**2
 + 46*e**3 - 9*e**2*(d + e) - 72*e*(d + e)**2)/(10*d**3 + 69*d**2*e + 102*
d*e**2 + 35*e**3))/2 + (d + 2*e)*log(x + (26*d**3 + 66*d**2*e - 18*d**2*(d
 + 2*e) + 78*d*e**2 - 24*d*e*(d + 2*e) - 28*d*(d + 2*e)**2 + 46*e**3 + 6*e
**2*(d + 2*e) - 32*e*(d + 2*e)**2)/(10*d**3 + 69*d**2*e + 102*d*e**2 + 35*
e**3))/3
```

3.80.7 Maxima [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 32, normalized size of antiderivative = 0.76

$$\int \frac{(2+x)(d+ex)}{4-5x^2+x^4}\,dx = \frac{1}{6}(d-e)\log(x+1) - \frac{1}{2}(d+e)\log(x-1) + \frac{1}{3}(d+2e)\log(x-2)$$

input
```
integrate((2+x)*(e*x+d)/(x^4-5*x^2+4),x, algorithm="maxima")
```

output
```
1/6*(d - e)*log(x + 1) - 1/2*(d + e)*log(x - 1) + 1/3*(d + 2*e)*log(x - 2)
```

3.80.8 Giac [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 35, normalized size of antiderivative = 0.83

$$\int \frac{(2+x)(d+ex)}{4-5x^2+x^4}\,dx = \frac{1}{6}\,(d-e)\log\left(|x+1|\right) - \frac{1}{2}\,(d+e)\log\left(|x-1|\right) + \frac{1}{3}\,(d+2\,e)\log\left(|x-2|\right)$$

input
```
integrate((2+x)*(e*x+d)/(x^4-5*x^2+4),x, algorithm="giac")
```

output
```
1/6*(d - e)*log(abs(x + 1)) - 1/2*(d + e)*log(abs(x - 1)) + 1/3*(d + 2*e)*
log(abs(x - 2))
```

3.80.9 Mupad [B] (verification not implemented)

Time = 7.78 (sec) , antiderivative size = 38, normalized size of antiderivative = 0.90

$$\int \frac{(2+x)(d+ex)}{4-5x^2+x^4}\,dx = \ln(x-2)\left(\frac{d}{3}+\frac{2\,e}{3}\right) - \ln(x-1)\left(\frac{d}{2}+\frac{e}{2}\right) + \ln(x+1)\left(\frac{d}{6}-\frac{e}{6}\right)$$

input
```
int(((x + 2)*(d + e*x))/(x^4 - 5*x^2 + 4),x)
```

output
```
log(x - 2)*(d/3 + (2*e)/3) - log(x - 1)*(d/2 + e/2) + log(x + 1)*(d/6 - e/
6)
```

3.81 $\int \frac{(2+x)(d+ex+fx^2)}{4-5x^2+x^4}\,dx$

3.81.1 Optimal result

Integrand size = 26, antiderivative size = 47

$$\int \frac{(2+x)(d+ex+fx^2)}{4-5x^2+x^4}\,dx = -\frac{1}{2}(d+e+f)\log(1-x)+\frac{1}{3}(d+2e+4f)\log(2-x)+\frac{1}{6}(d-e+f)\log(1+x)$$

output
```
-1/2*(d+e+f)*ln(1-x)+1/3*(d+2*e+4*f)*ln(2-x)+1/6*(d-e+f)*ln(1+x)
```

3.81.2 Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 44, normalized size of antiderivative = 0.94

$$\int \frac{(2+x)(d+ex+fx^2)}{4-5x^2+x^4}\,dx = \frac{1}{6}(-3(d+e+f)\log(1-x)+2(d+2e+4f)\log(2-x)+(d-e+f)\log(1+x))$$

input
```
Integrate[((2 + x)*(d + e*x + f*x^2))/(4 - 5*x^2 + x^4),x]
```

output
```
(-3*(d + e + f)*Log[1 - x] + 2*(d + 2*e + 4*f)*Log[2 - x] + (d - e + f)*Lo
g[1 + x])/6
```

3.81.3 Rubi [A] (verified)

Time = 0.25 (sec) , antiderivative size = 47, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.115$, Rules used = {2019, 2462, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x+2)\left(d+ex+fx^2\right)}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{2019}$$

$$\int \frac{d+ex+fx^2}{x^3-2x^2-x+2}dx$$

$$\Big\downarrow \text{2462}$$

$$\int \left(\frac{-d-e-f}{2(x-1)}+\frac{d+2e+4f}{3(x-2)}+\frac{d-e+f}{6(x+1)}\right)dx$$

$$\Big\downarrow \text{2009}$$

$$-\frac{1}{2}\log(1-x)(d+e+f)+\frac{1}{3}\log(2-x)(d+2e+4f)+\frac{1}{6}\log(x+1)(d-e+f)$$

input
```
Int[((2 + x)*(d + e*x + f*x^2))/(4 - 5*x^2 + x^4),x]
```

output
```
-1/2*((d + e + f)*Log[1 - x]) + ((d + 2*e + 4*f)*Log[2 - x])/3 + ((d - e +
 f)*Log[1 + x])/6
```

3.81.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2462
```
Int[(u_.)*(Px_)^(p_), x_Symbol] :> With[{Qx = Factor[Px]}, Int[ExpandIntegr
and[u*Qx^p, x], x] /;  !SumQ[NonfreeFactors[Qx, x]]] /; PolyQ[Px, x] && GtQ
[Expon[Px, x], 2] &&  !BinomialQ[Px, x] &&  !TrinomialQ[Px, x] && ILtQ[p, 0
] && RationalFunctionQ[u, x]
```

3.81.4 Maple [A] (verified)

Time = 0.06 (sec) , antiderivative size = 47, normalized size of antiderivative = 1.00

method	result
default	$\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}\right)\ln\left(x+1\right)+\left(-\frac{d}{2}-\frac{e}{2}-\frac{f}{2}\right)\ln\left(x-1\right)+\left(\frac{d}{3}+\frac{2e}{3}+\frac{4f}{3}\right)\ln\left(x-2\right)$
norman	$\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}\right)\ln\left(x+1\right)+\left(-\frac{d}{2}-\frac{e}{2}-\frac{f}{2}\right)\ln\left(x-1\right)+\left(\frac{d}{3}+\frac{2e}{3}+\frac{4f}{3}\right)\ln\left(x-2\right)$
parallelrisch	$\frac{\ln(x-2)d}{3}+\frac{2\ln(x-2)e}{3}+\frac{4\ln(x-2)f}{3}-\frac{\ln(x-1)d}{2}-\frac{\ln(x-1)e}{2}-\frac{\ln(x-1)f}{2}+\frac{\ln(x+1)d}{6}-\frac{\ln(x+1)e}{6}+\frac{\ln(x+1)f}{6}$
risch	$\frac{\ln(2-x)d}{3}+\frac{2\ln(2-x)e}{3}+\frac{4\ln(2-x)f}{3}-\frac{\ln(1-x)d}{2}-\frac{\ln(1-x)e}{2}-\frac{\ln(1-x)f}{2}+\frac{\ln(x+1)d}{6}-\frac{\ln(x+1)e}{6}+\frac{\ln(x+1)f}{6}$

input
```
int((x+2)*(f*x^2+e*x+d)/(x^4-5*x^2+4),x,method=_RETURNVERBOSE)
```

output
```
(1/6*d-1/6*e+1/6*f)*ln(x+1)+(-1/2*d-1/2*e-1/2*f)*ln(x-1)+(1/3*d+2/3*e+4/3*
f)*ln(x-2)
```

3.81.5 Fricas [A] (verification not implemented)

Time = 0.25 (sec) , antiderivative size = 37, normalized size of antiderivative = 0.79

$$\int\frac{(2+x)\left(d+ex+fx^2\right)}{4-5x^2+x^4}\,dx=\frac{1}{6}\left(d-e+f\right)\log\left(x+1\right)-\frac{1}{2}\left(d+e+f\right)\log\left(x-1\right)+\frac{1}{3}\left(d+2\,e+4\,f\right)\log\left(x-2\right)$$

input
```
integrate((2+x)*(f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="fricas")
```

output
```
1/6*(d - e + f)*log(x + 1) - 1/2*(d + e + f)*log(x - 1) + 1/3*(d + 2*e + 4
*f)*log(x - 2)
```

3.81.6 Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 716 vs. $2(49) = 98$.

Time = 7.75 (sec) , antiderivative size = 716, normalized size of antiderivative = 15.23

$$\int \frac{(2+x)\,(d+ex+fx^2)}{4-5x^2+x^4}\,dx$$
$$= \frac{(d-e+f)\log\left(x+\frac{26d^3+66d^2e+132d^2f-9d^2(d-e+f)+78de^2+276def-12de(d-e+f)+222df^2+6df(d-e+f)-7d(d-e+f)^2+46e^3+2\cdots}{10d^3+69d^2e+102d^2f+102de^2+318def+246df^2\cdots}\right.}{6}$$
$$-\frac{(d+e+f)\log\left(x+\frac{26d^3+66d^2e+132d^2f+27d^2(d+e+f)+78de^2+276def+36de(d+e+f)+222df^2-18df(d+e+f)-63d(d+e+f)^2+4\cdots}{10d^3+69d^2e+102d^2f+102de^2+318def+\cdots}\right.}{2}$$
$$+\frac{(d+2e+4f)\log\left(x+\frac{26d^3+66d^2e+132d^2f-18d^2(d+2e+4f)+78de^2+276def-24de(d+2e+4f)+222df^2+12df(d+2e+4f)-28d(d-\cdots}{10d^3+69d^2e+102d^2f+102d\cdots}\right.}{3}$$

input
```
integrate((2+x)*(f*x**2+e*x+d)/(x**4-5*x**2+4),x)
```

output
```
(d - e + f)*log(x + (26*d**3 + 66*d**2*e + 132*d**2*f - 9*d**2*(d - e + f)
 + 78*d*e**2 + 276*d*e*f - 12*d*e*(d - e + f) + 222*d*f**2 + 6*d*f*(d - e
+ f) - 7*d*(d - e + f)**2 + 46*e**3 + 204*e**2*f + 3*e**2*(d - e + f) + 28
2*e*f**2 + 36*e*f*(d - e + f) - 8*e*(d - e + f)**2 + 116*f**3 + 51*f**2*(d
 - e + f) - 13*f*(d - e + f)**2)/(10*d**3 + 69*d**2*e + 102*d**2*f + 102*d
*e**2 + 318*d*e*f + 246*d*f**2 + 35*e**3 + 174*e**2*f + 285*e*f**2 + 154*f
**3))/6 - (d + e + f)*log(x + (26*d**3 + 66*d**2*e + 132*d**2*f + 27*d**2*
(d + e + f) + 78*d*e**2 + 276*d*e*f + 36*d*e*(d + e + f) + 222*d*f**2 - 18
*d*f*(d + e + f) - 63*d*(d + e + f)**2 + 46*e**3 + 204*e**2*f - 9*e**2*(d
+ e + f) + 282*e*f**2 - 108*e*f*(d + e + f) - 72*e*(d + e + f)**2 + 116*f*
*3 - 153*f**2*(d + e + f) - 117*f*(d + e + f)**2)/(10*d**3 + 69*d**2*e + 1
02*d**2*f + 102*d*e**2 + 318*d*e*f + 246*d*f**2 + 35*e**3 + 174*e**2*f + 2
85*e*f**2 + 154*f**3))/2 + (d + 2*e + 4*f)*log(x + (26*d**3 + 66*d**2*e +
132*d**2*f - 18*d**2*(d + 2*e + 4*f) + 78*d*e**2 + 276*d*e*f - 24*d*e*(d +
 2*e + 4*f) + 222*d*f**2 + 12*d*f*(d + 2*e + 4*f) - 28*d*(d + 2*e + 4*f)**
2 + 46*e**3 + 204*e**2*f + 6*e**2*(d + 2*e + 4*f) + 282*e*f**2 + 72*e*f*(d
 + 2*e + 4*f) - 32*e*(d + 2*e + 4*f)**2 + 116*f**3 + 102*f**2*(d + 2*e + 4
*f) - 52*f*(d + 2*e + 4*f)**2)/(10*d**3 + 69*d**2*e + 102*d**2*f + 102*d*e
**2 + 318*d*e*f + 246*d*f**2 + 35*e**3 + 174*e**2*f + 285*e*f**2 + 154*f**
3))/3
```

3.81.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 37, normalized size of antiderivative = 0.79

$$\int \frac{(2+x)(d+ex+fx^2)}{4-5x^2+x^4}\,dx = \frac{1}{6}(d-e+f)\log(x+1) - \frac{1}{2}(d+e+f)\log(x-1) + \frac{1}{3}(d+2\,e+4\,f)\log(x-2)$$

input
```
integrate((2+x)*(f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="maxima")
```

output
```
1/6*(d - e + f)*log(x + 1) - 1/2*(d + e + f)*log(x - 1) + 1/3*(d + 2*e + 4
*f)*log(x - 2)
```

3.81.8 Giac [A] (verification not implemented)

Time = 0.31 (sec) , antiderivative size = 40, normalized size of antiderivative = 0.85

$$\int \frac{(2+x)(d+ex+fx^2)}{4-5x^2+x^4}\,dx = \frac{1}{6}(d-e+f)\log(|x+1|) - \frac{1}{2}(d+e+f)\log(|x-1|) + \frac{1}{3}(d+2\,e+4\,f)\log(|x-2|)$$

input
```
integrate((2+x)*(f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="giac")
```

output
```
1/6*(d - e + f)*log(abs(x + 1)) - 1/2*(d + e + f)*log(abs(x - 1)) + 1/3*(d
 + 2*e + 4*f)*log(abs(x - 2))
```

3.81.9 Mupad [B] (verification not implemented)

Time = 0.07 (sec) , antiderivative size = 47, normalized size of antiderivative = 1.00

$$\int \frac{(2+x)(d+ex+fx^2)}{4-5x^2+x^4}\,dx = \ln(x-2)\left(\frac{d}{3}+\frac{2\,e}{3}+\frac{4\,f}{3}\right) - \ln(x-1)\left(\frac{d}{2}+\frac{e}{2}+\frac{f}{2}\right) + \ln(x+1)\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}\right)$$

input
```
int(((x + 2)*(d + e*x + f*x^2))/(x^4 - 5*x^2 + 4),x)
```

output
```
log(x - 2)*(d/3 + (2*e)/3 + (4*f)/3) - log(x - 1)*(d/2 + e/2 + f/2) + log(
x + 1)*(d/6 - e/6 + f/6)
```

3.81. $\int \frac{(2+x)(d+ex+fx^2)}{4-5x^2+x^4}\,dx$

3.82 $\int \frac{(2+x)(d+ex+fx^2+gx^3)}{4-5x^2+x^4} \, dx$

3.82.1 Optimal result

Integrand size = 31, antiderivative size = 57

$$\int \frac{(2+x)(d+ex+fx^2+gx^3)}{4-5x^2+x^4} \, dx = gx - \frac{1}{2}(d+e+f+g)\log(1-x) + \frac{1}{3}(d+2e+4f+8g)\log(2-x) + \frac{1}{6}(d-e+f-g)\log(1+x)$$

output
```
g*x-1/2*(d+e+f+g)*ln(1-x)+1/3*(d+2*e+4*f+8*g)*ln(2-x)+1/6*(d-e+f-g)*ln(1+x
)
```

3.82.2 Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 55, normalized size of antiderivative = 0.96

$$\int \frac{(2+x)(d+ex+fx^2+gx^3)}{4-5x^2+x^4} \, dx = \frac{1}{6}(6gx - 3(d+e+f+g)\log(1-x) + 2(d+2e+4f+8g)\log(2-x) + (d-e+f-g)\log(1+x))$$

input
```
Integrate[((2 + x)*(d + e*x + f*x^2 + g*x^3))/(4 - 5*x^2 + x^4),x]
```

output
```
(6*g*x - 3*(d + e + f + g)*Log[1 - x] + 2*(d + 2*e + 4*f + 8*g)*Log[2 - x]
 + (d - e + f - g)*Log[1 + x])/6
```

3.82.3 Rubi [A] (verified)

Time = 0.27 (sec) , antiderivative size = 57, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.097$, Rules used = {2019, 2462, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x+2)\left(d+ex+fx^2+gx^3\right)}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{ 2019}$$

$$\int \frac{d+ex+fx^2+gx^3}{x^3-2x^2-x+2}dx$$

$$\Big\downarrow \text{ 2462}$$

$$\int \left(\frac{-d-e-f-g}{2(x-1)}+\frac{d+2e+4f+8g}{3(x-2)}+\frac{d-e+f-g}{6(x+1)}+g\right)dx$$

$$\Big\downarrow \text{ 2009}$$

$$-\frac{1}{2}\log(1-x)(d+e+f+g)+\frac{1}{3}\log(2-x)(d+2e+4f+8g)+\frac{1}{6}\log(x+1)(d-e+f-g)+gx$$

input
```
Int[((2 + x)*(d + e*x + f*x^2 + g*x^3))/(4 - 5*x^2 + x^4),x]
```

output
```
g*x - ((d + e + f + g)*Log[1 - x])/2 + ((d + 2*e + 4*f + 8*g)*Log[2 - x])/
3 + ((d - e + f - g)*Log[1 + x])/6
```

3.82.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2462
```
Int[(u_.)*(Px_)^(p_), x_Symbol] :> With[{Qx = Factor[Px]}, Int[ExpandIntegr
and[u*Qx^p, x], x] /;  !SumQ[NonfreeFactors[Qx, x]]] /; PolyQ[Px, x] && GtQ
[Expon[Px, x], 2] &&  !BinomialQ[Px, x] &&  !TrinomialQ[Px, x] && ILtQ[p, 0
] && RationalFunctionQ[u, x]
```

3.82.4 Maple [A] (verified)

Time = 0.06 (sec) , antiderivative size = 59, normalized size of antiderivative = 1.04

method	result
default	$gx + \left(\frac{d}{6} - \frac{e}{6} + \frac{f}{6} - \frac{g}{6}\right)\ln(x+1) + \left(-\frac{d}{2} - \frac{e}{2} - \frac{f}{2} - \frac{g}{2}\right)\ln(x-1) + \left(\frac{d}{3} + \frac{2e}{3} + \frac{4f}{3} + \frac{8g}{3}\right)\ln(x$
norman	$gx + \left(\frac{d}{6} - \frac{e}{6} + \frac{f}{6} - \frac{g}{6}\right)\ln(x+1) + \left(-\frac{d}{2} - \frac{e}{2} - \frac{f}{2} - \frac{g}{2}\right)\ln(x-1) + \left(\frac{d}{3} + \frac{2e}{3} + \frac{4f}{3} + \frac{8g}{3}\right)\ln(x$
parallelrisch	$gx + \frac{\ln(x-2)d}{3} + \frac{2\ln(x-2)e}{3} + \frac{4\ln(x-2)f}{3} + \frac{8\ln(x-2)g}{3} - \frac{\ln(x-1)d}{2} - \frac{\ln(x-1)e}{2} - \frac{\ln(x-1)f}{2} - \frac{\ln(x-1)g}{2} +$
risch	$gx + \frac{\ln(x+1)d}{6} - \frac{\ln(x+1)e}{6} + \frac{\ln(x+1)f}{6} - \frac{\ln(x+1)g}{6} - \frac{\ln(1-x)d}{2} - \frac{\ln(1-x)e}{2} - \frac{\ln(1-x)f}{2} - \frac{\ln(1-x)g}{2} +$

input
```
int((x+2)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x,method=_RETURNVERBOSE)
```

output
```
g*x+(1/6*d-1/6*e+1/6*f-1/6*g)*ln(x+1)+(-1/2*d-1/2*e-1/2*f-1/2*g)*ln(x-1)+(
1/3*d+2/3*e+4/3*f+8/3*g)*ln(x-2)
```

3.82.5 Fricas [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 47, normalized size of antiderivative = 0.82

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3\right)}{4-5x^2+x^4}\,dx = gx + \frac{1}{6}(d-e+f-g)\log(x+1) - \frac{1}{2}(d+e+f+g)\log(x-1) + \frac{1}{3}(d+2\,e+4\,f+8\,g)\log(x-2)$$

input
```
integrate((2+x)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="fricas")
```

output
```
g*x + 1/6*(d - e + f - g)*log(x + 1) - 1/2*(d + e + f + g)*log(x - 1) + 1/
3*(d + 2*e + 4*f + 8*g)*log(x - 2)
```

3.82.6 Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 1389 vs. $2(63) = 126$.

Time = 54.94 (sec) , antiderivative size = 1389, normalized size of antiderivative = 24.37

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3\right)}{4-5x^2+x^4}\,dx = \text{Too large to display}$$

input
```
integrate((2+x)*(g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4),x)
```

output
```
g*x + (d - e + f - g)*log(x + (26*d**3 + 66*d**2*e + 132*d**2*f + 174*d**2
*g - 9*d**2*(d - e + f - g) + 78*d*e**2 + 276*d*e*f + 444*d*e*g - 12*d*e*(
d - e + f - g) + 222*d*f**2 + 636*d*f*g + 6*d*f*(d - e + f - g) + 510*d*g*
*2 + 36*d*g*(d - e + f - g) - 7*d*(d - e + f - g)**2 + 46*e**3 + 204*e**2*
f + 390*e**2*g + 3*e**2*(d - e + f - g) + 282*e*f**2 + 984*e*f*g + 36*e*f*
(d - e + f - g) + 930*e*g**2 + 102*e*g*(d - e + f - g) - 8*e*(d - e + f -
g)**2 + 116*f**3 + 534*f**2*g + 51*f**2*(d - e + f - g) + 924*f*g**2 + 228
*f*g*(d - e + f - g) - 13*f*(d - e + f - g)**2 + 586*g**3 + 243*g**2*(d -
e + f - g) - 20*g*(d - e + f - g)**2)/(10*d**3 + 69*d**2*e + 102*d**2*f +
213*d**2*g + 102*d*e**2 + 318*d*e*f + 564*d*e*g + 246*d*f**2 + 894*d*f*g +
 750*d*g**2 + 35*e**3 + 174*e**2*f + 249*e**2*g + 285*e*f**2 + 852*e*f*g +
 537*e*g**2 + 154*f**3 + 717*f**2*g + 966*f*g**2 + 323*g**3))/6 - (d + e +
 f + g)*log(x + (26*d**3 + 66*d**2*e + 132*d**2*f + 174*d**2*g + 27*d**2*(
d + e + f + g) + 78*d*e**2 + 276*d*e*f + 444*d*e*g + 36*d*e*(d + e + f + g
) + 222*d*f**2 + 636*d*f*g - 18*d*f*(d + e + f + g) + 510*d*g**2 - 108*d*g
*(d + e + f + g) - 63*d*(d + e + f + g)**2 + 46*e**3 + 204*e**2*f + 390*e*
*2*g - 9*e**2*(d + e + f + g) + 282*e*f**2 + 984*e*f*g - 108*e*f*(d + e +
f + g) + 930*e*g**2 - 306*e*g*(d + e + f + g) - 72*e*(d + e + f + g)**2 +
116*f**3 + 534*f**2*g - 153*f**2*(d + e + f + g) + 924*f*g**2 - 684*f*g*(d
 + e + f + g) - 117*f*(d + e + f + g)**2 + 586*g**3 - 729*g**2*(d + e +...
```

3.82.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 47, normalized size of antiderivative = 0.82

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3\right)}{4-5x^2+x^4}\,dx = gx + \frac{1}{6}\,(d-e+f-g)\log\left(x+1\right) - \frac{1}{2}\,(d+e+f+g)\log\left(x-1\right) + \frac{1}{3}\,(d+2\,e+4\,f+8\,g)\log\left(x-2\right)$$

input
```
integrate((2+x)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="maxima")
```

output
```
g*x + 1/6*(d - e + f - g)*log(x + 1) - 1/2*(d + e + f + g)*log(x - 1) + 1/
3*(d + 2*e + 4*f + 8*g)*log(x - 2)
```

3.82.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 50, normalized size of antiderivative = 0.88

$$\int \frac{(2+x)\,(d+ex+fx^2+gx^3)}{4-5x^2+x^4}\,dx = gx + \frac{1}{6}\,(d-e+f-g)\log\left(|x+1|\right) - \frac{1}{2}\,(d+e+f+g)\log\left(|x-1|\right) + \frac{1}{3}\,(d+2\,e+4\,f+8\,g)\log\left(|x-2|\right)$$

input
```
integrate((2+x)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="giac")
```

output
```
g*x + 1/6*(d - e + f - g)*log(abs(x + 1)) - 1/2*(d + e + f + g)*log(abs(x
- 1)) + 1/3*(d + 2*e + 4*f + 8*g)*log(abs(x - 2))
```

3.82.9 Mupad [B] (verification not implemented)

Time = 7.87 (sec) , antiderivative size = 59, normalized size of antiderivative = 1.04

$$\int \frac{(2+x)\,(d+ex+fx^2+gx^3)}{4-5x^2+x^4}\,dx = \ln(x+1)\,\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}\right) - \ln(x-1)\,\left(\frac{d}{2}+\frac{e}{2}+\frac{f}{2}+\frac{g}{2}\right) + \ln(x-2)\,\left(\frac{d}{3}+\frac{2\,e}{3}+\frac{4\,f}{3}+\frac{8\,g}{3}\right) + g\,x$$

input
```
int(((x + 2)*(d + e*x + f*x^2 + g*x^3))/(x^4 - 5*x^2 + 4),x)
```

output
```
log(x + 1)*(d/6 - e/6 + f/6 - g/6) - log(x - 1)*(d/2 + e/2 + f/2 + g/2) +
log(x - 2)*(d/3 + (2*e)/3 + (4*f)/3 + (8*g)/3) + g*x
```

3.83 $\int \frac{(2+x)(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx$

3.83.1 Optimal result

Integrand size = 36, antiderivative size = 74

$$\int \frac{(2+x)(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx = (g+2h)x+\frac{hx^2}{2}-\frac{1}{2}(d+e+f+g+h)\log(1-x) + \frac{1}{3}(d+2e+4f+8g+16h)\log(2-x) + \frac{1}{6}(d-e+f-g+h)\log(1+x)$$

output
```
(g+2*h)*x+1/2*h*x^2-1/2*(d+e+f+g+h)*ln(1-x)+1/3*(d+2*e+4*f+8*g+16*h)*ln(2-
x)+1/6*(d-e+f-g+h)*ln(1+x)
```

3.83.2 Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 71, normalized size of antiderivative = 0.96

$$\int \frac{(2+x)(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx = \frac{1}{6}\big(6(g+2h)x+3hx^2 - 3(d+e+f+g+h)\log(1-x) + 2(d+2(e+2f+4g+8h))\log(2-x) + (d-e+f-g+h)\log(1+x)\big)$$

input
```
Integrate[((2 + x)*(d + e*x + f*x^2 + g*x^3 + h*x^4))/(4 - 5*x^2 + x^4),x]
```

output
```
(6*(g + 2*h)*x + 3*h*x^2 - 3*(d + e + f + g + h)*Log[1 - x] + 2*(d + 2*(e
+ 2*f + 4*g + 8*h))*Log[2 - x] + (d - e + f - g + h)*Log[1 + x])/6
```

3.83.3 Rubi [A] (verified)

Time = 0.30 (sec) , antiderivative size = 74, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.083$, Rules used = {2019, 2462, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x+2)\left(d+ex+fx^2+gx^3+hx^4\right)}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{ 2019}$$

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{x^3-2x^2-x+2}dx$$

$$\Big\downarrow \text{ 2462}$$

$$\int \left(\frac{-d-e-f-g-h}{2(x-1)}+\frac{d+2e+4f+8g+16h}{3(x-2)}+\frac{d-e+f-g+h}{6(x+1)}+g\left(\frac{2h}{g}+1\right)+hx\right)dx$$

$$\Big\downarrow \text{ 2009}$$

$$-\frac{1}{2}\log(1-x)(d+e+f+g+h)+\frac{1}{3}\log(2-x)(d+2e+4f+8g+16h)+\frac{1}{6}\log(x+1)(d-e+f-g+h)+x(g+2h)+\frac{hx^2}{2}$$

input
```
Int[((2 + x)*(d + e*x + f*x^2 + g*x^3 + h*x^4))/(4 - 5*x^2 + x^4),x]
```

output
```
(g + 2*h)*x + (h*x^2)/2 - ((d + e + f + g + h)*Log[1 - x])/2 + ((d + 2*e +
 4*f + 8*g + 16*h)*Log[2 - x])/3 + ((d - e + f - g + h)*Log[1 + x])/6
```

3.83.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2462
```
Int[(u_.)*(Px_)^(p_), x_Symbol] :> With[{Qx = Factor[Px]}, Int[ExpandIntegr
and[u*Qx^p, x], x] /;  !SumQ[NonfreeFactors[Qx, x]]] /; PolyQ[Px, x] && GtQ
[Expon[Px, x], 2] &&  !BinomialQ[Px, x] &&  !TrinomialQ[Px, x] && ILtQ[p, 0
] && RationalFunctionQ[u, x]
```

3.83.4 Maple [A] (verified)

Time = 0.07 (sec) , antiderivative size = 78, normalized size of antiderivative = 1.05

method	result
default	$\frac{h\,x^2}{2}+gx+2hx+\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}+\frac{h}{6}\right)\ln\left(x+1\right)+\left(-\frac{d}{2}-\frac{e}{2}-\frac{f}{2}-\frac{g}{2}-\frac{h}{2}\right)\ln\left(x-1\right)+\left(\frac{d}{3}\right.$
norman	$(g+2h)\,x+\frac{h\,x^2}{2}+\left(-\frac{d}{2}-\frac{e}{2}-\frac{f}{2}-\frac{g}{2}-\frac{h}{2}\right)\ln\left(x-1\right)+\left(\frac{d}{3}+\frac{2e}{3}+\frac{4f}{3}+\frac{8g}{3}+\frac{16h}{3}\right)\ln\left(x-2\right)$
parallelrisch	$\frac{h\,x^2}{2}+gx+2hx+\frac{\ln(x-2)d}{3}+\frac{2\ln(x-2)e}{3}+\frac{4\ln(x-2)f}{3}+\frac{8\ln(x-2)g}{3}+\frac{16\ln(x-2)h}{3}-\frac{\ln(x-1)d}{2}-\frac{\ln(x-1)}{2}$
risch	$\frac{h\,x^2}{2}+gx+2hx+\frac{\ln(x+1)d}{6}-\frac{\ln(x+1)e}{6}+\frac{\ln(x+1)f}{6}-\frac{\ln(x+1)g}{6}+\frac{\ln(x+1)h}{6}+\frac{\ln(2-x)d}{3}+\frac{2\ln(2-x)e}{3}+$

input
```
int((x+2)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x,method=_RETURNVERBOSE)
```

output
```
1/2*h*x^2+g*x+2*h*x+(1/6*d-1/6*e+1/6*f-1/6*g+1/6*h)*ln(x+1)+(-1/2*d-1/2*e-
1/2*f-1/2*g-1/2*h)*ln(x-1)+(1/3*d+2/3*e+4/3*f+8/3*g+16/3*h)*ln(x-2)
```

3.83.5 Fricas [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 62, normalized size of antiderivative = 0.84

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4\right)}{4-5x^2+x^4}\,dx = \frac{1}{2}\,hx^2 + (g+2\,h)x + \frac{1}{6}\,(d-e+f-g+h)\log\left(x+1\right) - \frac{1}{2}\,(d+e+f+g+h)\log\left(x-1\right) + \frac{1}{3}\,(d+2\,e+4\,f+8\,g+16\,h)\log\left(x-2\right)$$

input
```
integrate((2+x)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="fric
as")
```

output
```
1/2*h*x^2 + (g + 2*h)*x + 1/6*(d - e + f - g + h)*log(x + 1) - 1/2*(d + e
+ f + g + h)*log(x - 1) + 1/3*(d + 2*e + 4*f + 8*g + 16*h)*log(x - 2)
```

3.83.6 Sympy [F(-1)]

Timed out.

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4\right)}{4-5x^2+x^4}\,dx = \text{Timed out}$$

input
```
integrate((2+x)*(h*x**4+g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4),x)
```

output
```
Timed out
```

3.83.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 62, normalized size of antiderivative = 0.84

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4\right)}{4-5x^2+x^4}\,dx = \frac{1}{2}\,hx^2 + (g+2\,h)x + \frac{1}{6}\,(d-e+f-g+h)\log\left(x+1\right) - \frac{1}{2}\,(d+e+f+g+h)\log\left(x-1\right) + \frac{1}{3}\,(d+2\,e+4\,f+8\,g+16\,h)\log\left(x-2\right)$$

input
```
integrate((2+x)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="maxi
ma")
```

output
```
1/2*h*x^2 + (g + 2*h)*x + 1/6*(d - e + f - g + h)*log(x + 1) - 1/2*(d + e
+ f + g + h)*log(x - 1) + 1/3*(d + 2*e + 4*f + 8*g + 16*h)*log(x - 2)
```

3.83.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 65, normalized size of antiderivative = 0.88

$$\int \frac{(2+x)(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx = \frac{1}{2}\,hx^2+gx+2\,hx + \frac{1}{6}\,(d-e+f-g+h)\log\left(|x+1|\right) - \frac{1}{2}\,(d+e+f+g+h)\log\left(|x-1|\right) + \frac{1}{3}\,(d+2\,e+4\,f+8\,g+16\,h)\log\left(|x-2|\right)$$

input
```
integrate((2+x)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm="giac
")
```

output
```
1/2*h*x^2 + g*x + 2*h*x + 1/6*(d - e + f - g + h)*log(abs(x + 1)) - 1/2*(d
 + e + f + g + h)*log(abs(x - 1)) + 1/3*(d + 2*e + 4*f + 8*g + 16*h)*log(a
bs(x - 2))
```

3.83.9 Mupad [B] (verification not implemented)

Time = 8.03 (sec) , antiderivative size = 78, normalized size of antiderivative = 1.05

$$\int \frac{(2+x)(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx = x\,(g+2\,h)+\frac{h\,x^2}{2} - \ln\left(x-1\right)\,\left(\frac{d}{2}+\frac{e}{2}+\frac{f}{2}+\frac{g}{2}+\frac{h}{2}\right) + \ln\left(x+1\right)\,\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}+\frac{h}{6}\right) + \ln\left(x-2\right)\,\left(\frac{d}{3}+\frac{2\,e}{3}+\frac{4\,f}{3}+\frac{8\,g}{3}+\frac{16\,h}{3}\right)$$

input
```
int(((x + 2)*(d + e*x + f*x^2 + g*x^3 + h*x^4))/(x^4 - 5*x^2 + 4),x)
```

output
```
x*(g + 2*h) + (h*x^2)/2 - log(x - 1)*(d/2 + e/2 + f/2 + g/2 + h/2) + log(x
 + 1)*(d/6 - e/6 + f/6 - g/6 + h/6) + log(x - 2)*(d/3 + (2*e)/3 + (4*f)/3
+ (8*g)/3 + (16*h)/3)
```

3.83. $\int \frac{(2+x)(d+ex+fx^2+gx^3+hx^4)}{4-5x^2+x^4}\,dx$

3.84 $\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)}{4-5x^2+x^4} \, dx$

3.84.1 Optimal result

Integrand size = 41, antiderivative size = 96

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)}{4-5x^2+x^4} \, dx = (g+2h+5i)x+\frac{1}{2}(h+2i)x^2+\frac{ix^3}{3}-\frac{1}{2}(d+e+f+g+h+i)\log(1-x) + \frac{1}{3}(d+2e+4f+8g+16h+32i)\log(2-x)+\frac{1}{6}(d-e+f-g+h-i)\log(1+x)$$

output
```
(g+2*h+5*i)*x+1/2*(h+2*i)*x^2+1/3*i*x^3-1/2*(d+e+f+g+h+i)*ln(1-x)+1/3*(d+2
*e+4*f+8*g+16*h+32*i)*ln(2-x)+1/6*(d-e+f-g+h-i)*ln(1+x)
```

3.84.2 Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 91, normalized size of antiderivative = 0.95

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)}{4-5x^2+x^4} \, dx = \frac{1}{6}(6(g+2h+5i)x+3(h+2i)x^2+2ix^3-3(d+e+f+g+h+i)\log(1-x) + 2(d+2e+4(f+2g+4h+8i))\log(2-x)+(d-e+f-g+h-i)\log(1+x))$$

input
```
Integrate[((2 + x)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5))/(4 - 5*x^2 +
 x^4),x]
```

output
```
(6*(g + 2*h + 5*i)*x + 3*(h + 2*i)*x^2 + 2*i*x^3 - 3*(d + e + f + g + h +
i)*Log[1 - x] + 2*(d + 2*e + 4*(f + 2*g + 4*h + 8*i))*Log[2 - x] + (d - e
+ f - g + h - i)*Log[1 + x])/6
```

3.84.3 Rubi [A] (verified)

Time = 0.35 (sec) , antiderivative size = 96, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.073$, Rules used = {2019, 2462, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x+2)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)}{x^4-5x^2+4}dx$$

$$\Big\downarrow \text{ 2019}$$

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{x^3-2x^2-x+2}dx$$

$$\Big\downarrow \text{ 2462}$$

$$\int \left(\frac{d+2e+4f+8g+16h+32i}{3(x-2)}+\frac{-d-e-f-g-h-i}{2(x-1)}+\frac{d-e+f-g+h-i}{6(x+1)}+g\left(\frac{2h+5i}{g}+1\right)+x(h-$$

$$\Big\downarrow \text{ 2009}$$

$$-\frac{1}{2}\log(1-x)(d+e+f+g+h+i)+\frac{1}{3}\log(2-x)(d+2e+4f+8g+16h+32i)+\frac{1}{6}\log(x+1)(d-e+f-g+h-i)+x(g+2h+5i)+\frac{1}{2}x^2(h+2i)+\frac{ix^3}{3}$$

input
```
Int[((2 + x)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5))/(4 - 5*x^2 + x^4),
x]
```

output
```
(g + 2*h + 5*i)*x + ((h + 2*i)*x^2)/2 + (i*x^3)/3 - ((d + e + f + g + h +
i)*Log[1 - x])/2 + ((d + 2*e + 4*f + 8*g + 16*h + 32*i)*Log[2 - x])/3 + ((
d - e + f - g + h - i)*Log[1 + x])/6
```

3.84.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2462
```
Int[(u_.)*(Px_)^(p_), x_Symbol] :> With[{Qx = Factor[Px]}, Int[ExpandIntegr
and[u*Qx^p, x], x] /;  !SumQ[NonfreeFactors[Qx, x]]] /; PolyQ[Px, x] && GtQ
[Expon[Px, x], 2] &&  !BinomialQ[Px, x] &&  !TrinomialQ[Px, x] && ILtQ[p, 0
] && RationalFunctionQ[u, x]
```

3.84.4 Maple [A] (verified)

Time = 0.08 (sec) , antiderivative size = 99, normalized size of antiderivative = 1.03

method	result
norman	$\left(\frac{h}{2}+i\right)x^2+\left(g+2h+5i\right)x+\frac{i\,x^3}{3}+\left(-\frac{d}{2}-\frac{e}{2}-\frac{f}{2}-\frac{g}{2}-\frac{h}{2}-\frac{i}{2}\right)\ln\left(x-1\right)+\left(\frac{d}{3}+\frac{2e}{3}+\frac{4f}{3}-\right.$
default	$\frac{i\,x^3}{3}+\frac{h\,x^2}{2}+i\,x^2+gx+2hx+5ix+\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}+\frac{h}{6}-\frac{i}{6}\right)\ln\left(x+1\right)+\left(-\frac{d}{2}-\frac{e}{2}-\frac{f}{2}-\right.$
parallelrisch	$gx+i\,x^2+\frac{\ln(x-2)d}{3}+\frac{2\ln(x-2)e}{3}-\frac{\ln(x-1)d}{2}-\frac{\ln(x-1)e}{2}+\frac{h\,x^2}{2}+\frac{i\,x^3}{3}-\frac{\ln(x+1)i}{6}+\frac{\ln(x+1)f}{6}+2hx$
risch	$\frac{i\,x^3}{3}+\frac{h\,x^2}{2}+i\,x^2+gx+2hx+5ix+\frac{\ln(x+1)d}{6}-\frac{\ln(x+1)e}{6}+\frac{\ln(x+1)f}{6}-\frac{\ln(x+1)g}{6}+\frac{\ln(x+1)h}{6}-\ln($

input
```
int((x+2)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x,method=_RETURNVE
RBOSE)
```

output
```
(1/2*h+i)*x^2+(g+2*h+5*i)*x+1/3*i*x^3+(-1/2*d-1/2*e-1/2*f-1/2*g-1/2*h-1/2*
i)*ln(x-1)+(1/3*d+2/3*e+4/3*f+8/3*g+16/3*h+32/3*i)*ln(x-2)+(1/6*d-1/6*e+1/
6*f-1/6*g+1/6*h-1/6*i)*ln(x+1)
```

3.84.5 Fricas [A] (verification not implemented)

Time = 0.32 (sec) , antiderivative size = 82, normalized size of antiderivative = 0.85

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)}{4-5x^2+x^4}\,dx = \frac{1}{3}\,ix^3+\frac{1}{2}\,(h+2\,i)x^2+(g+2\,h+5\,i)x+\frac{1}{6}\,(d-e+f-g+h-i)\log\left(x+1\right) - \frac{1}{2}\,(d+e+f+g+h+i)\log\left(x-1\right)+\frac{1}{3}\,(d+2\,e+4\,f+8\,g+16\,h+32\,i)\log\left(x-2\right)$$

input
```
integrate((2+x)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm
="fricas")
```

output
```
1/3*i*x^3 + 1/2*(h + 2*i)*x^2 + (g + 2*h + 5*i)*x + 1/6*(d - e + f - g + h
 - i)*log(x + 1) - 1/2*(d + e + f + g + h + i)*log(x - 1) + 1/3*(d + 2*e +
 4*f + 8*g + 16*h + 32*i)*log(x - 2)
```

3.84.6 Sympy [F(-1)]

Timed out.

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)}{4-5x^2+x^4}\,dx = \text{Timed out}$$

input
```
integrate((2+x)*(i*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4),x)
```

output
```
Timed out
```

3.84.7 Maxima [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 82, normalized size of antiderivative = 0.85

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)}{4-5x^2+x^4}\,dx = \frac{1}{3}\,ix^3+\frac{1}{2}\,(h+2\,i)x^2+(g+2\,h+5\,i)x+\frac{1}{6}\,(d-e+f-g+h-i)\log\left(x+1\right) - \frac{1}{2}\,(d+e+f+g+h+i)\log\left(x-1\right)+\frac{1}{3}\,(d+2\,e+4\,f+8\,g+16\,h+32\,i)\log\left(x-2\right)$$

input
```
integrate((2+x)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm
="maxima")
```

output
```
1/3*i*x^3 + 1/2*(h + 2*i)*x^2 + (g + 2*h + 5*i)*x + 1/6*(d - e + f - g + h
 - i)*log(x + 1) - 1/2*(d + e + f + g + h + i)*log(x - 1) + 1/3*(d + 2*e +
 4*f + 8*g + 16*h + 32*i)*log(x - 2)
```

3.84.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 87, normalized size of antiderivative = 0.91

$$\begin{aligned}\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)}{4-5x^2+x^4}\,dx &= \frac{1}{3}\,ix^3+\frac{1}{2}\,hx^2+ix^2+gx+2\,hx+5\,ix+\frac{1}{6}\,(d-e+f-g+h-i)\log\left(|x+1|\right)\\&\quad-\frac{1}{2}\,(d+e+f+g+h+i)\log\left(|x-1|\right)\\&\quad+\frac{1}{3}\,(d+2\,e+4\,f+8\,g+16\,h+32\,i)\log\left(|x-2|\right)\end{aligned}$$

input
```
integrate((2+x)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4),x, algorithm
="giac")
```

output
```
1/3*i*x^3 + 1/2*h*x^2 + i*x^2 + g*x + 2*h*x + 5*i*x + 1/6*(d - e + f - g +
 h - i)*log(abs(x + 1)) - 1/2*(d + e + f + g + h + i)*log(abs(x - 1)) + 1/
3*(d + 2*e + 4*f + 8*g + 16*h + 32*i)*log(abs(x - 2))
```

3.84.9 Mupad [B] (verification not implemented)

Time = 7.93 (sec) , antiderivative size = 99, normalized size of antiderivative = 1.03

$$\begin{aligned}\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)}{4-5x^2+x^4}\,dx &= x\,(g+2\,h+5\,i)+\frac{i\,x^3}{3}-\ln\left(x-1\right)\left(\frac{d}{2}+\frac{e}{2}+\frac{f}{2}+\frac{g}{2}+\frac{h}{2}+\frac{i}{2}\right)\\&\quad+\ln\left(x+1\right)\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}+\frac{h}{6}-\frac{i}{6}\right)\\&\quad+\ln\left(x-2\right)\left(\frac{d}{3}+\frac{2\,e}{3}+\frac{4\,f}{3}+\frac{8\,g}{3}+\frac{16\,h}{3}+\frac{32\,i}{3}\right)+x^2\left(\frac{h}{2}+i\right)\end{aligned}$$

input
```
int(((x + 2)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5))/(x^4 - 5*x^2 + 4),
x)
```

output
```
x*(g + 2*h + 5*i) + (i*x^3)/3 - log(x - 1)*(d/2 + e/2 + f/2 + g/2 + h/2 +
i/2) + log(x + 1)*(d/6 - e/6 + f/6 - g/6 + h/6 - i/6) + log(x - 2)*(d/3 +
(2*e)/3 + (4*f)/3 + (8*g)/3 + (16*h)/3 + (32*i)/3) + x^2*(h/2 + i)
```

3.84. $\int \frac{(2+x)(d+ex+fx^2+gx^3+hx^4+ix^5)}{4-5x^2+x^4}\,dx$

3.85 $\int \frac{2-x-2x^2+x^3}{(4-5x^2+x^4)^2}\,dx$

3.85.1 Optimal result

Integrand size = 26, antiderivative size = 46

$$\int \frac{2-x-2x^2+x^3}{(4-5x^2+x^4)^2}\,dx = \frac{1}{12(2+x)}-\frac{1}{18}\log(1-x)+\frac{1}{48}\log(2-x)+\frac{1}{6}\log(1+x)-\frac{19}{144}\log(2+x)$$

output
```
1/12/(2+x)-1/18*ln(1-x)+1/48*ln(2-x)+1/6*ln(1+x)-19/144*ln(2+x)
```

3.85.2 Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 42, normalized size of antiderivative = 0.91

$$\int \frac{2-x-2x^2+x^3}{(4-5x^2+x^4)^2}\,dx = \frac{1}{144}\left(\frac{12}{2+x}+24\log(-1-x)-8\log(1-x)+3\log(2-x)-19\log(2+x)\right)$$

input
```
Integrate[(2 - x - 2*x^2 + x^3)/(4 - 5*x^2 + x^4)^2,x]
```

output
```
(12/(2 + x) + 24*Log[-1 - x] - 8*Log[1 - x] + 3*Log[2 - x] - 19*Log[2 + x]
)/144
```

3.85.3 Rubi [A] (verified)

Time = 0.23 (sec) , antiderivative size = 46, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.115, Rules used = {2019, 2462, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{x^3-2x^2-x+2}{(x^4-5x^2+4)^2}dx$$
$$\big\downarrow \text{ 2019}$$
$$\int \frac{1}{(x+2)^2(x^3-2x^2-x+2)}dx$$
$$\big\downarrow \text{ 2462}$$
$$\int \left(-\frac{1}{18(x-1)}+\frac{1}{6(x+1)}-\frac{19}{144(x+2)}-\frac{1}{12(x+2)^2}+\frac{1}{48(x-2)}\right)dx$$
$$\big\downarrow \text{ 2009}$$
$$\frac{1}{12(x+2)}-\frac{1}{18}\log(1-x)+\frac{1}{48}\log(2-x)+\frac{1}{6}\log(x+1)-\frac{19}{144}\log(x+2)$$

input
```
Int[(2 - x - 2*x^2 + x^3)/(4 - 5*x^2 + x^4)^2,x]
```

output
```
1/(12*(2 + x)) - Log[1 - x]/18 + Log[2 - x]/48 + Log[1 + x]/6 - (19*Log[2
+ x])/144
```

3.85.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2462
```
Int[(u_.)*(Px_)^(p_), x_Symbol] :> With[{Qx = Factor[Px]}, Int[ExpandIntegr
and[u*Qx^p, x], x] /;  !SumQ[NonfreeFactors[Qx, x]]] /; PolyQ[Px, x] && GtQ
[Expon[Px, x], 2] &&  !BinomialQ[Px, x] &&  !TrinomialQ[Px, x] && ILtQ[p, 0
] && RationalFunctionQ[u, x]
```

3.85.4 Maple [A] (verified)

Time = 0.06 (sec) , antiderivative size = 33, normalized size of antiderivative = 0.72

method	result	size
default	$\frac{1}{12x+24}-\frac{19\ln(x+2)}{144}+\frac{\ln(x+1)}{6}-\frac{\ln(x-1)}{18}+\frac{\ln(x-2)}{48}$	33
risch	$\frac{1}{12x+24}-\frac{19\ln(x+2)}{144}+\frac{\ln(x+1)}{6}-\frac{\ln(x-1)}{18}+\frac{\ln(x-2)}{48}$	33
norman	$\frac{-\frac{1}{6}x^2-\frac{1}{12}x+\frac{1}{12}x^3+\frac{1}{6}}{x^4-5x^2+4}+\frac{\ln(x-2)}{48}-\frac{\ln(x-1)}{18}+\frac{\ln(x+1)}{6}-\frac{19\ln(x+2)}{144}$	54
parallelrisch	$\frac{3\ln(x-2)x-8\ln(x-1)x+24\ln(x+1)x-19\ln(x+2)x+12+6\ln(x-2)-16\ln(x-1)+48\ln(x+1)-38\ln(x+2)}{144x+288}$	62

input
```
int((x^3-2*x^2-x+2)/(x^4-5*x^2+4)^2,x,method=_RETURNVERBOSE)
```

output
```
1/12/(x+2)-19/144*ln(x+2)+1/6*ln(x+1)-1/18*ln(x-1)+1/48*ln(x-2)
```

3.85.5 Fricas [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 45, normalized size of antiderivative = 0.98

$$\int \frac{2-x-2x^2+x^3}{(4-5x^2+x^4)^2}\,dx = -\frac{19\,(x+2)\log(x+2)-24\,(x+2)\log(x+1)+8\,(x+2)\log(x-1)-3\,(x+2)\log(x-2)-12}{144\,(x+2)}$$

input
```
integrate((x^3-2*x^2-x+2)/(x^4-5*x^2+4)^2,x, algorithm="fricas")
```

output
```
-1/144*(19*(x + 2)*log(x + 2) - 24*(x + 2)*log(x + 1) + 8*(x + 2)*log(x -
1) - 3*(x + 2)*log(x - 2) - 12)/(x + 2)
```

3.85.6 Sympy [A] (verification not implemented)

Time = 0.14 (sec) , antiderivative size = 34, normalized size of antiderivative = 0.74

$$\int \frac{2-x-2x^2+x^3}{(4-5x^2+x^4)^2}\,dx = \frac{\log(x-2)}{48} - \frac{\log(x-1)}{18} + \frac{\log(x+1)}{6} - \frac{19\log(x+2)}{144} + \frac{1}{12x+24}$$

input
```
integrate((x**3-2*x**2-x+2)/(x**4-5*x**2+4)**2,x)
```

output
```
log(x - 2)/48 - log(x - 1)/18 + log(x + 1)/6 - 19*log(x + 2)/144 + 1/(12*x
 + 24)
```

3.85.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 32, normalized size of antiderivative = 0.70

$$\int \frac{2-x-2x^2+x^3}{(4-5x^2+x^4)^2}\,dx = \frac{1}{12(x+2)} - \frac{19}{144}\log(x+2) + \frac{1}{6}\log(x+1) - \frac{1}{18}\log(x-1) + \frac{1}{48}\log(x-2)$$

input
```
integrate((x^3-2*x^2-x+2)/(x^4-5*x^2+4)^2,x, algorithm="maxima")
```

output
```
1/12/(x + 2) - 19/144*log(x + 2) + 1/6*log(x + 1) - 1/18*log(x - 1) + 1/48
*log(x - 2)
```

3.85.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 36, normalized size of antiderivative = 0.78

$$\int \frac{2-x-2x^2+x^3}{(4-5x^2+x^4)^2}\,dx = \frac{1}{12(x+2)} - \frac{19}{144}\log(|x+2|) + \frac{1}{6}\log(|x+1|) - \frac{1}{18}\log(|x-1|) + \frac{1}{48}\log(|x-2|)$$

input
```
integrate((x^3-2*x^2-x+2)/(x^4-5*x^2+4)^2,x, algorithm="giac")
```

output
```
1/12/(x + 2) - 19/144*log(abs(x + 2)) + 1/6*log(abs(x + 1)) - 1/18*log(abs
(x - 1)) + 1/48*log(abs(x - 2))
```

3.85.9 Mupad [B] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 32, normalized size of antiderivative = 0.70

$$\int \frac{2-x-2x^2+x^3}{(4-5x^2+x^4)^2}\,dx = \frac{\ln(x+1)}{6} - \frac{\ln(x-1)}{18} + \frac{\ln(x-2)}{48} - \frac{19\ln(x+2)}{144} + \frac{1}{12\,(x+2)}$$

input
```
int(-(x + 2*x^2 - x^3 - 2)/(x^4 - 5*x^2 + 4)^2,x)
```

output
```
log(x + 1)/6 - log(x - 1)/18 + log(x - 2)/48 - (19*log(x + 2))/144 + 1/(12
*(x + 2))
```

3.86 $\int \frac{(d+ex)(2-x-2x^2+x^3)}{(4-5x^2+x^4)^2}\,dx$

3.86.1 Optimal result

Integrand size = 31, antiderivative size = 71

$$\int \frac{(d+ex)(2-x-2x^2+x^3)}{(4-5x^2+x^4)^2}\,dx = \frac{d-2e}{12(2+x)} - \frac{1}{18}(d+e)\log(1-x) + \frac{1}{48}(d+2e)\log(2-x) + \frac{1}{6}(d-e)\log(1+x) - \frac{1}{144}(19d-26e)\log(2+x)$$

output
```
1/12*(d-2*e)/(2+x)-1/18*(d+e)*ln(1-x)+1/48*(d+2*e)*ln(2-x)+1/6*(d-e)*ln(1+
x)-1/144*(19*d-26*e)*ln(2+x)
```

3.86.2 Mathematica [A] (verified)

Time = 0.03 (sec) , antiderivative size = 66, normalized size of antiderivative = 0.93

$$\int \frac{(d+ex)(2-x-2x^2+x^3)}{(4-5x^2+x^4)^2}\,dx = \frac{1}{144}\left(\frac{12(d-2e)}{2+x} + 24(d-e)\log(-1-x) - 8(d+e)\log(1-x) + 3(d+2e)\log(2-x) + (-19d+26e)\log(2+x)\right)$$

input
```
Integrate[((d + e*x)*(2 - x - 2*x^2 + x^3))/(4 - 5*x^2 + x^4)^2,x]
```

output
```
((12*(d - 2*e))/(2 + x) + 24*(d - e)*Log[-1 - x] - 8*(d + e)*Log[1 - x] +
3*(d + 2*e)*Log[2 - x] + (-19*d + 26*e)*Log[2 + x])/144
```

3.86.3 Rubi [A] (verified)

Time = 0.31 (sec) , antiderivative size = 71, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.097, Rules used = {2019, 2462, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^3-2x^2-x+2)(d+ex)}{(x^4-5x^2+4)^2}dx$$
$$\Big\downarrow \text{2019}$$
$$\int \frac{d+ex}{(x+2)^2(x^3-2x^2-x+2)}dx$$
$$\Big\downarrow \text{2462}$$
$$\int \left(\frac{-d-e}{18(x-1)}+\frac{d+2e}{48(x-2)}+\frac{d-e}{6(x+1)}+\frac{26e-19d}{144(x+2)}+\frac{2e-d}{12(x+2)^2}\right)dx$$
$$\Big\downarrow \text{2009}$$
$$\frac{d-2e}{12(x+2)}-\frac{1}{18}(d+e)\log(1-x)+\frac{1}{48}(d+2e)\log(2-x)+\frac{1}{6}(d-e)\log(x+1)-\frac{1}{144}(19d-26e)\log(x+2)$$

input
```
Int[((d + e*x)*(2 - x - 2*x^2 + x^3))/(4 - 5*x^2 + x^4)^2,x]
```

output
```
(d - 2*e)/(12*(2 + x)) - ((d + e)*Log[1 - x])/18 + ((d + 2*e)*Log[2 - x])/
48 + ((d - e)*Log[1 + x])/6 - ((19*d - 26*e)*Log[2 + x])/144
```

3.86.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2462
```
Int[(u_.)*(Px_)^(p_), x_Symbol] :> With[{Qx = Factor[Px]}, Int[ExpandIntegr
and[u*Qx^p, x], x] /;  !SumQ[NonfreeFactors[Qx, x]]] /; PolyQ[Px, x] && GtQ
[Expon[Px, x], 2] &&  !BinomialQ[Px, x] &&  !TrinomialQ[Px, x] && ILtQ[p, 0
] && RationalFunctionQ[u, x]
```

3.86.4 Maple [A] (verified)

Time = 0.07 (sec) , antiderivative size = 64, normalized size of antiderivative = 0.90

method	result
default	$\left(-\frac{19d}{144}+\frac{13e}{72}\right)\ln\left(x+2\right)-\frac{-\frac{d}{12}+\frac{e}{6}}{x+2}+\left(\frac{d}{6}-\frac{e}{6}\right)\ln\left(x+1\right)+\left(-\frac{d}{18}-\frac{e}{18}\right)\ln\left(x-1\right)+\left(\frac{d}{48}+\frac{e}{24}\right)\text{l}$
risch	$\frac{d}{12x+24}-\frac{e}{6(x+2)}-\frac{\ln(x-1)d}{18}-\frac{\ln(x-1)e}{18}+\frac{\ln(x+1)d}{6}-\frac{\ln(x+1)e}{6}-\frac{19\ln(-x-2)d}{144}+\frac{13\ln(-x-2)e}{72}+\frac{\ln(2-x)}{48}$
norman	$\frac{\left(-\frac{d}{12}+\frac{e}{6}\right)x+\left(\frac{d}{12}-\frac{e}{6}\right)x^3+\left(-\frac{d}{6}+\frac{e}{3}\right)x^2+\frac{d}{6}-\frac{e}{3}}{x^4-5x^2+4}+\left(-\frac{19d}{144}+\frac{13e}{72}\right)\ln\left(x+2\right)+\left(-\frac{d}{18}-\frac{e}{18}\right)\ln\left(x-1\right)+\left(\frac{d}{6}\right.$
parallelrisch	$\frac{3\ln(x-2)xd+6\ln(x-2)xe-8\ln(x-1)xd-8\ln(x-1)xe+24\ln(x+1)xd-24\ln(x+1)xe-19\ln(x+2)xd+26\ln(x+2)xe+6\ln(x-2}{144x+288}$

input
```
int((e*x+d)*(x^3-2*x^2-x+2)/(x^4-5*x^2+4)^2,x,method=_RETURNVERBOSE)
```

output
```
(-19/144*d+13/72*e)*ln(x+2)-(-1/12*d+1/6*e)/(x+2)+(1/6*d-1/6*e)*ln(x+1)+(-
1/18*d-1/18*e)*ln(x-1)+(1/48*d+1/24*e)*ln(x-2)
```

3.86.5 Fricas [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 93, normalized size of antiderivative = 1.31

$$\int \frac{(d+ex)\left(2-x-2x^2+x^3\right)}{\left(4-5x^2+x^4\right)^2}\,dx = -\frac{((19\,d-26\,e)x+38\,d-52\,e)\log\left(x+2\right)-24\left((d-e)x+2\,d-2\,e\right)\log\left(x+1\right)+8\left((d+e)x+2\,d+2\right.}{144\,(x+2)}$$

input
```
integrate((e*x+d)*(x^3-2*x^2-x+2)/(x^4-5*x^2+4)^2,x, algorithm="fricas")
```

output
```
-1/144*(((19*d - 26*e)*x + 38*d - 52*e)*log(x + 2) - 24*((d - e)*x + 2*d -
 2*e)*log(x + 1) + 8*((d + e)*x + 2*d + 2*e)*log(x - 1) - 3*((d + 2*e)*x +
 2*d + 4*e)*log(x - 2) - 12*d + 24*e)/(x + 2)
```

3.86.6 Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 1188 vs. $2(60) = 120$.

Time = 7.10 (sec) , antiderivative size = 1188, normalized size of antiderivative = 16.73

$$\int \frac{(d+ex)(2-x-2x^2+x^3)}{(4-5x^2+x^4)^2}\,dx = \text{Too large to display}$$

input
```
integrate((e*x+d)*(x**3-2*x**2-x+2)/(x**4-5*x**2+4)**2,x)
```

output
```
(d - 2*e)/(12*x + 24) + (d - e)*log(x + (-1534775*d**6 + 8032360*d**5*e -
984027*d**5*(d - e) - 12991180*d**4*e**2 + 11797266*d**4*e*(d - e) + 35671
68*d**4*(d - e)**2 + 1075200*d**3*e**3 - 32721528*d**3*e**2*(d - e) - 8725
248*d**3*e*(d - e)**2 - 247104*d**3*(d - e)**3 + 16959280*d**2*e**4 + 3897
7296*d**2*e**3*(d - e) - 2820096*d**2*e**2*(d - e)**2 - 10357632*d**2*e*(d
 - e)**3 - 15836800*d*e**5 - 21294960*d*e**4*(d - e) + 15436800*d*e**3*(d
- e)**2 + 16277760*d*e**2*(d - e)**3 + 4283840*e**6 + 3876000*e**5*(d - e)
 - 6865920*e**4*(d - e)**2 - 4078080*e**3*(d - e)**3)/(801262*d**6 - 46622
51*d**5*e + 7296938*d**4*e**2 + 1388616*d**3*e**3 - 12447440*d**2*e**4 + 9
990800*d*e**5 - 2380000*e**6))/6 - (d + e)*log(x + (-1534775*d**6 + 803236
0*d**5*e + 328009*d**5*(d + e) - 12991180*d**4*e**2 - 3932422*d**4*e*(d +
e) + 396352*d**4*(d + e)**2 + 1075200*d**3*e**3 + 10907176*d**3*e**2*(d +
e) - 969472*d**3*e*(d + e)**2 + 9152*d**3*(d + e)**3 + 16959280*d**2*e**4
- 12992432*d**2*e**3*(d + e) - 313344*d**2*e**2*(d + e)**2 + 383616*d**2*e
*(d + e)**3 - 15836800*d*e**5 + 7098320*d*e**4*(d + e) + 1715200*d*e**3*(d
 + e)**2 - 602880*d*e**2*(d + e)**3 + 4283840*e**6 - 1292000*e**5*(d + e)
- 762880*e**4*(d + e)**2 + 151040*e**3*(d + e)**3)/(801262*d**6 - 4662251*
d**5*e + 7296938*d**4*e**2 + 1388616*d**3*e**3 - 12447440*d**2*e**4 + 9990
800*d*e**5 - 2380000*e**6))/18 + (d + 2*e)*log(x + (-1534775*d**6 + 803236
0*d**5*e - 984027*d**5*(d + 2*e)/8 - 12991180*d**4*e**2 + 5898633*d**4*...
```

3.86.7 Maxima [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 57, normalized size of antiderivative = 0.80

$$\begin{aligned}\int \frac{(d+ex)(2-x-2x^2+x^3)}{(4-5x^2+x^4)^2}\,dx = &-\frac{1}{144}\,(19\,d-26\,e)\log(x+2)\\ &+\frac{1}{6}\,(d-e)\log(x+1)-\frac{1}{18}\,(d+e)\log(x-1)\\ &+\frac{1}{48}\,(d+2\,e)\log(x-2)+\frac{d-2\,e}{12\,(x+2)}\end{aligned}$$

input
```
integrate((e*x+d)*(x^3-2*x^2-x+2)/(x^4-5*x^2+4)^2,x, algorithm="maxima")
```

output
```
-1/144*(19*d - 26*e)*log(x + 2) + 1/6*(d - e)*log(x + 1) - 1/18*(d + e)*lo
g(x - 1) + 1/48*(d + 2*e)*log(x - 2) + 1/12*(d - 2*e)/(x + 2)
```

3.86.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 61, normalized size of antiderivative = 0.86

$$\int \frac{(d+ex)(2-x-2x^2+x^3)}{(4-5x^2+x^4)^2}\,dx = -\frac{1}{144}(19\,d-26\,e)\log(|x+2|) + \frac{1}{6}(d-e)\log(|x+1|) - \frac{1}{18}(d+e)\log(|x-1|) + \frac{1}{48}(d+2\,e)\log(|x-2|) + \frac{d-2\,e}{12\,(x+2)}$$

input
```
integrate((e*x+d)*(x^3-2*x^2-x+2)/(x^4-5*x^2+4)^2,x, algorithm="giac")
```

output
```
-1/144*(19*d - 26*e)*log(abs(x + 2)) + 1/6*(d - e)*log(abs(x + 1)) - 1/18*
(d + e)*log(abs(x - 1)) + 1/48*(d + 2*e)*log(abs(x - 2)) + 1/12*(d - 2*e)/
(x + 2)
```

3.86.9 Mupad [B] (verification not implemented)

Time = 7.95 (sec) , antiderivative size = 64, normalized size of antiderivative = 0.90

$$\int \frac{(d+ex)(2-x-2x^2+x^3)}{(4-5x^2+x^4)^2}\,dx = \frac{\frac{d}{12}-\frac{e}{6}}{x+2} + \ln(x+1)\left(\frac{d}{6}-\frac{e}{6}\right) - \ln(x-1)\left(\frac{d}{18}+\frac{e}{18}\right) + \ln(x-2)\left(\frac{d}{48}+\frac{e}{24}\right) - \ln(x+2)\left(\frac{19\,d}{144}-\frac{13\,e}{72}\right)$$

input
```
int(-((d + e*x)*(x + 2*x^2 - x^3 - 2))/(x^4 - 5*x^2 + 4)^2,x)
```

output
```
(d/12 - e/6)/(x + 2) + log(x + 1)*(d/6 - e/6) - log(x - 1)*(d/18 + e/18) +
 log(x - 2)*(d/48 + e/24) - log(x + 2)*((19*d)/144 - (13*e)/72)
```

3.87 $\int \frac{(d+ex+fx^2)(2-x-2x^2+x^3)}{(4-5x^2+x^4)^2}\,dx$

3.87.1 Optimal result

Integrand size = 36, antiderivative size = 82

$$\int \frac{(d+ex+fx^2)(2-x-2x^2+x^3)}{(4-5x^2+x^4)^2}\,dx = \frac{d-2e+4f}{12(2+x)} - \frac{1}{18}(d+e+f)\log(1-x) + \frac{1}{48}(d+2e+4f)\log(2-x) + \frac{1}{6}(d-e+f)\log(1+x) - \frac{1}{144}(19d-26e+28f)\log(2+x)$$

output
```
1/12*(d-2*e+4*f)/(2+x)-1/18*(d+e+f)*ln(1-x)+1/48*(d+2*e+4*f)*ln(2-x)+1/6*(
d-e+f)*ln(1+x)-1/144*(19*d-26*e+28*f)*ln(2+x)
```

3.87.2 Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 77, normalized size of antiderivative = 0.94

$$\int \frac{(d+ex+fx^2)(2-x-2x^2+x^3)}{(4-5x^2+x^4)^2}\,dx = \frac{1}{144}\left(\frac{12(d-2e+4f)}{2+x} + 24(d-e+f)\log(-1-x) - 8(d+e+f)\log(1-x) + 3(d+2e+4f)\log(2-x) + (-19d+26e-28f)\log(2+x)\right)$$

input
```
Integrate[((d + e*x + f*x^2)*(2 - x - 2*x^2 + x^3))/(4 - 5*x^2 + x^4)^2,x]
```

output
```
((12*(d - 2*e + 4*f))/(2 + x) + 24*(d - e + f)*Log[-1 - x] - 8*(d + e + f)
*Log[1 - x] + 3*(d + 2*e + 4*f)*Log[2 - x] + (-19*d + 26*e - 28*f)*Log[2 +
 x])/144
```

3.87.3 Rubi [A] (verified)

Time = 0.34 (sec) , antiderivative size = 82, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.083$, Rules used = {2019, 2462, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^3-2x^2-x+2)(d+ex+fx^2)}{(x^4-5x^2+4)^2}\,dx$$

$$\Big\downarrow \; 2019$$

$$\int \frac{d+ex+fx^2}{(x+2)^2(x^3-2x^2-x+2)}dx$$

$$\Big\downarrow \; 2462$$

$$\int \left(\frac{-19d+26e-28f}{144(x+2)}+\frac{d+2e+4f}{48(x-2)}+\frac{-d-e-f}{18(x-1)}+\frac{d-e+f}{6(x+1)}+\frac{-d+2e-4f}{12(x+2)^2}\right)dx$$

$$\Big\downarrow \; 2009$$

$$\frac{d-2e+4f}{12(x+2)}-\frac{1}{18}\log(1-x)(d+e+f)+\frac{1}{48}\log(2-x)(d+2e+4f)+\frac{1}{6}\log(x+1)(d-e+f)-\frac{1}{144}\log(x+2)(19d-26e+28f)$$

input
```
Int[((d + e*x + f*x^2)*(2 - x - 2*x^2 + x^3))/(4 - 5*x^2 + x^4)^2,x]
```

output
```
(d - 2*e + 4*f)/(12*(2 + x)) - ((d + e + f)*Log[1 - x])/18 + ((d + 2*e + 4
*f)*Log[2 - x])/48 + ((d - e + f)*Log[1 + x])/6 - ((19*d - 26*e + 28*f)*Lo
g[2 + x])/144
```

3.87.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2462
```
Int[(u_.)*(Px_)^(p_), x_Symbol] :> With[{Qx = Factor[Px]}, Int[ExpandIntegr
and[u*Qx^p, x], x] /;  !SumQ[NonfreeFactors[Qx, x]]] /; PolyQ[Px, x] && GtQ
[Expon[Px, x], 2] &&  !BinomialQ[Px, x] &&  !TrinomialQ[Px, x] && ILtQ[p, 0
] && RationalFunctionQ[u, x]
```

3.87.4 Maple [A] (verified)

Time = 0.08 (sec) , antiderivative size = 79, normalized size of antiderivative = 0.96

method	result
default	$\left(\frac{13e}{72}-\frac{7f}{36}-\frac{19d}{144}\right)\ln\left(x+2\right)-\frac{-\frac{d}{12}+\frac{e}{6}-\frac{f}{3}}{x+2}+\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}\right)\ln\left(x+1\right)+\left(-\frac{d}{18}-\frac{e}{18}-\frac{f}{18}\right)\ln\left(x-\right.$
risch	$\frac{d}{12x+24}-\frac{e}{6(x+2)}+\frac{f}{3x+6}+\frac{13\ln(-x-2)e}{72}-\frac{7\ln(-x-2)f}{36}-\frac{19\ln(-x-2)d}{144}+\frac{\ln(x+1)d}{6}-\frac{\ln(x+1)e}{6}+\frac{\ln(x+1)}{6}$
norman	$\frac{\left(-\frac{d}{12}+\frac{e}{6}-\frac{f}{3}\right)x+\left(\frac{d}{12}-\frac{e}{6}+\frac{f}{3}\right)x^3+\left(\frac{e}{3}-\frac{2f}{3}-\frac{d}{6}\right)x^2-\frac{e}{3}+\frac{2f}{3}+\frac{d}{6}}{x^4-5x^2+4}+\left(-\frac{d}{18}-\frac{e}{18}-\frac{f}{18}\right)\ln\left(x-1\right)+\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}\right)\mathrm{l}$
parallelrisch	$48f+12d-24e+6\ln(x-2)d+12\ln(x-2)e-16\ln(x-1)d-16\ln(x-1)e-56\ln(x+2)f+48\ln(x+1)f+26\ln(x+2)xe+6\ln(x-2)xe$

input
```
int((f*x^2+e*x+d)*(x^3-2*x^2-x+2)/(x^4-5*x^2+4)^2,x,method=_RETURNVERBOSE)
```

output
```
(13/72*e-7/36*f-19/144*d)*ln(x+2)-(-1/12*d+1/6*e-1/3*f)/(x+2)+(1/6*d-1/6*e
+1/6*f)*ln(x+1)+(-1/18*d-1/18*e-1/18*f)*ln(x-1)+(1/48*d+1/24*e+1/12*f)*ln(
x-2)
```

3.87.5 Fricas [A] (verification not implemented)

Time = 0.34 (sec) , antiderivative size = 116, normalized size of antiderivative = 1.41

$$\int \frac{(d+ex+fx^2)(2-x-2x^2+x^3)}{(4-5x^2+x^4)^2}\,dx = -\frac{((19\,d-26\,e+28\,f)x+38\,d-52\,e+56\,f)\log(x+2)-24\,((d-e+f)x+2\,d-2\,e+2\,f)\log(x+1}{}$$

input
```
integrate((f*x^2+e*x+d)*(x^3-2*x^2-x+2)/(x^4-5*x^2+4)^2,x, algorithm="fric
as")
```

output
```
-1/144*(((19*d - 26*e + 28*f)*x + 38*d - 52*e + 56*f)*log(x + 2) - 24*((d
- e + f)*x + 2*d - 2*e + 2*f)*log(x + 1) + 8*((d + e + f)*x + 2*d + 2*e +
2*f)*log(x - 1) - 3*((d + 2*e + 4*f)*x + 2*d + 4*e + 8*f)*log(x - 2) - 12*
d + 24*e - 48*f)/(x + 2)
```

3.87.6 Sympy [F(-1)]

Timed out.

$$\int \frac{(d+ex+fx^2)(2-x-2x^2+x^3)}{(4-5x^2+x^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((f*x**2+e*x+d)*(x**3-2*x**2-x+2)/(x**4-5*x**2+4)**2,x)
```

output
```
Timed out
```

3.87.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 68, normalized size of antiderivative = 0.83

$$\int \frac{(d+ex+fx^2)(2-x-2x^2+x^3)}{(4-5x^2+x^4)^2}\,dx = -\frac{1}{144}\,(19\,d-26\,e+28\,f)\log(x+2) + \frac{1}{6}\,(d-e+f)\log(x+1) - \frac{1}{18}\,(d+e+f)\log(x-1) + \frac{1}{48}\,(d+2\,e+4\,f)\log(x-2) + \frac{d-2\,e+4\,f}{12\,(x+2)}$$

input
```
integrate((f*x^2+e*x+d)*(x^3-2*x^2-x+2)/(x^4-5*x^2+4)^2,x, algorithm="maxi
ma")
```

output
```
-1/144*(19*d - 26*e + 28*f)*log(x + 2) + 1/6*(d - e + f)*log(x + 1) - 1/18
*(d + e + f)*log(x - 1) + 1/48*(d + 2*e + 4*f)*log(x - 2) + 1/12*(d - 2*e
+ 4*f)/(x + 2)
```

3.87.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 72, normalized size of antiderivative = 0.88

$$\int \frac{(d+ex+fx^2)(2-x-2x^2+x^3)}{(4-5x^2+x^4)^2}\,dx = -\frac{1}{144}(19\,d-26\,e+28\,f)\log(|x+2|) + \frac{1}{6}(d-e+f)\log(|x+1|) - \frac{1}{18}(d+e+f)\log(|x-1|) + \frac{1}{48}(d+2\,e+4\,f)\log(|x-2|) + \frac{d-2\,e+4\,f}{12\,(x+2)}$$

input
```
integrate((f*x^2+e*x+d)*(x^3-2*x^2-x+2)/(x^4-5*x^2+4)^2,x, algorithm="giac
")
```

output
```
-1/144*(19*d - 26*e + 28*f)*log(abs(x + 2)) + 1/6*(d - e + f)*log(abs(x +
1)) - 1/18*(d + e + f)*log(abs(x - 1)) + 1/48*(d + 2*e + 4*f)*log(abs(x -
2)) + 1/12*(d - 2*e + 4*f)/(x + 2)
```

3.87.9 Mupad [B] (verification not implemented)

Time = 7.90 (sec) , antiderivative size = 79, normalized size of antiderivative = 0.96

$$\int \frac{(d+ex+fx^2)(2-x-2x^2+x^3)}{(4-5x^2+x^4)^2}\,dx = \frac{\frac{d}{12}-\frac{e}{6}+\frac{f}{3}}{x+2} + \ln(x+1)\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}\right) - \ln(x-1)\left(\frac{d}{18}+\frac{e}{18}+\frac{f}{18}\right) + \ln(x-2)\left(\frac{d}{48}+\frac{e}{24}+\frac{f}{12}\right) - \ln(x+2)\left(\frac{19\,d}{144}-\frac{13\,e}{72}+\frac{7\,f}{36}\right)$$

input
```
int(-((d + e*x + f*x^2)*(x + 2*x^2 - x^3 - 2))/(x^4 - 5*x^2 + 4)^2,x)
```

output
```
(d/12 - e/6 + f/3)/(x + 2) + log(x + 1)*(d/6 - e/6 + f/6) - log(x - 1)*(d/
18 + e/18 + f/18) + log(x - 2)*(d/48 + e/24 + f/12) - log(x + 2)*((19*d)/1
44 - (13*e)/72 + (7*f)/36)
```

3.87. $\int \frac{(d+ex+fx^2)(2-x-2x^2+x^3)}{(4-5x^2+x^4)^2}\,dx$

3.88 $\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx$

3.88.1 Optimal result

Integrand size = 41, antiderivative size = 95

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx = \frac{d-2e+4f-8g}{12(2+x)} - \frac{1}{18}(d+e+f+g)\log(1-x) + \frac{1}{48}(d+2e+4f+8g)\log(2-x) + \frac{1}{6}(d-e+f-g)\log(1+x) - \frac{1}{144}(19d-26e+28f-8g)\log(2+x)$$

output
```
1/12*(d-2*e+4*f-8*g)/(2+x)-1/18*(d+e+f+g)*ln(1-x)+1/48*(d+2*e+4*f+8*g)*ln(
2-x)+1/6*(d-e+f-g)*ln(1+x)-1/144*(19*d-26*e+28*f-8*g)*ln(2+x)
```

3.88.2 Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 90, normalized size of antiderivative = 0.95

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx = \frac{1}{144}\left(\frac{12(d-2e+4f-8g)}{2+x} + 24(d-e+f-g)\log(-1-x) - 8(d+e+f+g)\log(1-x) + 3(d+2e+4f+8g)\log(2-x) + (-19d+26e-28f+8g)\log(2+x)\right)$$

input
```
Integrate[((2 - x - 2*x^2 + x^3)*(d + e*x + f*x^2 + g*x^3))/(4 - 5*x^2 + x
^4)^2,x]
```

output
```
((12*(d - 2*e + 4*f - 8*g))/(2 + x) + 24*(d - e + f - g)*Log[-1 - x] - 8*(
d + e + f + g)*Log[1 - x] + 3*(d + 2*e + 4*f + 8*g)*Log[2 - x] + (-19*d +
26*e - 28*f + 8*g)*Log[2 + x])/144
```

3.88.3 Rubi [A] (verified)

Time = 0.37 (sec) , antiderivative size = 95, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.073$, Rules used = {2019, 2462, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^3-2x^2-x+2)(d+ex+fx^2+gx^3)}{(x^4-5x^2+4)^2}dx$$

$$\Big\downarrow \text{2019}$$

$$\int \frac{d+ex+fx^2+gx^3}{(x+2)^2(x^3-2x^2-x+2)}dx$$

$$\Big\downarrow \text{2462}$$

$$\int \left(\frac{-d-e-f-g}{18(x-1)}+\frac{d+2e+4f+8g}{48(x-2)}+\frac{d-e+f-g}{6(x+1)}+\frac{-19d+26e-28f+8g}{144(x+2)}+\frac{-d+2e-4f+8g}{12(x+2)^2}\right)dx$$

$$\Big\downarrow \text{2009}$$

$$\frac{d-2e+4f-8g}{12(x+2)}-\frac{1}{18}\log(1-x)(d+e+f+g)+\frac{1}{48}\log(2-x)(d+2e+4f+8g)+\frac{1}{6}\log(x+1)(d-e+f-g)-\frac{1}{144}\log(x+2)(19d-26e+28f-8g)$$

input
```
Int[((2 - x - 2*x^2 + x^3)*(d + e*x + f*x^2 + g*x^3))/(4 - 5*x^2 + x^4)^2,
x]
```

output
```
(d - 2*e + 4*f - 8*g)/(12*(2 + x)) - ((d + e + f + g)*Log[1 - x])/18 + ((d
 + 2*e + 4*f + 8*g)*Log[2 - x])/48 + ((d - e + f - g)*Log[1 + x])/6 - ((19
*d - 26*e + 28*f - 8*g)*Log[2 + x])/144
```

3.88.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2462
```
Int[(u_.)*(Px_)^(p_), x_Symbol] :> With[{Qx = Factor[Px]}, Int[ExpandIntegr
and[u*Qx^p, x], x] /;  !SumQ[NonfreeFactors[Qx, x]]] /; PolyQ[Px, x] && GtQ
[Expon[Px, x], 2] &&  !BinomialQ[Px, x] &&  !TrinomialQ[Px, x] && ILtQ[p, 0
] && RationalFunctionQ[u, x]
```

3.88.4 Maple [A] (verified)

Time = 0.08 (sec) , antiderivative size = 94, normalized size of antiderivative = 0.99

method	result
default	$\left(\frac{13e}{72}-\frac{7f}{36}+\frac{g}{18}-\frac{19d}{144}\right)\ln\left(x+2\right)-\frac{-\frac{d}{12}+\frac{e}{6}-\frac{f}{3}+\frac{2g}{3}}{x+2}+\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}\right)\ln\left(x+1\right)+\left(-\frac{d}{18}-\frac{e}{18}-\right.$
norman	$\frac{\left(-\frac{d}{12}+\frac{e}{6}-\frac{f}{3}+\frac{2g}{3}\right)x+\left(\frac{d}{12}-\frac{e}{6}+\frac{f}{3}-\frac{2g}{3}\right)x^3+\left(\frac{e}{3}-\frac{2f}{3}+\frac{4g}{3}-\frac{d}{6}\right)x^2-\frac{e}{3}+\frac{2f}{3}-\frac{4g}{3}+\frac{d}{6}}{x^4-5x^2+4}+\left(-\frac{d}{18}-\frac{e}{18}-\frac{f}{18}-\frac{g}{18}\right)\ln\left(x-\right.$
risch	$\frac{d}{12x+24}-\frac{e}{6(x+2)}+\frac{f}{3x+6}-\frac{2g}{3(x+2)}+\frac{\ln(x+1)d}{6}-\frac{\ln(x+1)e}{6}+\frac{\ln(x+1)f}{6}-\frac{\ln(x+1)g}{6}-\frac{\ln(x-1)d}{18}-\frac{\ln(x-1)}{18}$
parallelrisch	$48f-96g+12d-24e+6\ln(x-2)d+12\ln(x-2)e-16\ln(x-1)d-16\ln(x-1)e+24\ln(x-2)xg-8\ln(x-1)xg-24\ln(x+1)xg+8\ln($

input
```
int((x^3-2*x^2-x+2)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x,method=_RETURNVE
RBOSE)
```

output
```
(13/72*e-7/36*f+1/18*g-19/144*d)*ln(x+2)-(-1/12*d+1/6*e-1/3*f+2/3*g)/(x+2)
+(1/6*d-1/6*e+1/6*f-1/6*g)*ln(x+1)+(-1/18*d-1/18*e-1/18*f-1/18*g)*ln(x-1)+
(1/48*d+1/24*e+1/12*f+1/6*g)*ln(x-2)
```

3.88.5 Fricas [A] (verification not implemented)

Time = 0.72 (sec) , antiderivative size = 141, normalized size of antiderivative = 1.48

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx =$$

$$-\frac{((19\,d-26\,e+28\,f-8\,g)x+38\,d-52\,e+56\,f-16\,g)\log(x+2)-24\,((d-e+f-g)x+2\,d-2\,e}{}$$

input
```
integrate((x^3-2*x^2-x+2)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm
="fricas")
```

output
```
-1/144*((((19*d - 26*e + 28*f - 8*g)*x + 38*d - 52*e + 56*f - 16*g)*log(x +
 2) - 24*((d - e + f - g)*x + 2*d - 2*e + 2*f - 2*g)*log(x + 1) + 8*((d +
e + f + g)*x + 2*d + 2*e + 2*f + 2*g)*log(x - 1) - 3*((d + 2*e + 4*f + 8*g
)*x + 2*d + 4*e + 8*f + 16*g)*log(x - 2) - 12*d + 24*e - 48*f + 96*g)/(x +
 2)
```

3.88.6 Sympy [F(-1)]

Timed out.

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((x**3-2*x**2-x+2)*(g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4)**2,x)
```

output
```
Timed out
```

3.88.7 Maxima [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 81, normalized size of antiderivative = 0.85

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx$$
$$= -\frac{1}{144}\,(19\,d-26\,e+28\,f-8\,g)\log(x+2)+\frac{1}{6}\,(d-e+f-g)\log(x+1)$$
$$-\frac{1}{18}\,(d+e+f+g)\log(x-1)+\frac{1}{48}\,(d+2\,e+4\,f+8\,g)\log(x-2)+\frac{d-2\,e+4\,f-8\,g}{12\,(x+2)}$$

input
```
integrate((x^3-2*x^2-x+2)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm
="maxima")
```

output
```
-1/144*(19*d - 26*e + 28*f - 8*g)*log(x + 2) + 1/6*(d - e + f - g)*log(x +
 1) - 1/18*(d + e + f + g)*log(x - 1) + 1/48*(d + 2*e + 4*f + 8*g)*log(x -
 2) + 1/12*(d - 2*e + 4*f - 8*g)/(x + 2)
```

3.88.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 85, normalized size of antiderivative = 0.89

$$\begin{aligned}&\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx\\&=-\frac{1}{144}(19\,d-26\,e+28\,f-8\,g)\log(|x+2|)\\&\quad+\frac{1}{6}(d-e+f-g)\log(|x+1|)-\frac{1}{18}(d+e+f+g)\log(|x-1|)\\&\quad+\frac{1}{48}(d+2\,e+4\,f+8\,g)\log(|x-2|)+\frac{d-2\,e+4\,f-8\,g}{12\,(x+2)}\end{aligned}$$

input
```
integrate((x^3-2*x^2-x+2)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm
="giac")
```

output
```
-1/144*(19*d - 26*e + 28*f - 8*g)*log(abs(x + 2)) + 1/6*(d - e + f - g)*lo
g(abs(x + 1)) - 1/18*(d + e + f + g)*log(abs(x - 1)) + 1/48*(d + 2*e + 4*f
 + 8*g)*log(abs(x - 2)) + 1/12*(d - 2*e + 4*f - 8*g)/(x + 2)
```

3.88.9 Mupad [B] (verification not implemented)

Time = 7.94 (sec) , antiderivative size = 94, normalized size of antiderivative = 0.99

$$\begin{aligned}&\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx\\&=\frac{\frac{d}{12}-\frac{e}{6}+\frac{f}{3}-\frac{2\,g}{3}}{x+2}+\ln(x+1)\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}\right)-\ln(x-1)\left(\frac{d}{18}+\frac{e}{18}+\frac{f}{18}+\frac{g}{18}\right)\\&\quad+\ln(x-2)\left(\frac{d}{48}+\frac{e}{24}+\frac{f}{12}+\frac{g}{6}\right)-\ln(x+2)\left(\frac{19\,d}{144}-\frac{13\,e}{72}+\frac{7\,f}{36}-\frac{g}{18}\right)\end{aligned}$$

input
```
int(-((d + e*x + f*x^2 + g*x^3)*(x + 2*x^2 - x^3 - 2))/(x^4 - 5*x^2 + 4)^2
,x)
```

output
```
(d/12 - e/6 + f/3 - (2*g)/3)/(x + 2) + log(x + 1)*(d/6 - e/6 + f/6 - g/6)
- log(x - 1)*(d/18 + e/18 + f/18 + g/18) + log(x - 2)*(d/48 + e/24 + f/12
+ g/6) - log(x + 2)*((19*d)/144 - (13*e)/72 + (7*f)/36 - g/18)
```

3.88. $\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx$

3.89 $\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx$

3.89.1 Optimal result

Integrand size = 46, antiderivative size = 106

$$\begin{aligned}&\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx\\&=\frac{d-2e+4f-8g+16h}{12(2+x)}-\frac{1}{18}(d+e+f+g+h)\log(1-x)\\&\quad+\frac{1}{48}(d+2e+4f+8g+16h)\log(2-x)+\frac{1}{6}(d-e+f-g+h)\log(1+x)\\&\quad-\frac{1}{144}(19d-26e+28f-8g-80h)\log(2+x)\end{aligned}$$

output
```
1/12*(d-2*e+4*f-8*g+16*h)/(2+x)-1/18*(d+e+f+g+h)*ln(1-x)+1/48*(d+2*e+4*f+8
*g+16*h)*ln(2-x)+1/6*(d-e+f-g+h)*ln(1+x)-1/144*(19*d-26*e+28*f-8*g-80*h)*l
n(2+x)
```

3.89.2 Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 102, normalized size of antiderivative = 0.96

$$\begin{aligned}&\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx\\&=\frac{1}{144}\left(\frac{12(d-2e+4f-8g+16h)}{2+x}+24(d-e+f-g+h)\log(-1-x)\right.\\&\quad-8(d+e+f+g+h)\log(1-x)+3(d+2(e+2f+4g+8h))\log(2-x)\\&\quad\left.+(-19d+26e-28f+8g+80h)\log(2+x)\right)\end{aligned}$$

input
```
Integrate[((2 - x - 2*x^2 + x^3)*(d + e*x + f*x^2 + g*x^3 + h*x^4))/(4 - 5
*x^2 + x^4)^2,x]
```

output
```
((12*(d - 2*e + 4*f - 8*g + 16*h))/(2 + x) + 24*(d - e + f - g + h)*Log[-1
 - x] - 8*(d + e + f + g + h)*Log[1 - x] + 3*(d + 2*(e + 2*f + 4*g + 8*h))
*Log[2 - x] + (-19*d + 26*e - 28*f + 8*g + 80*h)*Log[2 + x])/144
```

3.89.3 Rubi [A] (verified)

Time = 0.39 (sec) , antiderivative size = 106, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.065$, Rules used = {2019, 2462, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^3-2x^2-x+2)\,(d+ex+fx^2+gx^3+hx^4)}{(x^4-5x^2+4)^2}dx$$

$$\Big\downarrow \text{2019}$$

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(x+2)^2\,(x^3-2x^2-x+2)}dx$$

$$\Big\downarrow \text{2462}$$

$$\int \left(\frac{-d+2e-4f+8g-16h}{12(x+2)^2}+\frac{d+2e+4f+8g+16h}{48(x-2)}+\frac{-d-e-f-g-h}{18(x-1)}+\frac{d-e+f-g+h}{6(x+1)}+\frac{-19d+2\right.$$

$$\Big\downarrow \text{2009}$$

$$\frac{d-2e+4f-8g+16h}{12(x+2)}-\frac{1}{18}\log(1-x)(d+e+f+g+h)+\frac{1}{48}\log(2-x)(d+2e+4f+8g+16h)+\frac{1}{6}\log(x+1)(d-e+f-g+h)-\frac{1}{144}\log(x+2)(19d-26e+28f-8g-80h)$$

input
```
Int[((2 - x - 2*x^2 + x^3)*(d + e*x + f*x^2 + g*x^3 + h*x^4))/(4 - 5*x^2 +
 x^4)^2,x]
```

output
```
(d - 2*e + 4*f - 8*g + 16*h)/(12*(2 + x)) - ((d + e + f + g + h)*Log[1 - x
])/18 + ((d + 2*e + 4*f + 8*g + 16*h)*Log[2 - x])/48 + ((d - e + f - g + h
)*Log[1 + x])/6 - ((19*d - 26*e + 28*f - 8*g - 80*h)*Log[2 + x])/144
```

3.89.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2462
```
Int[(u_.)*(Px_)^(p_), x_Symbol] :> With[{Qx = Factor[Px]}, Int[ExpandIntegr
and[u*Qx^p, x], x] /;  !SumQ[NonfreeFactors[Qx, x]]] /; PolyQ[Px, x] && GtQ
[Expon[Px, x], 2] &&  !BinomialQ[Px, x] &&  !TrinomialQ[Px, x] && ILtQ[p, 0
] && RationalFunctionQ[u, x]
```

3.89.4 Maple [A] (verified)

Time = 0.10 (sec) , antiderivative size = 109, normalized size of antiderivative = 1.03

method	result
default	$\left(\frac{5h}{9}+\frac{g}{18}-\frac{7f}{36}+\frac{13e}{72}-\frac{19d}{144}\right)\ln\left(x+2\right)-\frac{-\frac{d}{12}+\frac{e}{6}-\frac{f}{3}+\frac{2g}{3}-\frac{4h}{3}}{x+2}+\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}+\frac{h}{6}\right)\ln\left(x+1\right)+$
norman	$\frac{\left(-\frac{d}{12}+\frac{e}{6}-\frac{f}{3}+\frac{2g}{3}-\frac{4h}{3}\right)x+\left(\frac{d}{12}-\frac{e}{6}+\frac{f}{3}-\frac{2g}{3}+\frac{4h}{3}\right)x^3+\left(-\frac{8h}{3}+\frac{4g}{3}-\frac{2f}{3}+\frac{e}{3}-\frac{d}{6}\right)x^2+\frac{8h}{3}-\frac{4g}{3}+\frac{2f}{3}-\frac{e}{3}+\frac{d}{6}}{x^4-5x^2+4}+\left(-\frac{d}{18}-\frac{e}{18}-\frac{f}{18}\right.$
risch	$-\frac{\ln(x-1)d}{18}-\frac{\ln(x-1)e}{18}+\frac{\ln(x+1)f}{6}+\frac{\ln(2-x)f}{12}+\frac{5\ln(-x-2)h}{9}+\frac{d}{12x+24}+\frac{\ln(2-x)d}{48}+\frac{\ln(2-x)e}{24}+\frac{\ln(x+1)d}{6}$
parallelrisch	$\frac{48f-96g+12d+192h-24e+48\ln(x-2)xh-8\ln(x-1)xh+24\ln(x+1)xh+80\ln(x+2)xh+6\ln(x-2)d+12\ln(x-2)e-16\ln(x-1)}{}$

input
```
int((x^3-2*x^2-x+2)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x,method=_RE
TURNVERBOSE)
```

output
```
(5/9*h+1/18*g-7/36*f+13/72*e-19/144*d)*ln(x+2)-(-1/12*d+1/6*e-1/3*f+2/3*g-
4/3*h)/(x+2)+(1/6*d-1/6*e+1/6*f-1/6*g+1/6*h)*ln(x+1)+(-1/18*d-1/18*e-1/18*
f-1/18*g-1/18*h)*ln(x-1)+(1/48*d+1/24*e+1/12*f+1/6*g+1/3*h)*ln(x-2)
```

3.89.5 Fricas [A] (verification not implemented)

Time = 3.26 (sec) , antiderivative size = 164, normalized size of antiderivative = 1.55

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx =$$
$$-\frac{((19\,d-26\,e+28\,f-8\,g-80\,h)x+38\,d-52\,e+56\,f-16\,g-160\,h)\log(x+2)-24\,((d-e+f-$$

input
```
integrate((x^3-2*x^2-x+2)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, alg
orithm="fricas")
```

output
```
-1/144*(((19*d - 26*e + 28*f - 8*g - 80*h)*x + 38*d - 52*e + 56*f - 16*g -
 160*h)*log(x + 2) - 24*((d - e + f - g + h)*x + 2*d - 2*e + 2*f - 2*g + 2
*h)*log(x + 1) + 8*((d + e + f + g + h)*x + 2*d + 2*e + 2*f + 2*g + 2*h)*l
og(x - 1) - 3*((d + 2*e + 4*f + 8*g + 16*h)*x + 2*d + 4*e + 8*f + 16*g + 3
2*h)*log(x - 2) - 12*d + 24*e - 48*f + 96*g - 192*h)/(x + 2)
```

3.89.6 Sympy [F(-1)]

Timed out.

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((x**3-2*x**2-x+2)*(h*x**4+g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4)**
2,x)
```

output
```
Timed out
```

3.89.7 Maxima [A] (verification not implemented)

Time = 0.18 (sec) , antiderivative size = 92, normalized size of antiderivative = 0.87

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx$$
$$= -\frac{1}{144}(19\,d-26\,e+28\,f-8\,g-80\,h)\log(x+2) + \frac{1}{6}(d-e+f-g+h)\log(x+1) - \frac{1}{18}(d+e+f+g+h)\log(x-1) + \frac{1}{48}(d+2\,e+4\,f+8\,g+16\,h)\log(x-2) + \frac{d-2\,e+4\,f-8\,g+16\,h}{12\,(x+2)}$$

input
```
integrate((x^3-2*x^2-x+2)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, alg
orithm="maxima")
```

output
```
-1/144*(19*d - 26*e + 28*f - 8*g - 80*h)*log(x + 2) + 1/6*(d - e + f - g +
 h)*log(x + 1) - 1/18*(d + e + f + g + h)*log(x - 1) + 1/48*(d + 2*e + 4*f
 + 8*g + 16*h)*log(x - 2) + 1/12*(d - 2*e + 4*f - 8*g + 16*h)/(x + 2)
```

3.89.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 96, normalized size of antiderivative = 0.91

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx$$
$$= -\frac{1}{144}(19\,d-26\,e+28\,f-8\,g-80\,h)\log(|x+2|) + \frac{1}{6}(d-e+f-g+h)\log(|x+1|) - \frac{1}{18}(d+e+f+g+h)\log(|x-1|) + \frac{1}{48}(d+2\,e+4\,f+8\,g+16\,h)\log(|x-2|) + \frac{d-2\,e+4\,f-8\,g+16\,h}{12\,(x+2)}$$

input
```
integrate((x^3-2*x^2-x+2)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, alg
orithm="giac")
```

output
```
-1/144*(19*d - 26*e + 28*f - 8*g - 80*h)*log(abs(x + 2)) + 1/6*(d - e + f
- g + h)*log(abs(x + 1)) - 1/18*(d + e + f + g + h)*log(abs(x - 1)) + 1/48
*(d + 2*e + 4*f + 8*g + 16*h)*log(abs(x - 2)) + 1/12*(d - 2*e + 4*f - 8*g
+ 16*h)/(x + 2)
```

3.89.9 Mupad [B] (verification not implemented)

Time = 8.11 (sec) , antiderivative size = 108, normalized size of antiderivative = 1.02

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx$$
$$= \frac{\frac{d}{12}-\frac{e}{6}+\frac{f}{3}-\frac{2\,g}{3}+\frac{4\,h}{3}}{x+2} + \ln(x+1)\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}+\frac{h}{6}\right) - \ln(x-1)\left(\frac{d}{18}+\frac{e}{18}+\frac{f}{18}+\frac{g}{18}+\frac{h}{18}\right) + \ln(x-2)\left(\frac{d}{48}+\frac{e}{24}+\frac{f}{12}+\frac{g}{6}+\frac{h}{3}\right) + \ln(x+2)\left(\frac{13\,e}{72}-\frac{19\,d}{144}-\frac{7\,f}{36}+\frac{g}{18}+\frac{5\,h}{9}\right)$$

input
```
int(-((x + 2*x^2 - x^3 - 2)*(d + e*x + f*x^2 + g*x^3 + h*x^4))/(x^4 - 5*x^
2 + 4)^2,x)
```

output
```
(d/12 - e/6 + f/3 - (2*g)/3 + (4*h)/3)/(x + 2) + log(x + 1)*(d/6 - e/6 + f
/6 - g/6 + h/6) - log(x - 1)*(d/18 + e/18 + f/18 + g/18 + h/18) + log(x -
2)*(d/48 + e/24 + f/12 + g/6 + h/3) + log(x + 2)*((13*e)/72 - (19*d)/144 -
 (7*f)/36 + g/18 + (5*h)/9)
```

3.89. $\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx$

3.90 $\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4+ix^5)}{(4-5x^2+x^4)^2}\,dx$

3.90.1 Optimal result

Integrand size = 51, antiderivative size = 122

$$\begin{aligned}\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4+ix^5)}{(4-5x^2+x^4)^2}\,dx &= ix+\frac{d-2e+4f-8g+16h-32i}{12(2+x)}-\frac{1}{18}(d+e+f+g+h+i)\log(1-x)\\ &\quad+\frac{1}{48}(d+2e+4f+8g+16h+32i)\log(2-x)+\frac{1}{6}(d-e+f-g+h-i)\log(1+x)\\ &\quad-\frac{1}{144}(19d-26e+28f-8g-80h+352i)\log(2+x)\end{aligned}$$

output
```
i*x+1/12*(d-2*e+4*f-8*g+16*h-32*i)/(2+x)-1/18*(d+e+f+g+h+i)*ln(1-x)+1/48*(
d+2*e+4*f+8*g+16*h+32*i)*ln(2-x)+1/6*(d-e+f-g+h-i)*ln(1+x)-1/144*(19*d-26*
e+28*f-8*g-80*h+352*i)*ln(2+x)
```

3.90.2 Mathematica [A] (verified)

Time = 0.05 (sec) , antiderivative size = 118, normalized size of antiderivative = 0.97

$$\begin{aligned}\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4+ix^5)}{(4-5x^2+x^4)^2}\,dx &= \frac{1}{144}\left(144ix+\frac{12(d-2(e-2f+4g-8h+16i))}{2+x}-8(d+e+f+g+h+i)\log(1-x)\right.\\ &\quad+3(d+2e+4(f+2g+4h+8i))\log(2-x)+24(d-e+f-g+h-i)\log(1+x)\\ &\quad\left.+(-19d+26e-28f+8g+80h-352i)\log(2+x)\right)\end{aligned}$$

input
```
Integrate[((2 - x - 2*x^2 + x^3)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5)
)/(4 - 5*x^2 + x^4)^2,x]
```

output
```
(144*i*x + (12*(d - 2*(e - 2*f + 4*g - 8*h + 16*i)))/(2 + x) - 8*(d + e +
f + g + h + i)*Log[1 - x] + 3*(d + 2*e + 4*(f + 2*g + 4*h + 8*i))*Log[2 -
x] + 24*(d - e + f - g + h - i)*Log[1 + x] + (-19*d + 26*e - 28*f + 8*g +
80*h - 352*i)*Log[2 + x])/144
```

3.90.3 Rubi [A] (verified)

Time = 0.44 (sec) , antiderivative size = 122, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.059, Rules used = {2019, 2462, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^3-2x^2-x+2)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)}{(x^4-5x^2+4)^2}dx$$

$$\Big\downarrow \text{2019}$$

$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(x+2)^2(x^3-2x^2-x+2)}dx$$

$$\Big\downarrow \text{2462}$$

$$\int \left(\frac{-19d+26e-28f+8g+80h-352i}{144(x+2)}+\frac{d+2e+4f+8g+16h+32i}{48(x-2)}+\frac{-d-e-f-g-h-i}{18(x-1)}+\frac{d-e+}{6(}\right.$$

$$\Big\downarrow \text{2009}$$

$$\frac{d-2e+4f-8g+16h-32i}{12(x+2)}-\frac{1}{18}\log(1-x)(d+e+f+g+h+i)+\frac{1}{48}\log(2-x)(d+2e+4f+8g+16h+32i)+\frac{1}{6}\log(x+1)(d-e+f-g+h-i)-\frac{1}{144}\log(x+2)(19d-26e+28f-8g-80h+352i)+ix$$

input
```
Int[((2 - x - 2*x^2 + x^3)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5))/(4 -
 5*x^2 + x^4)^2,x]
```

output
```
i*x + (d - 2*e + 4*f - 8*g + 16*h - 32*i)/(12*(2 + x)) - ((d + e + f + g +
 h + i)*Log[1 - x])/18 + ((d + 2*e + 4*f + 8*g + 16*h + 32*i)*Log[2 - x])/
48 + ((d - e + f - g + h - i)*Log[1 + x])/6 - ((19*d - 26*e + 28*f - 8*g -
 80*h + 352*i)*Log[2 + x])/144
```

3.90.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2462
```
Int[(u_.)*(Px_)^(p_), x_Symbol] :> With[{Qx = Factor[Px]}, Int[ExpandIntegr
and[u*Qx^p, x], x] /;  !SumQ[NonfreeFactors[Qx, x]]] /; PolyQ[Px, x] && GtQ
[Expon[Px, x], 2] &&  !BinomialQ[Px, x] &&  !TrinomialQ[Px, x] && ILtQ[p, 0
] && RationalFunctionQ[u, x]
```

3.90.4 Maple [A] (verified)

Time = 0.12 (sec) , antiderivative size = 127, normalized size of antiderivative = 1.04

method	result
default	$ix+\left(\frac{5h}{9}-\frac{22i}{9}+\frac{g}{18}-\frac{7f}{36}+\frac{13e}{72}-\frac{19d}{144}\right)\ln\left(x+2\right)-\frac{-\frac{d}{12}+\frac{e}{6}-\frac{f}{3}+\frac{2g}{3}-\frac{4h}{3}+\frac{8i}{3}}{x+2}+\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}+\ldots\right.$
norman	$\frac{i\,x^5+\left(-\frac{4h}{3}+\frac{20i}{3}+\frac{2g}{3}-\frac{f}{3}+\frac{e}{6}-\frac{d}{12}\right)x+\left(-\frac{23i}{3}+\frac{4h}{3}-\frac{2g}{3}+\frac{f}{3}-\frac{e}{6}+\frac{d}{12}\right)x^3+\left(-\frac{d}{6}+\frac{e}{3}-\frac{2f}{3}+\frac{4g}{3}-\frac{8h}{3}+\frac{16i}{3}\right)x^2+\frac{8h}{3}-\frac{16i}{3}+\frac{d}{6}-\frac{e}{3}+\frac{2f}{3}\ldots}{x^4-5x^2+4}$
risch	$-\frac{\ln(x-1)d}{18}-\frac{\ln(x-1)e}{18}-\frac{\ln(x+1)i}{6}+\frac{\ln(x+1)f}{6}+\frac{\ln(2-x)f}{12}+\frac{5\ln(-x-2)h}{9}-\frac{22\ln(-x-2)i}{9}+\frac{d}{12x+24}+\ldots$
parallelrisch	$\frac{-960i+48f-96g-352\ln(x+2)xi+96\ln(x-2)xi+12d+192h-24e+48\ln(x-2)xh-8\ln(x-1)xh+24\ln(x+1)xh+80\ln(x+2)xh\ldots}{\ldots}$

input
```
int((x^3-2*x^2-x+2)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x,meth
od=_RETURNVERBOSE)
```

output
```
i*x+(5/9*h-22/9*i+1/18*g-7/36*f+13/72*e-19/144*d)*ln(x+2)-(-1/12*d+1/6*e-1
/3*f+2/3*g-4/3*h+8/3*i)/(x+2)+(1/6*d-1/6*e+1/6*f-1/6*g+1/6*h-1/6*i)*ln(x+1
)+(-1/18*d-1/18*e-1/18*f-1/18*g-1/18*h-1/18*i)*ln(x-1)+(1/48*d+1/24*e+1/12
*f+1/6*g+1/3*h+2/3*i)*ln(x-2)
```

3.90.5 Fricas [A] (verification not implemented)

Time = 19.69 (sec) , antiderivative size = 200, normalized size of antiderivative = 1.64

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4+ix^5)}{(4-5x^2+x^4)^2}\,dx$$
$$= \frac{144\,ix^2+288\,ix-((19\,d-26\,e+28\,f-8\,g-80\,h+352\,i)x+38\,d-52\,e+56\,f-16\,g-160\,h+704\,i}{}$$

input
```
integrate((x^3-2*x^2-x+2)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,
x, algorithm="fricas")
```

output
```
1/144*(144*i*x^2 + 288*i*x - ((19*d - 26*e + 28*f - 8*g - 80*h + 352*i)*x
+ 38*d - 52*e + 56*f - 16*g - 160*h + 704*i)*log(x + 2) + 24*((d - e + f -
 g + h - i)*x + 2*d - 2*e + 2*f - 2*g + 2*h - 2*i)*log(x + 1) - 8*((d + e
+ f + g + h + i)*x + 2*d + 2*e + 2*f + 2*g + 2*h + 2*i)*log(x - 1) + 3*((d
 + 2*e + 4*f + 8*g + 16*h + 32*i)*x + 2*d + 4*e + 8*f + 16*g + 32*h + 64*i
)*log(x - 2) + 12*d - 24*e + 48*f - 96*g + 192*h - 384*i)/(x + 2)
```

3.90.6 Sympy [F(-1)]

Timed out.

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4+ix^5)}{(4-5x^2+x^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((x**3-2*x**2-x+2)*(i*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(x**4-5*x*
*2+4)**2,x)
```

output
```
Timed out
```

3.90.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 108, normalized size of antiderivative = 0.89

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4+ix^5)}{(4-5x^2+x^4)^2}\,dx$$
$$= ix - \frac{1}{144}(19\,d-26\,e+28\,f-8\,g-80\,h+352\,i)\log(x+2) + \frac{1}{6}(d-e+f-g+h-i)\log(x+1) - \frac{1}{18}(d+e+f+g+h+i)\log(x-1) + \frac{1}{48}(d+2\,e+4\,f+8\,g+16\,h+32\,i)\log(x-2) + \frac{d-2\,e+4\,f-8\,g+16\,h-32\,i}{12\,(x+2)}$$

input
```
integrate((x^3-2*x^2-x+2)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,
x, algorithm="maxima")
```

output
```
i*x - 1/144*(19*d - 26*e + 28*f - 8*g - 80*h + 352*i)*log(x + 2) + 1/6*(d
- e + f - g + h - i)*log(x + 1) - 1/18*(d + e + f + g + h + i)*log(x - 1)
+ 1/48*(d + 2*e + 4*f + 8*g + 16*h + 32*i)*log(x - 2) + 1/12*(d - 2*e + 4*
f - 8*g + 16*h - 32*i)/(x + 2)
```

3.90.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 112, normalized size of antiderivative = 0.92

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4+ix^5)}{(4-5x^2+x^4)^2}\,dx$$
$$= ix - \frac{1}{144}(19\,d-26\,e+28\,f-8\,g-80\,h+352\,i)\log(|x+2|) + \frac{1}{6}(d-e+f-g+h-i)\log(|x+1|) - \frac{1}{18}(d+e+f+g+h+i)\log(|x-1|) + \frac{1}{48}(d+2\,e+4\,f+8\,g+16\,h+32\,i)\log(|x-2|) + \frac{d-2\,e+4\,f-8\,g+16\,h-32\,i}{12\,(x+2)}$$

input
```
integrate((x^3-2*x^2-x+2)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,
x, algorithm="giac")
```

output
```
i*x - 1/144*(19*d - 26*e + 28*f - 8*g - 80*h + 352*i)*log(abs(x + 2)) + 1/
6*(d - e + f - g + h - i)*log(abs(x + 1)) - 1/18*(d + e + f + g + h + i)*l
og(abs(x - 1)) + 1/48*(d + 2*e + 4*f + 8*g + 16*h + 32*i)*log(abs(x - 2))
+ 1/12*(d - 2*e + 4*f - 8*g + 16*h - 32*i)/(x + 2)
```

3.90.9 Mupad [B] (verification not implemented)

Time = 8.37 (sec) , antiderivative size = 127, normalized size of antiderivative = 1.04

$$\int \frac{(2-x-2x^2+x^3)(d+ex+fx^2+gx^3+hx^4+ix^5)}{(4-5x^2+x^4)^2}\,dx$$
$$= i\,x+\frac{\frac{d}{12}-\frac{e}{6}+\frac{f}{3}-\frac{2\,g}{3}+\frac{4\,h}{3}-\frac{8\,i}{3}}{x+2}+\ln(x+1)\left(\frac{d}{6}-\frac{e}{6}+\frac{f}{6}-\frac{g}{6}+\frac{h}{6}-\frac{i}{6}\right)$$
$$+\ln(x-2)\left(\frac{d}{48}+\frac{e}{24}+\frac{f}{12}+\frac{g}{6}+\frac{h}{3}+\frac{2\,i}{3}\right)-\ln(x-1)\left(\frac{d}{18}+\frac{e}{18}+\frac{f}{18}+\frac{g}{18}+\frac{h}{18}+\frac{i}{18}\right)$$
$$-\ln(x+2)\left(\frac{19\,d}{144}-\frac{13\,e}{72}+\frac{7\,f}{36}-\frac{g}{18}-\frac{5\,h}{9}+\frac{22\,i}{9}\right)$$

input
```
int(-((x + 2*x^2 - x^3 - 2)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5))/(x^
4 - 5*x^2 + 4)^2,x)
```

output
```
i*x + (d/12 - e/6 + f/3 - (2*g)/3 + (4*h)/3 - (8*i)/3)/(x + 2) + log(x + 1
)*(d/6 - e/6 + f/6 - g/6 + h/6 - i/6) + log(x - 2)*(d/48 + e/24 + f/12 + g
/6 + h/3 + (2*i)/3) - log(x - 1)*(d/18 + e/18 + f/18 + g/18 + h/18 + i/18)
 - log(x + 2)*((19*d)/144 - (13*e)/72 + (7*f)/36 - g/18 - (5*h)/9 + (22*i)
/9)
```

3.91 $\int \frac{2-3x+x^2}{(4-5x^2+x^4)^2} \, dx$

3.91.1 Optimal result

Integrand size = 21, antiderivative size = 56

$$\int \frac{2-3x+x^2}{(4-5x^2+x^4)^2} \, dx = -\frac{5+3x}{12(2+3x+x^2)} - \frac{1}{36}\log(1-x) + \frac{1}{144}\log(2-x) - \frac{7}{36}\log(1+x) + \frac{31}{144}\log(2+x)$$

output
```
1/12*(-5-3*x)/(x^2+3*x+2)-1/36*ln(1-x)+1/144*ln(2-x)-7/36*ln(1+x)+31/144*l
n(2+x)
```

3.91.2 Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 48, normalized size of antiderivative = 0.86

$$\int \frac{2-3x+x^2}{(4-5x^2+x^4)^2} \, dx = \frac{1}{144}\left(-\frac{12(5+3x)}{2+3x+x^2} - 4\log(1-x) + \log(2-x) - 28\log(1+x) + 31\log(2+x)\right)$$

input
```
Integrate[(2 - 3*x + x^2)/(4 - 5*x^2 + x^4)^2,x]
```

output
```
((-12*(5 + 3*x))/(2 + 3*x + x^2) - 4*Log[1 - x] + Log[2 - x] - 28*Log[1 +
x] + 31*Log[2 + x])/144
```

3.91.3 Rubi [A] (verified)

Time = 0.22 (sec) , antiderivative size = 55, normalized size of antiderivative = 0.98, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.143, Rules used = {2019, 1299, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{x^2-3x+2}{(x^4-5x^2+4)^2}dx$$

$$\Big\downarrow \text{ 2019}$$

$$\int \frac{1}{(x^2-3x+2)(x^2+3x+2)^2}dx$$

$$\Big\downarrow \text{ 1299}$$

$$\int \left(-\frac{1}{144(2-x)}-\frac{7}{36(x+1)}+\frac{31}{144(x+2)}+\frac{1}{6(x+1)^2}+\frac{1}{12(x+2)^2}+\frac{1}{36(1-x)}\right)dx$$

$$\Big\downarrow \text{ 2009}$$

$$-\frac{1}{6(x+1)}-\frac{1}{12(x+2)}-\frac{1}{36}\log(1-x)+\frac{1}{144}\log(2-x)-\frac{7}{36}\log(x+1)+\frac{31}{144}\log(x+2)$$

input
```
Int[(2 - 3*x + x^2)/(4 - 5*x^2 + x^4)^2,x]
```

output
```
-1/6*1/(1 + x) - 1/(12*(2 + x)) - Log[1 - x]/36 + Log[2 - x]/144 - (7*Log[
1 + x])/36 + (31*Log[2 + x])/144
```

3.91.3.1 Defintions of rubi rules used

rule 1299
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_.) + (e_.)*(x_) + (f_.)*(x
_)^2)^(q_), x_Symbol] :> With[{r = Rt[b^2 - 4*a*c, 2]}, Simp[1/c^p   Int[Ex
pandIntegrand[(b/2 - r/2 + c*x)^p*(b/2 + r/2 + c*x)^p*(d + e*x + f*x^2)^q,
x], x], x] /; EqQ[p, -1] ||  !FractionalPowerFactorQ[r]] /; FreeQ[{a, b, c,
 d, e, f}, x] && ILtQ[p, 0] && IntegerQ[q] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

3.91.4 Maple [A] (verified)

Time = 0.06 (sec) , antiderivative size = 40, normalized size of antiderivative = 0.71

method	result
default	$-\frac{1}{12(x+2)}+\frac{31\ln(x+2)}{144}-\frac{1}{6(x+1)}-\frac{7\ln(x+1)}{36}-\frac{\ln(x-1)}{36}+\frac{\ln(x-2)}{144}$
risch	$\frac{-\frac{x}{4}-\frac{5}{12}}{x^2+3x+2}+\frac{\ln(x-2)}{144}-\frac{\ln(x-1)}{36}-\frac{7\ln(x+1)}{36}+\frac{31\ln(x+2)}{144}$
norman	$\frac{\frac{1}{3}x^2+\frac{3}{4}x-\frac{1}{4}x^3-\frac{5}{6}}{x^4-5x^2+4}+\frac{\ln(x-2)}{144}-\frac{\ln(x-1)}{36}-\frac{7\ln(x+1)}{36}+\frac{31\ln(x+2)}{144}$
parallelrisch	$\frac{\ln(x-2)x^2-4\ln(x-1)x^2-28\ln(x+1)x^2+31\ln(x+2)x^2-60+3\ln(x-2)x-12\ln(x-1)x-84\ln(x+1)x+93\ln(x+2)x+2\ln(x-2)}{144x^2+432x+288}$

input
```
int((x^2-3*x+2)/(x^4-5*x^2+4)^2,x,method=_RETURNVERBOSE)
```

output
```
-1/12/(x+2)+31/144*ln(x+2)-1/6/(x+1)-7/36*ln(x+1)-1/36*ln(x-1)+1/144*ln(x-
2)
```

3.91.5 Fricas [A] (verification not implemented)

Time = 0.25 (sec) , antiderivative size = 72, normalized size of antiderivative = 1.29

$$\int \frac{2-3x+x^2}{(4-5x^2+x^4)^2}\,dx = \frac{31\,(x^2+3\,x+2)\log(x+2)-28\,(x^2+3\,x+2)\log(x+1)-4\,(x^2+3\,x+2)\log(x-1)+(x^2+3\,x+2)}{144\,(x^2+3\,x+2)}$$

input
```
integrate((x^2-3*x+2)/(x^4-5*x^2+4)^2,x, algorithm="fricas")
```

output
```
1/144*(31*(x^2 + 3*x + 2)*log(x + 2) - 28*(x^2 + 3*x + 2)*log(x + 1) - 4*(
x^2 + 3*x + 2)*log(x - 1) + (x^2 + 3*x + 2)*log(x - 2) - 36*x - 60)/(x^2 +
 3*x + 2)
```

3.91.6 Sympy [A] (verification not implemented)

Time = 0.15 (sec) , antiderivative size = 46, normalized size of antiderivative = 0.82

$$\int \frac{2-3x+x^2}{(4-5x^2+x^4)^2}\,dx = \frac{-3x-5}{12x^2+36x+24} + \frac{\log(x-2)}{144} - \frac{\log(x-1)}{36} - \frac{7\log(x+1)}{36} + \frac{31\log(x+2)}{144}$$

input
```
integrate((x**2-3*x+2)/(x**4-5*x**2+4)**2,x)
```

output
```
(-3*x - 5)/(12*x**2 + 36*x + 24) + log(x - 2)/144 - log(x - 1)/36 - 7*log(
x + 1)/36 + 31*log(x + 2)/144
```

3.91.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 42, normalized size of antiderivative = 0.75

$$\int \frac{2-3x+x^2}{(4-5x^2+x^4)^2}\,dx = -\frac{3\,x+5}{12\,(x^2+3\,x+2)} + \frac{31}{144}\log(x+2) - \frac{7}{36}\log(x+1) - \frac{1}{36}\log(x-1) + \frac{1}{144}\log(x-2)$$

input
```
integrate((x^2-3*x+2)/(x^4-5*x^2+4)^2,x, algorithm="maxima")
```

output
```
-1/12*(3*x + 5)/(x^2 + 3*x + 2) + 31/144*log(x + 2) - 7/36*log(x + 1) - 1/
36*log(x - 1) + 1/144*log(x - 2)
```

3.91.8 Giac [A] (verification not implemented)

Time = 0.31 (sec) , antiderivative size = 46, normalized size of antiderivative = 0.82

$$\int \frac{2-3x+x^2}{(4-5x^2+x^4)^2}\,dx = -\frac{3\,x+5}{12\,(x+2)(x+1)} + \frac{31}{144}\log(|x+2|) - \frac{7}{36}\log(|x+1|) - \frac{1}{36}\log(|x-1|) + \frac{1}{144}\log(|x-2|)$$

input
```
integrate((x^2-3*x+2)/(x^4-5*x^2+4)^2,x, algorithm="giac")
```

output
```
-1/12*(3*x + 5)/((x + 2)*(x + 1)) + 31/144*log(abs(x + 2)) - 7/36*log(abs(
x + 1)) - 1/36*log(abs(x - 1)) + 1/144*log(abs(x - 2))
```

3.91.9 Mupad [B] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 42, normalized size of antiderivative = 0.75

$$\int \frac{2-3x+x^2}{(4-5x^2+x^4)^2}\,dx = \frac{\ln(x-2)}{144} - \frac{7\ln(x+1)}{36} - \frac{\ln(x-1)}{36} + \frac{31\ln(x+2)}{144} - \frac{\frac{x}{4}+\frac{5}{12}}{x^2+3x+2}$$

input
```
int((x^2 - 3*x + 2)/(x^4 - 5*x^2 + 4)^2,x)
```

output
```
log(x - 2)/144 - (7*log(x + 1))/36 - log(x - 1)/36 + (31*log(x + 2))/144 -
 (x/4 + 5/12)/(3*x + x^2 + 2)
```

3.92 $\int \frac{(d+ex)(2-3x+x^2)}{(4-5x^2+x^4)^2} \, dx$

3.92.1 Optimal result

Integrand size = 26, antiderivative size = 89

$$\int \frac{(d+ex)(2-3x+x^2)}{(4-5x^2+x^4)^2} \, dx = -\frac{5d-6e+(3d-4e)x}{12(2+3x+x^2)} - \frac{1}{36}(d+e)\log(1-x) + \frac{1}{144}(d+2e)\log(2-x) - \frac{1}{36}(7d-13e)\log(1+x) + \frac{1}{144}(31d-50e)\log(2+x)$$

output
```
1/12*(-5*d+6*e-(3*d-4*e)*x)/(x^2+3*x+2)-1/36*(d+e)*ln(1-x)+1/144*(d+2*e)*l
n(2-x)-1/36*(7*d-13*e)*ln(1+x)+1/144*(31*d-50*e)*ln(2+x)
```

3.92.2 Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 80, normalized size of antiderivative = 0.90

$$\int \frac{(d+ex)(2-3x+x^2)}{(4-5x^2+x^4)^2} \, dx = \frac{1}{144}\left(\frac{12(-5d+6e-3dx+4ex)}{2+3x+x^2} - 4(d+e)\log(1-x) + (d+2e)\log(2-x) + 4(-7d+13e)\log(1+x) + (31d-50e)\log(2+x)\right)$$

input
```
Integrate[((d + e*x)*(2 - 3*x + x^2))/(4 - 5*x^2 + x^4)^2,x]
```

output
```
((12*(-5*d + 6*e - 3*d*x + 4*e*x))/(2 + 3*x + x^2) - 4*(d + e)*Log[1 - x]
+ (d + 2*e)*Log[2 - x] + 4*(-7*d + 13*e)*Log[1 + x] + (31*d - 50*e)*Log[2
+ x])/144
```

3.92.3 Rubi [A] (verified)

Time = 0.44 (sec) , antiderivative size = 94, normalized size of antiderivative = 1.06, number of steps used = 7, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.269, Rules used = {2019, 1349, 27, 2141, 27, 1141, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^2-3x+2)(d+ex)}{(x^4-5x^2+4)^2}dx$$

$$\Big\downarrow \text{ 2019}$$

$$\int \frac{d+ex}{(x^2-3x+2)(x^2+3x+2)^2}dx$$

$$\Big\downarrow \text{ 1349}$$

$$-\frac{1}{72}\int \frac{6\left((3d-4e)x^2-4(2d-3e)x+3d-10e\right)}{(x^2-3x+2)(x^2+3x+2)}dx-\frac{x(3d-4e)+5d-6e}{12(x^2+3x+2)}$$

$$\Big\downarrow \text{ 27}$$

$$-\frac{1}{12}\int \frac{(3d-4e)x^2-4(2d-3e)x+3d-10e}{(x^2-3x+2)(x^2+3x+2)}dx-\frac{x(3d-4e)+5d-6e}{12(x^2+3x+2)}$$

$$\Big\downarrow \text{ 2141}$$

$$\frac{1}{12}\left(-\frac{1}{72}\int -\frac{6(7d+6e-(3d+2e)x)}{x^2-3x+2}dx-\frac{1}{72}\int \frac{6(25d-54e-(3d+2e)x)}{x^2+3x+2}dx\right)-\frac{x(3d-4e)+5d-6e}{12(x^2+3x+2)}$$

$$\Big\downarrow \text{ 27}$$

$$\frac{1}{12}\left(\frac{1}{12}\int \frac{7d+6e-(3d+2e)x}{x^2-3x+2}dx-\frac{1}{12}\int \frac{25d-54e-(3d+2e)x}{x^2+3x+2}dx\right)-\frac{x(3d-4e)+5d-6e}{12(x^2+3x+2)}$$

$$\Big\downarrow \text{ 1141}$$

$$\frac{1}{12}\left(\frac{1}{12}\int\left(\frac{4(d+e)}{1-x}-\frac{d+2e}{2-x}\right)dx-\frac{1}{12}\int\left(\frac{4(7d-13e)}{x+1}-\frac{31d-50e}{x+2}\right)dx\right)-\frac{x(3d-4e)+5d-6e}{12\left(x^2+3x+2\right)}$$

$\big\downarrow$ 2009

$$\frac{1}{12}\left(\frac{1}{12}((d+2e)\log(2-x)-4(d+e)\log(1-x))+\frac{1}{12}((31d-50e)\log(x+2)-4(7d-13e)\log(x+1))\right)-\frac{x(3d-4e)+5d-6e}{12\left(x^2+3x+2\right)}$$

input
```
Int[((d + e*x)*(2 - 3*x + x^2))/(4 - 5*x^2 + x^4)^2,x]
```

output
```
-1/12*(5*d - 6*e + (3*d - 4*e)*x)/(2 + 3*x + x^2) + ((-4*(d + e)*Log[1 - x
] + (d + 2*e)*Log[2 - x])/12 + (-4*(7*d - 13*e)*Log[1 + x] + (31*d - 50*e)
*Log[2 + x])/12)/12
```

3.92.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 1141
```
Int[((d_.) + (e_.)*(x_))^(m_.)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_
Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[1/c^p   Int[ExpandIntegrand[
(d + e*x)^m*(b/2 - q/2 + c*x)^p*(b/2 + q/2 + c*x)^p, x], x], x] /; EqQ[p, -
1] ||  !FractionalPowerFactorQ[q]] /; FreeQ[{a, b, c, d, e}, x] && ILtQ[p,
0] && IntegerQ[m] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 1349
```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (e
_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1)*
((d + e*x + f*x^2)^(q + 1)/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f))*(p + 1)))*(g*c*(2*a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d +
b^2*f - c*(b*e + 2*a*f)) + c*(g*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)) - h*(b*
c*d - 2*a*c*e + a*b*f))*x), x] + Simp[1/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*
d - a*e)*(c*e - b*f))*(p + 1))   Int[(a + b*x + c*x^2)^(p + 1)*(d + e*x + f
*x^2)^q*Simp[(b*h - 2*g*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*(p + 1
) + (b^2*(g*f) - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*((-h)*c
*e)))*(a*f*(p + 1) - c*d*(p + 2)) - e*((g*c)*(2*a*c*e - b*(c*d + a*f)) + (g
*b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(p + q + 2) - (2*f*((g*c)*(2
*a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*
(p + q + 2) - (b^2*g*f - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a
*((-h)*c*e)))*(b*f*(p + 1) - c*e*(2*p + q + 4)))*x - c*f*(b^2*(g*f) - b*(h*
c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) + a*h*c*e))*(2*p + 2*q + 5)*x^2,
x], x], x] /; FreeQ[{a, b, c, d, e, f, g, h, q}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1] && NeQ[(c*d - a*f)^2 - (b*d - a*e)*(c*
e - b*f), 0] &&  !( !IntegerQ[p] && ILtQ[q, -1])
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2141
```
Int[(Px_)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*((d_) + (e_.)*(x_) + (f_.)*(x
_)^2)), x_Symbol] :> With[{A = Coeff[Px, x, 0], B = Coeff[Px, x, 1], C = Co
eff[Px, x, 2], q = c^2*d^2 - b*c*d*e + a*c*e^2 + b^2*d*f - 2*a*c*d*f - a*b*
e*f + a^2*f^2}, Simp[1/q   Int[(A*c^2*d - a*c*C*d - A*b*c*e + a*B*c*e + A*b
^2*f - a*b*B*f - a*A*c*f + a^2*C*f + c*(B*c*d - b*C*d - A*c*e + a*C*e + A*b
*f - a*B*f)*x)/(a + b*x + c*x^2), x], x] + Simp[1/q   Int[(c*C*d^2 - B*c*d*
e + A*c*e^2 + b*B*d*f - A*c*d*f - a*C*d*f - A*b*e*f + a*A*f^2 - f*(B*c*d -
b*C*d - A*c*e + a*C*e + A*b*f - a*B*f)*x)/(d + e*x + f*x^2), x], x] /; NeQ[
q, 0]] /; FreeQ[{a, b, c, d, e, f}, x] && PolyQ[Px, x, 2]
```

3.92. $\int \frac{(d+ex)(2-3x+x^2)}{(4-5x^2+x^4)^2}\,dx$

3.92.4 Maple [A] (verified)

Time = 0.07 (sec) , antiderivative size = 78, normalized size of antiderivative = 0.88

method	result
default	$-\frac{\frac{d}{12}-\frac{e}{6}}{x+2}+\left(\frac{31d}{144}-\frac{25e}{72}\right)\ln(x+2)+\left(-\frac{7d}{36}+\frac{13e}{36}\right)\ln(x+1)-\frac{\frac{d}{6}-\frac{e}{6}}{x+1}+\left(-\frac{d}{36}-\frac{e}{36}\right)\ln(x-1)+($
risch	$\frac{\left(-\frac{d}{4}+\frac{e}{3}\right)x-\frac{5d}{12}+\frac{e}{2}}{x^2+3x+2}+\frac{\ln(2-x)d}{144}+\frac{\ln(2-x)e}{72}+\frac{31\ln(x+2)d}{144}-\frac{25\ln(x+2)e}{72}-\frac{\ln(x-1)d}{36}-\frac{\ln(x-1)e}{36}-\frac{7\ln(-x-1)d}{36}$
norman	$\frac{\left(-\frac{d}{4}+\frac{e}{3}\right)x^3+\left(\frac{3d}{4}-\frac{5e}{6}\right)x+\left(\frac{d}{3}-\frac{e}{2}\right)x^2-\frac{5d}{6}+e}{x^4-5x^2+4}+\left(-\frac{7d}{36}+\frac{13e}{36}\right)\ln(x+1)+\left(-\frac{d}{36}-\frac{e}{36}\right)\ln(x-1)+\left(\frac{d}{144}\right.$
parallelrisch	$\frac{-60d+72e-36dx+2\ln(x-2)d+4\ln(x-2)e-8\ln(x-1)d-8\ln(x-1)e-150\ln(x+2)xe+6\ln(x-2)xe-12\ln(x-1)xd-12\ln(x-1}{}$

input
```
int((e*x+d)*(x^2-3*x+2)/(x^4-5*x^2+4)^2,x,method=_RETURNVERBOSE)
```

output
```
-(1/12*d-1/6*e)/(x+2)+(31/144*d-25/72*e)*ln(x+2)+(-7/36*d+13/36*e)*ln(x+1)
-(1/6*d-1/6*e)/(x+1)+(-1/36*d-1/36*e)*ln(x-1)+(1/144*d+1/72*e)*ln(x-2)
```

3.92.5 Fricas [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 153, normalized size of antiderivative = 1.72

$$\int \frac{(d+ex)(2-3x+x^2)}{(4-5x^2+x^4)^2}\,dx = -\frac{12(3d-4e)x-((31d-50e)x^2+3(31d-50e)x+62d-100e)\log(x+2)+4((7d-13e)x^2+3(}{}$$

input
```
integrate((e*x+d)*(x^2-3*x+2)/(x^4-5*x^2+4)^2,x, algorithm="fricas")
```

output
```
-1/144*(12*(3*d - 4*e)*x - ((31*d - 50*e)*x^2 + 3*(31*d - 50*e)*x + 62*d -
 100*e)*log(x + 2) + 4*((7*d - 13*e)*x^2 + 3*(7*d - 13*e)*x + 14*d - 26*e)
*log(x + 1) + 4*((d + e)*x^2 + 3*(d + e)*x + 2*d + 2*e)*log(x - 1) - ((d +
 2*e)*x^2 + 3*(d + 2*e)*x + 2*d + 4*e)*log(x - 2) + 60*d - 72*e)/(x^2 + 3*
x + 2)
```

3.92.6 Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 1255 vs. $2(80) = 160$.

Time = 6.98 (sec) , antiderivative size = 1255, normalized size of antiderivative = 14.10

$$\int \frac{(d+ex)(2-3x+x^2)}{(4-5x^2+x^4)^2}\,dx = \text{Too large to display}$$

input
```
integrate((e*x+d)*(x**2-3*x+2)/(x**4-5*x**2+4)**2,x)
```

output
```
-(d + e)*log(x + (-24383100*d**6 + 187408066*d**5*e + 10439775*d**5*(d + e
) - 511591980*d**4*e**2 - 94132290*d**4*e*(d + e) + 667200*d**4*(d + e)**2
 + 469491120*d**3*e**3 + 333672552*d**3*e**2*(d + e) - 2703328*d**3*e*(d +
 e)**2 - 198000*d**3*(d + e)**3 + 322778400*d**2*e**4 - 582497712*d**2*e**
3*(d + e) + 1752768*d**2*e**2*(d + e)**2 + 1107552*d**2*e*(d + e)**3 - 863
493856*d*e**5 + 500776560*d*e**4*(d + e) + 4226944*d*e**3*(d + e)**2 - 188
0640*d*e**2*(d + e)**3 + 429000000*e**6 - 169242912*e**5*(d + e) - 4538112
*e**4*(d + e)**2 + 964224*e**3*(d + e)**3)/(13474125*d**6 - 102860175*d**5
*e + 274190390*d**4*e**2 - 224142072*d**3*e**3 - 245084096*d**2*e**4 + 535
797456*d*e**5 - 256183200*e**6))/36 + (d + 2*e)*log(x + (-24383100*d**6 +
187408066*d**5*e - 10439775*d**5*(d + 2*e)/4 - 511591980*d**4*e**2 + 47066
145*d**4*e*(d + 2*e)/2 + 41700*d**4*(d + 2*e)**2 + 469491120*d**3*e**3 - 8
3418138*d**3*e**2*(d + 2*e) - 168958*d**3*e*(d + 2*e)**2 + 12375*d**3*(d +
 2*e)**3/4 + 322778400*d**2*e**4 + 145624428*d**2*e**3*(d + 2*e) + 109548*
d**2*e**2*(d + 2*e)**2 - 34611*d**2*e*(d + 2*e)**3/2 - 863493856*d*e**5 -
125194140*d*e**4*(d + 2*e) + 264184*d*e**3*(d + 2*e)**2 + 29385*d*e**2*(d
+ 2*e)**3 + 429000000*e**6 + 42310728*e**5*(d + 2*e) - 283632*e**4*(d + 2*
e)**2 - 15066*e**3*(d + 2*e)**3)/(13474125*d**6 - 102860175*d**5*e + 27419
0390*d**4*e**2 - 224142072*d**3*e**3 - 245084096*d**2*e**4 + 535797456*d*e
**5 - 256183200*e**6))/144 - (7*d - 13*e)*log(x + (-24383100*d**6 + 187...
```

3.92.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 75, normalized size of antiderivative = 0.84

$$\begin{aligned}\int \frac{(d+ex)(2-3x+x^2)}{(4-5x^2+x^4)^2}\,dx = {} & \frac{1}{144}\,(31\,d-50\,e)\log(x+2) \\ & -\frac{1}{36}\,(7\,d-13\,e)\log(x+1)-\frac{1}{36}\,(d+e)\log(x-1) \\ & +\frac{1}{144}\,(d+2\,e)\log(x-2)-\frac{(3\,d-4\,e)x+5\,d-6\,e}{12\,(x^2+3\,x+2)}\end{aligned}$$

input
```
integrate((e*x+d)*(x^2-3*x+2)/(x^4-5*x^2+4)^2,x, algorithm="maxima")
```

output
```
1/144*(31*d - 50*e)*log(x + 2) - 1/36*(7*d - 13*e)*log(x + 1) - 1/36*(d +
e)*log(x - 1) + 1/144*(d + 2*e)*log(x - 2) - 1/12*((3*d - 4*e)*x + 5*d - 6
*e)/(x^2 + 3*x + 2)
```

3.92.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 79, normalized size of antiderivative = 0.89

$$\int \frac{(d+ex)(2-3x+x^2)}{(4-5x^2+x^4)^2}\,dx = \frac{1}{144}\,(31\,d-50\,e)\log\left(|x+2|\right) - \frac{1}{36}\,(7\,d-13\,e)\log\left(|x+1|\right) - \frac{1}{36}\,(d+e)\log\left(|x-1|\right) + \frac{1}{144}\,(d+2\,e)\log\left(|x-2|\right) - \frac{(3\,d-4\,e)x+5\,d-6\,e}{12\,(x+2)(x+1)}$$

input
```
integrate((e*x+d)*(x^2-3*x+2)/(x^4-5*x^2+4)^2,x, algorithm="giac")
```

output
```
1/144*(31*d - 50*e)*log(abs(x + 2)) - 1/36*(7*d - 13*e)*log(abs(x + 1)) -
1/36*(d + e)*log(abs(x - 1)) + 1/144*(d + 2*e)*log(abs(x - 2)) - 1/12*((3*
d - 4*e)*x + 5*d - 6*e)/((x + 2)*(x + 1))
```

3.92.9 Mupad [B] (verification not implemented)

Time = 0.06 (sec) , antiderivative size = 79, normalized size of antiderivative = 0.89

$$\int \frac{(d+ex)(2-3x+x^2)}{(4-5x^2+x^4)^2}\,dx = \ln\left(x-2\right)\left(\frac{d}{144}+\frac{e}{72}\right) - \ln\left(x-1\right)\left(\frac{d}{36}+\frac{e}{36}\right) - \ln\left(x+1\right)\left(\frac{7\,d}{36}-\frac{13\,e}{36}\right) - \frac{\frac{5\,d}{12}-\frac{e}{2}+x\left(\frac{d}{4}-\frac{e}{3}\right)}{x^2+3\,x+2} + \ln\left(x+2\right)\left(\frac{31\,d}{144}-\frac{25\,e}{72}\right)$$

input
```
int(((d + e*x)*(x^2 - 3*x + 2))/(x^4 - 5*x^2 + 4)^2,x)
```

output
```
log(x - 2)*(d/144 + e/72) - log(x - 1)*(d/36 + e/36) - log(x + 1)*((7*d)/3
6 - (13*e)/36) - ((5*d)/12 - e/2 + x*(d/4 - e/3))/(3*x + x^2 + 2) + log(x
+ 2)*((31*d)/144 - (25*e)/72)
```

3.92. $\int \frac{(d+ex)(2-3x+x^2)}{(4-5x^2+x^4)^2}\,dx$

3.93 $\int \frac{(2-3x+x^2)(d+ex+fx^2)}{(4-5x^2+x^4)^2}\,dx$

3.93.1 Optimal result

Integrand size = 31, antiderivative size = 105

$$\int \frac{(2-3x+x^2)(d+ex+fx^2)}{(4-5x^2+x^4)^2}\,dx = -\frac{5d-6e+8f+(3d-4e+6f)x}{12(2+3x+x^2)} - \frac{1}{36}(d+e+f)\log(1-x) + \frac{1}{144}(d+2e+4f)\log(2-x) - \frac{1}{36}(7d-13e+19f)\log(1+x) + \frac{1}{144}(31d-50e+76f)\log(2+x)$$

output
```
1/12*(-5*d+6*e-8*f-(3*d-4*e+6*f)*x)/(x^2+3*x+2)-1/36*(d+e+f)*ln(1-x)+1/144
*(d+2*e+4*f)*ln(2-x)-1/36*(7*d-13*e+19*f)*ln(1+x)+1/144*(31*d-50*e+76*f)*l
n(2+x)
```

3.93.2 Mathematica [A] (verified)

Time = 0.05 (sec) , antiderivative size = 97, normalized size of antiderivative = 0.92

$$\int \frac{(2-3x+x^2)(d+ex+fx^2)}{(4-5x^2+x^4)^2}\,dx = \frac{1}{144}\left(-\frac{12(-6e+8f-4ex+6fx+d(5+3x))}{2+3x+x^2} - 4(d+e+f)\log(1-x) + (d+2e+4f)\log(2-x) - 4(7d-13e+19f)\log(1+x) + (31d-50e+76f)\log(2+x)\right)$$

input
```
Integrate[((2 - 3*x + x^2)*(d + e*x + f*x^2))/(4 - 5*x^2 + x^4)^2,x]
```

output
```
((-12*(-6*e + 8*f - 4*e*x + 6*f*x + d*(5 + 3*x)))/(2 + 3*x + x^2) - 4*(d +
 e + f)*Log[1 - x] + (d + 2*e + 4*f)*Log[2 - x] - 4*(7*d - 13*e + 19*f)*Lo
g[1 + x] + (31*d - 50*e + 76*f)*Log[2 + x])/144
```

3.93.3 Rubi [A] (verified)

Time = 0.53 (sec) , antiderivative size = 110, normalized size of antiderivative = 1.05, number of steps used = 7, number of rules used = 7, $\frac{\text{number of rules}}{\text{integrand size}} = 0.226$, Rules used = {2019, 2135, 27, 2141, 27, 1141, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^2-3x+2)(d+ex+fx^2)}{(x^4-5x^2+4)^2}dx$$

$$\Big\downarrow \text{2019}$$

$$\int \frac{d+ex+fx^2}{(x^2-3x+2)(x^2+3x+2)^2}dx$$

$$\Big\downarrow \text{2135}$$

$$-\frac{1}{72}\int \frac{6((3d-4e+6f)x^2-4(2d-3e+5f)x+3d-10e+12f)}{(x^2-3x+2)(x^2+3x+2)}dx-\frac{x(3d-4e+6f)+5d-6e+8f}{12(x^2+3x+2)}$$

$$\Big\downarrow \text{27}$$

$$-\frac{1}{12}\int \frac{(3d-4e+6f)x^2-4(2d-3e+5f)x+3d-10e+12f}{(x^2-3x+2)(x^2+3x+2)}dx-\frac{x(3d-4e+6f)+5d-6e+8f}{12(x^2+3x+2)}$$

$$\Big\downarrow \text{2141}$$

$$\frac{1}{12}\left(-\frac{1}{72}\int -\frac{6(7d+6e+4f-(3d+2e)x)}{x^2-3x+2}dx-\frac{1}{72}\int \frac{6(25d-54e+76f-(3d+2e)x)}{x^2+3x+2}dx\right)-\frac{x(3d-4e+6f)+5d-6e+8f}{12(x^2+3x+2)}$$

$$\Big\downarrow \text{27}$$

$$\frac{1}{12}\left(\frac{1}{12}\int \frac{7d+6e+4f-(3d+2e)x}{x^2-3x+2}dx-\frac{1}{12}\int \frac{25d-54e+76f-(3d+2e)x}{x^2+3x+2}dx\right)-\frac{x(3d-4e+6f)+5d-6e+8f}{12\left(x^2+3x+2\right)}$$

$$\Big\downarrow \; 1141$$

$$\frac{1}{12}\left(\frac{1}{12}\int \left(\frac{4(d+e+f)}{1-x}-\frac{d+2e+4f}{2-x}\right)dx-\frac{1}{12}\int \left(\frac{4(7d-13e+19f)}{x+1}-\frac{31d-50e+76f}{x+2}\right)dx\right)-\frac{x(3d-4e+6f)+5d-6e+8f}{12\left(x^2+3x+2\right)}$$

$$\Big\downarrow \; 2009$$

$$\frac{1}{12}\left(\frac{1}{12}(\log(2-x)(d+2e+4f)-4\log(1-x)(d+e+f))+\frac{1}{12}(\log(x+2)(31d-50e+76f)-4\log(x+1)(7d\right.$$
$$-\frac{x(3d-4e+6f)+5d-6e+8f}{12\left(x^2+3x+2\right)}$$

input
```
Int[((2 - 3*x + x^2)*(d + e*x + f*x^2))/(4 - 5*x^2 + x^4)^2,x]
```

output
```
-1/12*(5*d - 6*e + 8*f + (3*d - 4*e + 6*f)*x)/(2 + 3*x + x^2) + ((-4*(d +
e + f)*Log[1 - x] + (d + 2*e + 4*f)*Log[2 - x])/12 + (-4*(7*d - 13*e + 19*
f)*Log[1 + x] + (31*d - 50*e + 76*f)*Log[2 + x])/12)/12
```

3.93.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 1141
```
Int[((d_.) + (e_.)*(x_))^(m_.)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_
Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Simp[1/c^p   Int[ExpandIntegrand[
(d + e*x)^m*(b/2 - q/2 + c*x)^p*(b/2 + q/2 + c*x)^p, x], x], x] /; EqQ[p, -
1] ||  !FractionalPowerFactorQ[q]] /; FreeQ[{a, b, c, d, e}, x] && ILtQ[p,
0] && IntegerQ[m] && NiceSqrtQ[b^2 - 4*a*c]
```

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2135
```
Int[(Px_)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (e_.)*(x_) + (f_.
)*(x_)^2)^(q_), x_Symbol] :> With[{A = Coeff[Px, x, 0], B = Coeff[Px, x, 1]
, C = Coeff[Px, x, 2]}, Simp[(a + b*x + c*x^2)^(p + 1)*((d + e*x + f*x^2)^(
q + 1)/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*(p + 1)))*(
(A*c - a*C)*(2*a*c*e - b*(c*d + a*f)) + (A*b - a*B)*(2*c^2*d + b^2*f - c*(b
*e + 2*a*f)) + c*(A*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)) - B*(b*c*d - 2*a*c*
e + a*b*f) + C*(b^2*d - a*b*e - 2*a*(c*d - a*f)))*x), x] + Simp[1/((b^2 - 4
*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*(p + 1))   Int[(a + b*x + c
*x^2)^(p + 1)*(d + e*x + f*x^2)^q*Simp[(b*B - 2*A*c - 2*a*C)*((c*d - a*f)^2
 - (b*d - a*e)*(c*e - b*f))*(p + 1) + (b^2*(C*d + A*f) - b*(B*c*d + A*c*e +
 a*C*e + a*B*f) + 2*(A*c*(c*d - a*f) - a*(c*C*d - B*c*e - a*C*f)))*(a*f*(p
+ 1) - c*d*(p + 2)) - e*((A*c - a*C)*(2*a*c*e - b*(c*d + a*f)) + (A*b - a*B
)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(p + q + 2) - (2*f*((A*c - a*C)*(2*a
*c*e - b*(c*d + a*f)) + (A*b - a*B)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(p
 + q + 2) - (b^2*(C*d + A*f) - b*(B*c*d + A*c*e + a*C*e + a*B*f) + 2*(A*c*(
c*d - a*f) - a*(c*C*d - B*c*e - a*C*f)))*(b*f*(p + 1) - c*e*(2*p + q + 4)))
*x - c*f*(b^2*(C*d + A*f) - b*(B*c*d + A*c*e + a*C*e + a*B*f) + 2*(A*c*(c*d
 - a*f) - a*(c*C*d - B*c*e - a*C*f)))*(2*p + 2*q + 5)*x^2, x], x], x]] /; F
reeQ[{a, b, c, d, e, f, q}, x] && PolyQ[Px, x, 2] && LtQ[p, -1] && NeQ[(c*d
 - a*f)^2 - (b*d - a*e)*(c*e - b*f), 0] &&  !( !IntegerQ[p] && ILtQ[q, -1])
 &&  !IGtQ[q, 0]
```

rule 2141
```
Int[(Px_)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*((d_) + (e_.)*(x_) + (f_.)*(x
_)^2)), x_Symbol] :> With[{A = Coeff[Px, x, 0], B = Coeff[Px, x, 1], C = Co
eff[Px, x, 2], q = c^2*d^2 - b*c*d*e + a*c*e^2 + b^2*d*f - 2*a*c*d*f - a*b*
e*f + a^2*f^2}, Simp[1/q   Int[(A*c^2*d - a*c*C*d - A*b*c*e + a*B*c*e + A*b
^2*f - a*b*B*f - a*A*c*f + a^2*C*f + c*(B*c*d - b*C*d - A*c*e + a*C*e + A*b
*f - a*B*f)*x)/(a + b*x + c*x^2), x], x] + Simp[1/q   Int[(c*C*d^2 - B*c*d*
e + A*c*e^2 + b*B*d*f - A*c*d*f - a*C*d*f - A*b*e*f + a*A*f^2 - f*(B*c*d -
b*C*d - A*c*e + a*C*e + A*b*f - a*B*f)*x)/(d + e*x + f*x^2), x], x] /; NeQ[
q, 0]] /; FreeQ[{a, b, c, d, e, f}, x] && PolyQ[Px, x, 2]
```

3.93. $\int \frac{(2-3x+x^2)(d+ex+fx^2)}{(4-5x^2+x^4)^2}\,dx$

3.93.4 Maple [A] (verified)

Time = 0.08 (sec) , antiderivative size = 96, normalized size of antiderivative = 0.91

method	result
default	$-\frac{\frac{d}{12}-\frac{e}{6}+\frac{f}{3}}{x+2}+\left(\frac{31d}{144}-\frac{25e}{72}+\frac{19f}{36}\right)\ln\left(x+2\right)+\left(-\frac{7d}{36}+\frac{13e}{36}-\frac{19f}{36}\right)\ln\left(x+1\right)-\frac{\frac{d}{6}-\frac{e}{6}+\frac{f}{6}}{x+1}+\left(-\frac{d}{36}-\right.$
norman	$\frac{\left(-\frac{d}{4}+\frac{e}{3}-\frac{f}{2}\right)x^3+\left(\frac{3d}{4}-\frac{5e}{6}+f\right)x+\left(\frac{d}{3}-\frac{e}{2}+\frac{5f}{6}\right)x^2-\frac{5d}{6}+e-\frac{4f}{3}}{x^4-5x^2+4}+\left(-\frac{7d}{36}+\frac{13e}{36}-\frac{19f}{36}\right)\ln\left(x+1\right)+\left(-\frac{d}{36}-\frac{e}{36}-\right.$
risch	$\frac{\left(-\frac{d}{4}+\frac{e}{3}-\frac{f}{2}\right)x-\frac{5d}{12}+\frac{e}{2}-\frac{2f}{3}}{x^2+3x+2}-\frac{\ln(x-1)d}{36}-\frac{\ln(x-1)e}{36}-\frac{\ln(x-1)f}{36}+\frac{31\ln(x+2)d}{144}-\frac{25\ln(x+2)e}{72}+\frac{19\ln(x+2)f}{36}-$
parallelrisch	$\frac{-96f-60d+72e-36dx+2\ln(x-2)d+4\ln(x-2)e-8\ln(x-1)d-8\ln(x-1)e+152\ln(x+2)f-152\ln(x+1)f-150\ln(x+2)xe+6\ln}{}$

input
```
int((x^2-3*x+2)*(f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x,method=_RETURNVERBOSE)
```

output
```
-(1/12*d-1/6*e+1/3*f)/(x+2)+(31/144*d-25/72*e+19/36*f)*ln(x+2)+(-7/36*d+13
/36*e-19/36*f)*ln(x+1)-(1/6*d-1/6*e+1/6*f)/(x+1)+(-1/36*d-1/36*e-1/36*f)*l
n(x-1)+(1/144*d+1/72*e+1/36*f)*ln(x-2)
```

3.93.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 191 vs. $2(95) = 190$.

Time = 0.34 (sec) , antiderivative size = 191, normalized size of antiderivative = 1.82

$$\int \frac{(2-3x+x^2)\left(d+ex+fx^2\right)}{(4-5x^2+x^4)^2}\,dx =$$
$$-\frac{12\,(3\,d-4\,e+6\,f)x-((31\,d-50\,e+76\,f)x^2+3\,(31\,d-50\,e+76\,f)x+62\,d-100\,e+152\,f)\log\left(x\right.}{}$$

input
```
integrate((x^2-3*x+2)*(f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="fricas")
```

output
```
-1/144*(12*(3*d - 4*e + 6*f)*x - ((31*d - 50*e + 76*f)*x^2 + 3*(31*d - 50*
e + 76*f)*x + 62*d - 100*e + 152*f)*log(x + 2) + 4*((7*d - 13*e + 19*f)*x^
2 + 3*(7*d - 13*e + 19*f)*x + 14*d - 26*e + 38*f)*log(x + 1) + 4*((d + e +
 f)*x^2 + 3*(d + e + f)*x + 2*d + 2*e + 2*f)*log(x - 1) - ((d + 2*e + 4*f)
*x^2 + 3*(d + 2*e + 4*f)*x + 2*d + 4*e + 8*f)*log(x - 2) + 60*d - 72*e + 9
6*f)/(x^2 + 3*x + 2)
```

3.93.6 Sympy [F(-1)]

Timed out.

$$\int \frac{(2-3x+x^2)(d+ex+fx^2)}{(4-5x^2+x^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((x**2-3*x+2)*(f*x**2+e*x+d)/(x**4-5*x**2+4)**2,x)
```

output
```
Timed out
```

3.93.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 91, normalized size of antiderivative = 0.87

$$\begin{aligned}\int \frac{(2-3x+x^2)(d+ex+fx^2)}{(4-5x^2+x^4)^2}\,dx &= \frac{1}{144}\,(31\,d-50\,e+76\,f)\log(x+2) \\ &\quad - \frac{1}{36}\,(7\,d-13\,e+19\,f)\log(x+1) \\ &\quad - \frac{1}{36}\,(d+e+f)\log(x-1) \\ &\quad + \frac{1}{144}\,(d+2\,e+4\,f)\log(x-2) \\ &\quad - \frac{(3\,d-4\,e+6\,f)x+5\,d-6\,e+8\,f}{12\,(x^2+3\,x+2)}\end{aligned}$$

input
```
integrate((x^2-3*x+2)*(f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="maxima")
```

output
```
1/144*(31*d - 50*e + 76*f)*log(x + 2) - 1/36*(7*d - 13*e + 19*f)*log(x + 1
) - 1/36*(d + e + f)*log(x - 1) + 1/144*(d + 2*e + 4*f)*log(x - 2) - 1/12*
((3*d - 4*e + 6*f)*x + 5*d - 6*e + 8*f)/(x^2 + 3*x + 2)
```

3.93.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 95, normalized size of antiderivative = 0.90

$$\int \frac{(2-3x+x^2)(d+ex+fx^2)}{(4-5x^2+x^4)^2}\,dx = \frac{1}{144}(31\,d-50\,e+76\,f)\log(|x+2|) - \frac{1}{36}(7\,d-13\,e+19\,f)\log(|x+1|) - \frac{1}{36}(d+e+f)\log(|x-1|) + \frac{1}{144}(d+2\,e+4\,f)\log(|x-2|) - \frac{(3\,d-4\,e+6\,f)x+5\,d-6\,e+8\,f}{12\,(x+2)(x+1)}$$

input
```
integrate((x^2-3*x+2)*(f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="giac")
```

output
```
1/144*(31*d - 50*e + 76*f)*log(abs(x + 2)) - 1/36*(7*d - 13*e + 19*f)*log(
abs(x + 1)) - 1/36*(d + e + f)*log(abs(x - 1)) + 1/144*(d + 2*e + 4*f)*log
(abs(x - 2)) - 1/12*((3*d - 4*e + 6*f)*x + 5*d - 6*e + 8*f)/((x + 2)*(x +
1))
```

3.93.9 Mupad [B] (verification not implemented)

Time = 0.08 (sec) , antiderivative size = 97, normalized size of antiderivative = 0.92

$$\int \frac{(2-3x+x^2)(d+ex+fx^2)}{(4-5x^2+x^4)^2}\,dx = \ln(x-2)\left(\frac{d}{144}+\frac{e}{72}+\frac{f}{36}\right) - \ln(x+1)\left(\frac{7\,d}{36}-\frac{13\,e}{36}+\frac{19\,f}{36}\right) - \ln(x-1)\left(\frac{d}{36}+\frac{e}{36}+\frac{f}{36}\right) + \ln(x+2)\left(\frac{31\,d}{144}-\frac{25\,e}{72}+\frac{19\,f}{36}\right) - \frac{\frac{5\,d}{12}-\frac{e}{2}+\frac{2\,f}{3}+x\left(\frac{d}{4}-\frac{e}{3}+\frac{f}{2}\right)}{x^2+3\,x+2}$$

input
```
int(((x^2 - 3*x + 2)*(d + e*x + f*x^2))/(x^4 - 5*x^2 + 4)^2,x)
```

output
```
log(x - 2)*(d/144 + e/72 + f/36) - log(x + 1)*((7*d)/36 - (13*e)/36 + (19*
f)/36) - log(x - 1)*(d/36 + e/36 + f/36) + log(x + 2)*((31*d)/144 - (25*e)
/72 + (19*f)/36) - ((5*d)/12 - e/2 + (2*f)/3 + x*(d/4 - e/3 + f/2))/(3*x +
 x^2 + 2)
```

3.93. $\int \frac{(2-3x+x^2)(d+ex+fx^2)}{(4-5x^2+x^4)^2} \, dx$

3.94 $\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx$

3.94.1 Optimal result

Integrand size = 36, antiderivative size = 117

$$\begin{aligned}\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx = &-\frac{d-e+f-g}{6(1+x)}-\frac{d-2e+4f-8g}{12(2+x)}\\ &-\frac{1}{36}(d+e+f+g)\log(1-x)\\ &+\frac{1}{144}(d+2e+4f+8g)\log(2-x)\\ &-\frac{1}{36}(7d-13e+19f-25g)\log(1+x)\\ &+\frac{1}{144}(31d-50e+76f-104g)\log(2+x)\end{aligned}$$

output
```
1/6*(-d+e-f+g)/(1+x)+1/12*(-d+2*e-4*f+8*g)/(2+x)-1/36*(d+e+f+g)*ln(1-x)+1/
144*(d+2*e+4*f+8*g)*ln(2-x)-1/36*(7*d-13*e+19*f-25*g)*ln(1+x)+1/144*(31*d-
50*e+76*f-104*g)*ln(2+x)
```

3.94.2 Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 114, normalized size of antiderivative = 0.97

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx$$
$$=\frac{1}{144}\left(\frac{12(-5d+6e-8f+12g-3dx+4ex-6fx+10gx)}{2+3x+x^2}-4(d+e+f+g)\log(1-x)+(d+2e+4f+8g)\log(2-x)+4(-7d+13e-19f+25g)\log(1+x)+(31d-50e+76f-104g)\log(2+x)\right)$$

input
```
Integrate[((2 - 3*x + x^2)*(d + e*x + f*x^2 + g*x^3))/(4 - 5*x^2 + x^4)^2,
x]
```

output
```
((12*(-5*d + 6*e - 8*f + 12*g - 3*d*x + 4*e*x - 6*f*x + 10*g*x))/(2 + 3*x
+ x^2) - 4*(d + e + f + g)*Log[1 - x] + (d + 2*e + 4*f + 8*g)*Log[2 - x] +
 4*(-7*d + 13*e - 19*f + 25*g)*Log[1 + x] + (31*d - 50*e + 76*f - 104*g)*L
og[2 + x])/144
```

3.94.3 Rubi [A] (verified)

Time = 0.48 (sec) , antiderivative size = 117, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.083, Rules used = {2019, 7279, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^2-3x+2)(d+ex+fx^2+gx^3)}{(x^4-5x^2+4)^2}dx$$

$$\Big\downarrow \text{2019}$$

$$\int \frac{d+ex+fx^2+gx^3}{(x^2-3x+2)(x^2+3x+2)^2}dx$$

$$\Big\downarrow \text{7279}$$

$$\int \left(\frac{31d-50e+76f-104g}{144(x+2)}+\frac{d+2e+4f+8g}{144(x-2)}+\frac{-d-e-f-g}{36(x-1)}+\frac{-7d+13e-19f+25g}{36(x+1)}+\frac{d-e+f-g}{6(x+1)^2}\right.$$

$$\Big\downarrow \text{2009}$$

$$-\frac{d-2e+4f-8g}{12(x+2)}-\frac{d-e+f-g}{6(x+1)}-\frac{1}{36}\log(1-x)(d+e+f+g)+\frac{1}{144}\log(2-x)(d+2e+4f+8g)-\frac{1}{36}\log(x+1)(7d-13e+19f-25g)+\frac{1}{144}\log(x+2)(31d-50e+76f-104g)$$

input
```
Int[((2 - 3*x + x^2)*(d + e*x + f*x^2 + g*x^3))/(4 - 5*x^2 + x^4)^2,x]
```

output
```
-1/6*(d - e + f - g)/(1 + x) - (d - 2*e + 4*f - 8*g)/(12*(2 + x)) - ((d +
e + f + g)*Log[1 - x])/36 + ((d + 2*e + 4*f + 8*g)*Log[2 - x])/144 - ((7*d
 - 13*e + 19*f - 25*g)*Log[1 + x])/36 + ((31*d - 50*e + 76*f - 104*g)*Log[
2 + x])/144
```

3.94.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 7279
```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

3.94. $\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx$

3.94.4 Maple [A] (verified)

Time = 0.10 (sec) , antiderivative size = 114, normalized size of antiderivative = 0.97

method	result
default	$-\frac{\frac{d}{12}-\frac{e}{6}+\frac{f}{3}-\frac{2g}{3}}{x+2}+\left(\frac{31d}{144}-\frac{25e}{72}+\frac{19f}{36}-\frac{13g}{18}\right)\ln(x+2)+\left(-\frac{7d}{36}+\frac{13e}{36}-\frac{19f}{36}+\frac{25g}{36}\right)\ln(x+1)-\frac{d}{6}$
norman	$\frac{\left(-\frac{d}{4}+\frac{e}{3}-\frac{f}{2}+\frac{5g}{6}\right)x^3+\left(\frac{3d}{4}-\frac{5e}{6}+f-\frac{4g}{3}\right)x+\left(\frac{d}{3}-\frac{e}{2}+\frac{5f}{6}-\frac{3g}{2}\right)x^2-\frac{5d}{6}+e+2g-\frac{4f}{3}}{x^4-5x^2+4}+\left(-\frac{7d}{36}+\frac{13e}{36}-\frac{19f}{36}+\frac{25g}{36}\right)\ln(x$
risch	$\frac{\left(-\frac{d}{4}+\frac{e}{3}-\frac{f}{2}+\frac{5g}{6}\right)x-\frac{5d}{12}+\frac{e}{2}-\frac{2f}{3}+g}{x^2+3x+2}-\frac{\ln(x-1)d}{36}-\frac{\ln(x-1)e}{36}-\frac{\ln(x-1)f}{36}-\frac{\ln(x-1)g}{36}-\frac{7\ln(-x-1)d}{36}+\frac{13\ln(-x-1}{36}$
parallelrisch	$\frac{-96f+144g+120gx-60d+72e-36dx+2\ln(x-2)d+4\ln(x-2)e-8\ln(x-1)d-8\ln(x-1)e+24\ln(x-2)xg-12\ln(x-1)xg+300\ln}{}$

input
```
int((x^2-3*x+2)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x,method=_RETURNVERBOS
E)
```

output
```
-(1/12*d-1/6*e+1/3*f-2/3*g)/(x+2)+(31/144*d-25/72*e+19/36*f-13/18*g)*ln(x+
2)+(-7/36*d+13/36*e-19/36*f+25/36*g)*ln(x+1)-(1/6*d-1/6*e+1/6*f-1/6*g)/(x+
1)+(-1/36*d-1/36*e-1/36*f-1/36*g)*ln(x-1)+(1/144*d+1/72*e+1/36*f+1/18*g)*l
n(x-2)
```

3.94.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 229 vs. $2(105) = 210$.

Time = 0.72 (sec) , antiderivative size = 229, normalized size of antiderivative = 1.96

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx =$$
$$-\frac{12(3d-4e+6f-10g)x-((31d-50e+76f-104g)x^2+3(31d-50e+76f-104g)x+62d}{}$$

input
```
integrate((x^2-3*x+2)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="fr
icas")
```

output
```
-1/144*(12*(3*d - 4*e + 6*f - 10*g)*x - ((31*d - 50*e + 76*f - 104*g)*x^2
+ 3*(31*d - 50*e + 76*f - 104*g)*x + 62*d - 100*e + 152*f - 208*g)*log(x +
 2) + 4*((7*d - 13*e + 19*f - 25*g)*x^2 + 3*(7*d - 13*e + 19*f - 25*g)*x +
 14*d - 26*e + 38*f - 50*g)*log(x + 1) + 4*((d + e + f + g)*x^2 + 3*(d + e
 + f + g)*x + 2*d + 2*e + 2*f + 2*g)*log(x - 1) - ((d + 2*e + 4*f + 8*g)*x
^2 + 3*(d + 2*e + 4*f + 8*g)*x + 2*d + 4*e + 8*f + 16*g)*log(x - 2) + 60*d
 - 72*e + 96*f - 144*g)/(x^2 + 3*x + 2)
```

3.94.6 Sympy [F(-1)]

Timed out.

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((x**2-3*x+2)*(g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4)**2,x)
```

output
```
Timed out
```

3.94.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 107, normalized size of antiderivative = 0.91

$$\begin{aligned}
&\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx \\
&= \frac{1}{144}(31d-50e+76f-104g)\log(x+2) - \frac{1}{36}(7d-13e+19f-25g)\log(x+1) \\
&\quad - \frac{1}{36}(d+e+f+g)\log(x-1) + \frac{1}{144}(d+2e+4f+8g)\log(x-2) \\
&\quad - \frac{(3d-4e+6f-10g)x+5d-6e+8f-12g}{12(x^2+3x+2)}
\end{aligned}$$

input
```
integrate((x^2-3*x+2)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="ma
xima")
```

output
```
1/144*(31*d - 50*e + 76*f - 104*g)*log(x + 2) - 1/36*(7*d - 13*e + 19*f -
25*g)*log(x + 1) - 1/36*(d + e + f + g)*log(x - 1) + 1/144*(d + 2*e + 4*f
+ 8*g)*log(x - 2) - 1/12*((3*d - 4*e + 6*f - 10*g)*x + 5*d - 6*e + 8*f - 1
2*g)/(x^2 + 3*x + 2)
```

3.94.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 111, normalized size of antiderivative = 0.95

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx$$
$$= \frac{1}{144}(31\,d-50\,e+76\,f-104\,g)\log(|x+2|)-\frac{1}{36}(7\,d-13\,e+19\,f-25\,g)\log(|x+1|) - \frac{1}{36}(d+e+f+g)\log(|x-1|)+\frac{1}{144}(d+2\,e+4\,f+8\,g)\log(|x-2|) - \frac{(3\,d-4\,e+6\,f-10\,g)x+5\,d-6\,e+8\,f-12\,g}{12\,(x+2)(x+1)}$$

input
```
integrate((x^2-3*x+2)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="gi
ac")
```

output
```
1/144*(31*d - 50*e + 76*f - 104*g)*log(abs(x + 2)) - 1/36*(7*d - 13*e + 19
*f - 25*g)*log(abs(x + 1)) - 1/36*(d + e + f + g)*log(abs(x - 1)) + 1/144*
(d + 2*e + 4*f + 8*g)*log(abs(x - 2)) - 1/12*((3*d - 4*e + 6*f - 10*g)*x +
 5*d - 6*e + 8*f - 12*g)/((x + 2)*(x + 1))
```

3.94.9 Mupad [B] (verification not implemented)

Time = 7.98 (sec) , antiderivative size = 115, normalized size of antiderivative = 0.98

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx = \ln(x-2)\left(\frac{d}{144}+\frac{e}{72}+\frac{f}{36}+\frac{g}{18}\right) - \ln(x+1)\left(\frac{7\,d}{36}-\frac{13\,e}{36}+\frac{19\,f}{36}-\frac{25\,g}{36}\right) - \ln(x-1)\left(\frac{d}{36}+\frac{e}{36}+\frac{f}{36}+\frac{g}{36}\right) + \ln(x+2)\left(\frac{31\,d}{144}-\frac{25\,e}{72}+\frac{19\,f}{36}-\frac{13\,g}{18}\right) - \frac{\frac{5\,d}{12}-\frac{e}{2}+\frac{2\,f}{3}-g+x\left(\frac{d}{4}-\frac{e}{3}+\frac{f}{2}-\frac{5\,g}{6}\right)}{x^2+3\,x+2}$$

input
```
int(((x^2 - 3*x + 2)*(d + e*x + f*x^2 + g*x^3))/(x^4 - 5*x^2 + 4)^2,x)
```

output

```
log(x - 2)*(d/144 + e/72 + f/36 + g/18) - log(x + 1)*((7*d)/36 - (13*e)/36
 + (19*f)/36 - (25*g)/36) - log(x - 1)*(d/36 + e/36 + f/36 + g/36) + log(x
 + 2)*((31*d)/144 - (25*e)/72 + (19*f)/36 - (13*g)/18) - ((5*d)/12 - e/2 +
 (2*f)/3 - g + x*(d/4 - e/3 + f/2 - (5*g)/6))/(3*x + x^2 + 2)
```

3.94. $\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx$

3.95 $\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx$

3.95.1 Optimal result

Integrand size = 41, antiderivative size = 131

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx = -\frac{d-e+f-g+h}{6(1+x)}-\frac{d-2e+4f-8g+16h}{12(2+x)}-\frac{1}{36}(d+e+f+g+h)\log(1-x) + \frac{1}{144}(d+2e+4f+8g+16h)\log(2-x)-\frac{1}{36}(7d-13e+19f-25g+31h)\log(1+x) + \frac{1}{144}(31d-50e+76f-104g+112h)\log(2+x)$$

output
```
1/6*(-d+e-f+g-h)/(1+x)+1/12*(-d+2*e-4*f+8*g-16*h)/(2+x)-1/36*(d+e+f+g+h)*l
n(1-x)+1/144*(d+2*e+4*f+8*g+16*h)*ln(2-x)-1/36*(7*d-13*e+19*f-25*g+31*h)*l
n(1+x)+1/144*(31*d-50*e+76*f-104*g+112*h)*ln(2+x)
```

3.95.2 Mathematica [A] (verified)

Time = 0.06 (sec) , antiderivative size = 136, normalized size of antiderivative = 1.04

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx = \frac{1}{144}\left(-\frac{12(d(5+3x)+2(4f-6g+10h+3fx-5gx+9hx-e(3+2x)))}{2+3x+x^2} - 4(d+e+f+g+h)\log(1-x)+(d+2(e+2f+4g+8h))\log(2-x) - 4(7d-13e+19f-25g+31h)\log(1+x)+(31d-50e+76f-104g+112h)\log(2+x)\right)$$

input
```
Integrate[((2 - 3*x + x^2)*(d + e*x + f*x^2 + g*x^3 + h*x^4))/(4 - 5*x^2 +
 x^4)^2,x]
```

output
```
((-12*(d*(5 + 3*x) + 2*(4*f - 6*g + 10*h + 3*f*x - 5*g*x + 9*h*x - e*(3 +
2*x))))/(2 + 3*x + x^2) - 4*(d + e + f + g + h)*Log[1 - x] + (d + 2*(e + 2
*f + 4*g + 8*h))*Log[2 - x] - 4*(7*d - 13*e + 19*f - 25*g + 31*h)*Log[1 +
x] + (31*d - 50*e + 76*f - 104*g + 112*h)*Log[2 + x])/144
```

3.95.3 Rubi [A] (verified)

Time = 0.52 (sec) , antiderivative size = 131, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.073, Rules used = {2019, 7279, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^2-3x+2)\,(d+ex+fx^2+gx^3+hx^4)}{(x^4-5x^2+4)^2}dx$$

$$\Big\downarrow \text{2019}$$

$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(x^2-3x+2)\,(x^2+3x+2)^2}dx$$

$$\Big\downarrow \text{7279}$$

$$\int \left(\frac{-7d+13e-19f+25g-31h}{36(x+1)}+\frac{d+2e+4f+8g+16h}{144(x-2)}+\frac{-d-e-f-g-h}{36(x-1)}+\frac{31d-50e+76f-104g}{144(x+2)}\right.$$

$$\Big\downarrow \text{2009}$$

$$-\frac{d-e+f-g+h}{6(x+1)}-\frac{d-2e+4f-8g+16h}{12(x+2)}-\frac{1}{36}\log(1-x)(d+e+f+g+h)+\frac{1}{144}\log(2-x)(d+2e+4f+8g+16h)-\frac{1}{36}\log(x+1)(7d-13e+19f-25g+31h)+\frac{1}{144}\log(x+2)(31d-50e+76f-104g+112h)$$

input
```
Int[((2 - 3*x + x^2)*(d + e*x + f*x^2 + g*x^3 + h*x^4))/(4 - 5*x^2 + x^4)^
2,x]
```

output
```
-1/6*(d - e + f - g + h)/(1 + x) - (d - 2*e + 4*f - 8*g + 16*h)/(12*(2 + x
)) - ((d + e + f + g + h)*Log[1 - x])/36 + ((d + 2*e + 4*f + 8*g + 16*h)*L
og[2 - x])/144 - ((7*d - 13*e + 19*f - 25*g + 31*h)*Log[1 + x])/36 + ((31*
d - 50*e + 76*f - 104*g + 112*h)*Log[2 + x])/144
```

3.95.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 7279
```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

3.95.4 Maple [A] (verified)

Time = 0.12 (sec) , antiderivative size = 132, normalized size of antiderivative = 1.01

method	result
default	$-\frac{\frac{d}{12}-\frac{e}{6}+\frac{f}{3}-\frac{2g}{3}+\frac{4h}{3}}{x+2}+\left(\frac{31d}{144}-\frac{25e}{72}+\frac{19f}{36}-\frac{13g}{18}+\frac{7h}{9}\right)\ln\left(x+2\right)+\left(-\frac{7d}{36}+\frac{13e}{36}-\frac{19f}{36}+\frac{25g}{36}-\frac{31h}{36}\right)$
norman	$\frac{\left(-\frac{d}{4}+\frac{e}{3}-\frac{f}{2}+\frac{5g}{6}-\frac{3h}{2}\right)x^3+\left(\frac{3d}{4}-\frac{5e}{6}+f-\frac{4g}{3}+2h\right)x+\left(\frac{d}{3}-\frac{e}{2}+\frac{5f}{6}-\frac{3g}{2}+\frac{17h}{6}\right)x^2-\frac{5d}{6}+e+2g-\frac{10h}{3}-\frac{4f}{3}}{x^4-5x^2+4}+\left(-\frac{7d}{36}+\frac{13e}{36}-\right.$
risch	$\frac{\left(-\frac{d}{4}+\frac{e}{3}-\frac{f}{2}+\frac{5g}{6}-\frac{3h}{2}\right)x-\frac{5d}{12}+\frac{e}{2}-\frac{2f}{3}+g-\frac{5h}{3}}{x^2+3x+2}+\frac{31\ln(x+2)d}{144}-\frac{25\ln(x+2)e}{72}+\frac{19\ln(x+2)f}{36}-\frac{13\ln(x+2)g}{18}+\frac{7\ln(x+2}{9}$
parallelrisch	$-96f+144g+120gx-60d-240h+72e+48\ln(x-2)xh-12\ln(x-1)xh-372\ln(x+1)xh+336\ln(x+2)xh-36dx+2\ln(x-2)d+4\ln$

input
```
int((x^2-3*x+2)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x,method=_RETURN
VERBOSE)
```

output
```
-(1/12*d-1/6*e+1/3*f-2/3*g+4/3*h)/(x+2)+(31/144*d-25/72*e+19/36*f-13/18*g+
7/9*h)*ln(x+2)+(-7/36*d+13/36*e-19/36*f+25/36*g-31/36*h)*ln(x+1)-(1/6*d-1/
6*e+1/6*f-1/6*g+1/6*h)/(x+1)+(-1/36*d-1/36*e-1/36*f-1/36*g-1/36*h)*ln(x-1)
+(1/144*d+1/72*e+1/36*f+1/18*g+1/9*h)*ln(x-2)
```

3.95.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 267 vs. $2(119) = 238$.

Time = 3.52 (sec) , antiderivative size = 267, normalized size of antiderivative = 2.04

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx = -\frac{12\,(3\,d-4\,e+6\,f-10\,g+18\,h)x-((31\,d-50\,e+76\,f-104\,g+112\,h)x^2+3\,(31\,d-50\,e+76\,f-}{}$$

input
```
integrate((x^2-3*x+2)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorit
hm="fricas")
```

output
```
-1/144*(12*(3*d - 4*e + 6*f - 10*g + 18*h)*x - ((31*d - 50*e + 76*f - 104*
g + 112*h)*x^2 + 3*(31*d - 50*e + 76*f - 104*g + 112*h)*x + 62*d - 100*e +
 152*f - 208*g + 224*h)*log(x + 2) + 4*((7*d - 13*e + 19*f - 25*g + 31*h)*
x^2 + 3*(7*d - 13*e + 19*f - 25*g + 31*h)*x + 14*d - 26*e + 38*f - 50*g +
62*h)*log(x + 1) + 4*((d + e + f + g + h)*x^2 + 3*(d + e + f + g + h)*x +
2*d + 2*e + 2*f + 2*g + 2*h)*log(x - 1) - ((d + 2*e + 4*f + 8*g + 16*h)*x^
2 + 3*(d + 2*e + 4*f + 8*g + 16*h)*x + 2*d + 4*e + 8*f + 16*g + 32*h)*log(
x - 2) + 60*d - 72*e + 96*f - 144*g + 240*h)/(x^2 + 3*x + 2)
```

3.95.6 Sympy [F(-1)]

Timed out.

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((x**2-3*x+2)*(h*x**4+g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4)**2,x)
```

output
```
Timed out
```

3.95.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 123, normalized size of antiderivative = 0.94

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx$$
$$= \frac{1}{144}(31\,d-50\,e+76\,f-104\,g+112\,h)\log(x+2) - \frac{1}{36}(7\,d-13\,e+19\,f-25\,g+31\,h)\log(x+1) - \frac{1}{36}(d+e+f+g+h)\log(x-1) + \frac{1}{144}(d+2\,e+4\,f+8\,g+16\,h)\log(x-2) - \frac{(3\,d-4\,e+6\,f-10\,g+18\,h)x+5\,d-6\,e+8\,f-12\,g+20\,h}{12\,(x^2+3\,x+2)}$$

input
```
integrate((x^2-3*x+2)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorit
hm="maxima")
```

output
```
1/144*(31*d - 50*e + 76*f - 104*g + 112*h)*log(x + 2) - 1/36*(7*d - 13*e +
 19*f - 25*g + 31*h)*log(x + 1) - 1/36*(d + e + f + g + h)*log(x - 1) + 1/
144*(d + 2*e + 4*f + 8*g + 16*h)*log(x - 2) - 1/12*((3*d - 4*e + 6*f - 10*
g + 18*h)*x + 5*d - 6*e + 8*f - 12*g + 20*h)/(x^2 + 3*x + 2)
```

3.95.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 127, normalized size of antiderivative = 0.97

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx$$
$$= \frac{1}{144}(31\,d-50\,e+76\,f-104\,g+112\,h)\log(|x+2|) - \frac{1}{36}(7\,d-13\,e+19\,f-25\,g+31\,h)\log(|x+1|) - \frac{1}{36}(d+e+f+g+h)\log(|x-1|) + \frac{1}{144}(d+2\,e+4\,f+8\,g+16\,h)\log(|x-2|) - \frac{(3\,d-4\,e+6\,f-10\,g+18\,h)x+5\,d-6\,e+8\,f-12\,g+20\,h}{12\,(x+2)(x+1)}$$

input
```
integrate((x^2-3*x+2)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorit
hm="giac")
```

output
```
1/144*(31*d - 50*e + 76*f - 104*g + 112*h)*log(abs(x + 2)) - 1/36*(7*d - 1
3*e + 19*f - 25*g + 31*h)*log(abs(x + 1)) - 1/36*(d + e + f + g + h)*log(a
bs(x - 1)) + 1/144*(d + 2*e + 4*f + 8*g + 16*h)*log(abs(x - 2)) - 1/12*((3
*d - 4*e + 6*f - 10*g + 18*h)*x + 5*d - 6*e + 8*f - 12*g + 20*h)/((x + 2)*
(x + 1))
```

3.95.9 Mupad [B] (verification not implemented)

Time = 8.24 (sec) , antiderivative size = 133, normalized size of antiderivative = 1.02

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx$$
$$= \ln(x-2)\left(\frac{d}{144}+\frac{e}{72}+\frac{f}{36}+\frac{g}{18}+\frac{h}{9}\right)-\ln(x-1)\left(\frac{d}{36}+\frac{e}{36}+\frac{f}{36}+\frac{g}{36}+\frac{h}{36}\right)$$
$$-\ln(x+1)\left(\frac{7\,d}{36}-\frac{13\,e}{36}+\frac{19\,f}{36}-\frac{25\,g}{36}+\frac{31\,h}{36}\right)$$
$$-\frac{\frac{5\,d}{12}-\frac{e}{2}+\frac{2\,f}{3}-g+\frac{5\,h}{3}+x\left(\frac{d}{4}-\frac{e}{3}+\frac{f}{2}-\frac{5\,g}{6}+\frac{3\,h}{2}\right)}{x^2+3\,x+2}$$
$$+\ln(x+2)\left(\frac{31\,d}{144}-\frac{25\,e}{72}+\frac{19\,f}{36}-\frac{13\,g}{18}+\frac{7\,h}{9}\right)$$

input
```
int(((x^2 - 3*x + 2)*(d + e*x + f*x^2 + g*x^3 + h*x^4))/(x^4 - 5*x^2 + 4)^
2,x)
```

output
```
log(x - 2)*(d/144 + e/72 + f/36 + g/18 + h/9) - log(x - 1)*(d/36 + e/36 +
f/36 + g/36 + h/36) - log(x + 1)*((7*d)/36 - (13*e)/36 + (19*f)/36 - (25*g
)/36 + (31*h)/36) - ((5*d)/12 - e/2 + (2*f)/3 - g + (5*h)/3 + x*(d/4 - e/3
 + f/2 - (5*g)/6 + (3*h)/2))/(3*x + x^2 + 2) + log(x + 2)*((31*d)/144 - (2
5*e)/72 + (19*f)/36 - (13*g)/18 + (7*h)/9)
```

3.96 $\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4+ix^5)}{(4-5x^2+x^4)^2}\,dx$

3.96.1 Optimal result

Integrand size = 46, antiderivative size = 147

$$\begin{aligned}
&\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4+ix^5)}{(4-5x^2+x^4)^2}\,dx \\
&= -\frac{d-e+f-g+h-i}{6(1+x)} - \frac{d-2e+4f-8g+16h-32i}{12(2+x)} \\
&\quad - \frac{1}{36}(d+e+f+g+h+i)\log(1-x) + \frac{1}{144}(d+2e+4f+8g+16h+32i)\log(2-x) \\
&\quad - \frac{1}{36}(7d-13e+19f-25g+31h-37i)\log(1+x) \\
&\quad + \frac{1}{144}(31d-50e+76f-104g+112h-32i)\log(2+x)
\end{aligned}$$

output
```
1/6*(-d+e-f+g-h+i)/(1+x)+1/12*(-d+2*e-4*f+8*g-16*h+32*i)/(2+x)-1/36*(d+e+f
+g+h+i)*ln(1-x)+1/144*(d+2*e+4*f+8*g+16*h+32*i)*ln(2-x)-1/36*(7*d-13*e+19*
f-25*g+31*h-37*i)*ln(1+x)+1/144*(31*d-50*e+76*f-104*g+112*h-32*i)*ln(2+x)
```

3.96.2 Mathematica [A] (verified)

Time = 0.06 (sec) , antiderivative size = 153, normalized size of antiderivative = 1.04

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4+ix^5)}{(4-5x^2+x^4)^2}\,dx$$
$$=\frac{1}{144}\left(\frac{12(-d(5+3x)+2(-4f+6g-10h+18i-3fx+5gx-9hx+17ix+e(3+2x)))}{2+3x+x^2}-4(d+e+f+g+h+i)\log(1-x)+(d+2e+4(f+2g+4h+8i))\log(2-x)+4(-7d+13e-19f+25g-31h+37i)\log(1+x)+(31d-50e+76f-104g+112h-32i)\log(2+x)\right)$$

input
```
Integrate[((2 - 3*x + x^2)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5))/(4 -
 5*x^2 + x^4)^2,x]
```

output
```
((12*(-(d*(5 + 3*x)) + 2*(-4*f + 6*g - 10*h + 18*i - 3*f*x + 5*g*x - 9*h*x
 + 17*i*x + e*(3 + 2*x))))/(2 + 3*x + x^2) - 4*(d + e + f + g + h + i)*Log
[1 - x] + (d + 2*e + 4*(f + 2*g + 4*h + 8*i))*Log[2 - x] + 4*(-7*d + 13*e
- 19*f + 25*g - 31*h + 37*i)*Log[1 + x] + (31*d - 50*e + 76*f - 104*g + 11
2*h - 32*i)*Log[2 + x])/144
```

3.96.3 Rubi [A] (verified)

Time = 0.55 (sec) , antiderivative size = 147, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.065, Rules used = {2019, 7279, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x^2-3x+2)(d+ex+fx^2+gx^3+hx^4+ix^5)}{(x^4-5x^2+4)^2}dx$$
$$\Big\downarrow \text{2019}$$
$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(x^2-3x+2)(x^2+3x+2)^2}dx$$
$$\Big\downarrow \text{7279}$$

$$\int \left(\frac{d-2e+4f-8g+16h-32i}{12(x+2)^2} + \frac{d+2e+4f+8g+16h+32i}{144(x-2)} + \frac{-d-e-f-g-h-i}{36(x-1)} + \frac{-7d+13e-19}{36}\right.$$

$$\Big\downarrow \; 2009$$

$$-\frac{d-2e+4f-8g+16h-32i}{12(x+2)} - \frac{d-e+f-g+h-i}{6(x+1)} - \frac{1}{36}\log(1-x)(d+e+f+g+h+i) + \frac{1}{144}\log(2-x)(d+2e+4f+8g+16h+32i) - \frac{1}{36}\log(x+1)(7d-13e+19f-25g+31h-37i) + \frac{1}{144}\log(x+2)(31d-50e+76f-104g+112h-32i)$$

input
```
Int[((2 - 3*x + x^2)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5))/(4 - 5*x^2
 + x^4)^2,x]
```

output
```
-1/6*(d - e + f - g + h - i)/(1 + x) - (d - 2*e + 4*f - 8*g + 16*h - 32*i)
/(12*(2 + x)) - ((d + e + f + g + h + i)*Log[1 - x])/36 + ((d + 2*e + 4*f
+ 8*g + 16*h + 32*i)*Log[2 - x])/144 - ((7*d - 13*e + 19*f - 25*g + 31*h -
 37*i)*Log[1 + x])/36 + ((31*d - 50*e + 76*f - 104*g + 112*h - 32*i)*Log[2
 + x])/144
```

3.96.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 7279
```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

3.96.4 Maple [A] (verified)

Time = 0.12 (sec) , antiderivative size = 150, normalized size of antiderivative = 1.02

method	result
default	$-\frac{\frac{d}{12}-\frac{e}{6}+\frac{f}{3}-\frac{2g}{3}+\frac{4h}{3}-\frac{8i}{3}}{x+2}+\left(\frac{31d}{144}-\frac{25e}{72}+\frac{19f}{36}-\frac{13g}{18}-\frac{2i}{9}+\frac{7h}{9}\right)\ln\left(x+2\right)+\left(-\frac{7d}{36}+\frac{13e}{36}-\frac{19f}{36}+\frac{25g}{36}\right.$
norman	$\frac{\left(-\frac{d}{4}+\frac{e}{3}-\frac{f}{2}+\frac{5g}{6}-\frac{3h}{2}+\frac{17i}{6}\right)x^3+\left(\frac{3d}{4}-\frac{5e}{6}+f-\frac{4g}{3}+2h-\frac{10i}{3}\right)x+\left(\frac{d}{3}-\frac{e}{2}+\frac{5f}{6}-\frac{3g}{2}+\frac{17h}{6}-\frac{11i}{2}\right)x^2+6i-\frac{5d}{6}+e+2g-\frac{10h}{3}-\frac{4f}{3}}{x^4-5x^2+4}+($
risch	$\frac{25\ln(-x-1)g}{36}-\frac{\ln(x-1)d}{36}-\frac{\ln(x-1)e}{36}+\frac{19\ln(x+2)f}{36}-\frac{31\ln(-x-1)h}{36}+\frac{\ln(2-x)f}{36}+\frac{\ln(2-x)d}{144}+\frac{\ln(2-x)e}{72}+$
parallelrisch	$\frac{432i-96f+144g-96\ln(x+2)xi+96\ln(x-2)xi+120gx-60d-240h+72e+48\ln(x-2)xh-12\ln(x-1)xh-372\ln(x+1)xh+336\ln}{}$

input
```
int((x^2-3*x+2)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x,method=_
RETURNVERBOSE)
```

output
```
-(1/12*d-1/6*e+1/3*f-2/3*g+4/3*h-8/3*i)/(x+2)+(31/144*d-25/72*e+19/36*f-13
/18*g-2/9*i+7/9*h)*ln(x+2)+(-7/36*d+13/36*e-19/36*f+25/36*g-31/36*h+37/36*
i)*ln(x+1)-(1/6*d-1/6*e+1/6*f-1/6*g+1/6*h-1/6*i)/(x+1)+(-1/36*d-1/36*e-1/3
6*f-1/36*g-1/36*h-1/36*i)*ln(x-1)+(1/144*d+1/72*e+1/36*f+1/18*g+1/9*h+2/9*
i)*ln(x-2)
```

3.96.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 305 vs. 2(135) = 270.

Time = 21.46 (sec) , antiderivative size = 305, normalized size of antiderivative = 2.07

$$\int \frac{(2-3x+x^2)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)}{(4-5x^2+x^4)^2}\,dx =$$
$$-\frac{12\,(3\,d-4\,e+6\,f-10\,g+18\,h-34\,i)x-((31\,d-50\,e+76\,f-104\,g+112\,h-32\,i)x^2+3\,(31\,d-5}{}$$

input
```
integrate((x^2-3*x+2)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, a
lgorithm="fricas")
```

output
```
-1/144*(12*(3*d - 4*e + 6*f - 10*g + 18*h - 34*i)*x - ((31*d - 50*e + 76*f
 - 104*g + 112*h - 32*i)*x^2 + 3*(31*d - 50*e + 76*f - 104*g + 112*h - 32*
i)*x + 62*d - 100*e + 152*f - 208*g + 224*h - 64*i)*log(x + 2) + 4*((7*d -
 13*e + 19*f - 25*g + 31*h - 37*i)*x^2 + 3*(7*d - 13*e + 19*f - 25*g + 31*
h - 37*i)*x + 14*d - 26*e + 38*f - 50*g + 62*h - 74*i)*log(x + 1) + 4*((d
+ e + f + g + h + i)*x^2 + 3*(d + e + f + g + h + i)*x + 2*d + 2*e + 2*f +
 2*g + 2*h + 2*i)*log(x - 1) - ((d + 2*e + 4*f + 8*g + 16*h + 32*i)*x^2 +
3*(d + 2*e + 4*f + 8*g + 16*h + 32*i)*x + 2*d + 4*e + 8*f + 16*g + 32*h +
64*i)*log(x - 2) + 60*d - 72*e + 96*f - 144*g + 240*h - 432*i)/(x^2 + 3*x
+ 2)
```

3.96.6 Sympy [F(-1)]

Timed out.

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4+ix^5)}{(4-5x^2+x^4)^2}\,dx = \text{Timed out}$$

input
```
integrate((x**2-3*x+2)*(i*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4)
**2,x)
```

output
```
Timed out
```

3.96.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 139, normalized size of antiderivative = 0.95

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4+ix^5)}{(4-5x^2+x^4)^2}\,dx$$
$$= \frac{1}{144}(31\,d-50\,e+76\,f-104\,g+112\,h-32\,i)\log(x+2) - \frac{1}{36}(7\,d-13\,e+19\,f-25\,g+31\,h-37\,i)\log(x+1) - \frac{1}{36}(d+e+f+g+h+i)\log(x-1) + \frac{1}{144}(d+2\,e+4\,f+8\,g+16\,h+32\,i)\log(x-2) - \frac{(3\,d-4\,e+6\,f-10\,g+18\,h-34\,i)x+5\,d-6\,e+8\,f-12\,g+20\,h-36\,i}{12\,(x^2+3\,x+2)}$$

input
```
integrate((x^2-3*x+2)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, a
lgorithm="maxima")
```

output
```
1/144*(31*d - 50*e + 76*f - 104*g + 112*h - 32*i)*log(x + 2) - 1/36*(7*d -
 13*e + 19*f - 25*g + 31*h - 37*i)*log(x + 1) - 1/36*(d + e + f + g + h +
i)*log(x - 1) + 1/144*(d + 2*e + 4*f + 8*g + 16*h + 32*i)*log(x - 2) - 1/1
2*((3*d - 4*e + 6*f - 10*g + 18*h - 34*i)*x + 5*d - 6*e + 8*f - 12*g + 20*
h - 36*i)/(x^2 + 3*x + 2)
```

3.96.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 143, normalized size of antiderivative = 0.97

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4+ix^5)}{(4-5x^2+x^4)^2}\,dx$$
$$= \frac{1}{144}(31\,d-50\,e+76\,f-104\,g+112\,h-32\,i)\log(|x+2|) - \frac{1}{36}(7\,d-13\,e+19\,f-25\,g+31\,h-37\,i)\log(|x+1|) - \frac{1}{36}(d+e+f+g+h+i)\log(|x-1|) + \frac{1}{144}(d+2\,e+4\,f+8\,g+16\,h+32\,i)\log(|x-2|) - \frac{(3\,d-4\,e+6\,f-10\,g+18\,h-34\,i)x+5\,d-6\,e+8\,f-12\,g+20\,h-36\,i}{12\,(x+2)(x+1)}$$

input
```
integrate((x^2-3*x+2)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, a
lgorithm="giac")
```

output
```
1/144*(31*d - 50*e + 76*f - 104*g + 112*h - 32*i)*log(abs(x + 2)) - 1/36*(
7*d - 13*e + 19*f - 25*g + 31*h - 37*i)*log(abs(x + 1)) - 1/36*(d + e + f
+ g + h + i)*log(abs(x - 1)) + 1/144*(d + 2*e + 4*f + 8*g + 16*h + 32*i)*l
og(abs(x - 2)) - 1/12*((3*d - 4*e + 6*f - 10*g + 18*h - 34*i)*x + 5*d - 6*
e + 8*f - 12*g + 20*h - 36*i)/((x + 2)*(x + 1))
```

3.96.9 Mupad [B] (verification not implemented)

Time = 8.67 (sec) , antiderivative size = 151, normalized size of antiderivative = 1.03

$$\int \frac{(2-3x+x^2)(d+ex+fx^2+gx^3+hx^4+ix^5)}{(4-5x^2+x^4)^2}\,dx$$
$$=\ln(x-2)\left(\frac{d}{144}+\frac{e}{72}+\frac{f}{36}+\frac{g}{18}+\frac{h}{9}+\frac{2i}{9}\right)-\ln(x-1)\left(\frac{d}{36}+\frac{e}{36}+\frac{f}{36}+\frac{g}{36}+\frac{h}{36}+\frac{i}{36}\right)$$
$$-\ln(x+1)\left(\frac{7d}{36}-\frac{13e}{36}+\frac{19f}{36}-\frac{25g}{36}+\frac{31h}{36}-\frac{37i}{36}\right)$$
$$+\ln(x+2)\left(\frac{31d}{144}-\frac{25e}{72}+\frac{19f}{36}-\frac{13g}{18}+\frac{7h}{9}-\frac{2i}{9}\right)$$
$$-\frac{\frac{5d}{12}-\frac{e}{2}+\frac{2f}{3}-g+\frac{5h}{3}-3i+x\left(\frac{d}{4}-\frac{e}{3}+\frac{f}{2}-\frac{5g}{6}+\frac{3h}{2}-\frac{17i}{6}\right)}{x^2+3x+2}$$

input
```
int(((x^2 - 3*x + 2)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5))/(x^4 - 5*x
^2 + 4)^2,x)
```

output
```
log(x - 2)*(d/144 + e/72 + f/36 + g/18 + h/9 + (2*i)/9) - log(x - 1)*(d/36
 + e/36 + f/36 + g/36 + h/36 + i/36) - log(x + 1)*((7*d)/36 - (13*e)/36 +
(19*f)/36 - (25*g)/36 + (31*h)/36 - (37*i)/36) + log(x + 2)*((31*d)/144 -
(25*e)/72 + (19*f)/36 - (13*g)/18 + (7*h)/9 - (2*i)/9) - ((5*d)/12 - e/2 +
 (2*f)/3 - g + (5*h)/3 - 3*i + x*(d/4 - e/3 + f/2 - (5*g)/6 + (3*h)/2 - (1
7*i)/6))/(3*x + x^2 + 2)
```

3.97 $\int \frac{2+x}{(4-5x^2+x^4)^2} \, dx$

3.97.1 Optimal result

Integrand size = 16, antiderivative size = 68

$$\int \frac{2+x}{(4-5x^2+x^4)^2} \, dx = \frac{1}{12(1-x)} + \frac{1}{36(2-x)} - \frac{1}{36(1+x)} + \frac{1}{18}\log(1-x) - \frac{35}{432}\log(2-x) + \frac{1}{54}\log(1+x) + \frac{1}{144}\log(2+x)$$

output
```
1/12/(1-x)+1/36/(2-x)-1/36/(1+x)+1/18*ln(1-x)-35/432*ln(2-x)+1/54*ln(1+x)+
1/144*ln(2+x)
```

3.97.2 Mathematica [A] (verified)

Time = 0.02 (sec) , antiderivative size = 60, normalized size of antiderivative = 0.88

$$\int \frac{2+x}{(4-5x^2+x^4)^2} \, dx = \frac{1}{432}\left(\frac{12(5+6x-5x^2)}{2-x-2x^2+x^3} + 24\log(1-x) - 35\log(2-x) + 8\log(1+x) + 3\log(2+x)\right)$$

input
```
Integrate[(2 + x)/(4 - 5*x^2 + x^4)^2,x]
```

output
```
((12*(5 + 6*x - 5*x^2))/(2 - x - 2*x^2 + x^3) + 24*Log[1 - x] - 35*Log[2 -
 x] + 8*Log[1 + x] + 3*Log[2 + x])/432
```

3.97.3 Rubi [A] (verified)

Time = 0.25 (sec) , antiderivative size = 68, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.188, Rules used = {2019, 2462, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{x+2}{(x^4-5x^2+4)^2}dx$$

$$\Big\downarrow \text{2019}$$

$$\int \frac{1}{(x+2)(x^3-2x^2-x+2)^2}dx$$

$$\Big\downarrow \text{2462}$$

$$\int \left(\frac{1}{18(x-1)}+\frac{1}{54(x+1)}+\frac{1}{144(x+2)}+\frac{1}{12(x-1)^2}+\frac{1}{36(x+1)^2}-\frac{35}{432(x-2)}+\frac{1}{36(x-2)^2}\right)dx$$

$$\Big\downarrow \text{2009}$$

$$\frac{1}{12(1-x)}+\frac{1}{36(2-x)}-\frac{1}{36(x+1)}+\frac{1}{18}\log(1-x)-\frac{35}{432}\log(2-x)+\frac{1}{54}\log(x+1)+\frac{1}{144}\log(x+2)$$

input
```
Int[(2 + x)/(4 - 5*x^2 + x^4)^2,x]
```

output
```
1/(12*(1 - x)) + 1/(36*(2 - x)) - 1/(36*(1 + x)) + Log[1 - x]/18 - (35*Log
[2 - x])/432 + Log[1 + x]/54 + Log[2 + x]/144
```

3.97.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2462
```
Int[(u_.)*(Px_)^(p_), x_Symbol] :> With[{Qx = Factor[Px]}, Int[ExpandIntegr
and[u*Qx^p, x], x] /;  !SumQ[NonfreeFactors[Qx, x]]] /; PolyQ[Px, x] && GtQ
[Expon[Px, x], 2] &&  !BinomialQ[Px, x] &&  !TrinomialQ[Px, x] && ILtQ[p, 0
] && RationalFunctionQ[u, x]
```

3.97.4 Maple [A] (verified)

Time = 0.07 (sec) , antiderivative size = 47, normalized size of antiderivative = 0.69

method	result
default	$\frac{\ln(x+2)}{144}-\frac{1}{36(x+1)}+\frac{\ln(x+1)}{54}-\frac{1}{12(x-1)}+\frac{\ln(x-1)}{18}-\frac{1}{36(x-2)}-\frac{35\ln(x-2)}{432}$
risch	$\frac{-\frac{5}{36}x^2+\frac{1}{6}x+\frac{5}{36}}{x^3-2x^2-x+2}-\frac{35\ln(x-2)}{432}+\frac{\ln(x-1)}{18}+\frac{\ln(x+1)}{54}+\frac{\ln(x+2)}{144}$
norman	$\frac{-\frac{1}{9}x^2+\frac{17}{36}x-\frac{5}{36}x^3+\frac{5}{18}}{x^4-5x^2+4}-\frac{35\ln(x-2)}{432}+\frac{\ln(x-1)}{18}+\frac{\ln(x+1)}{54}+\frac{\ln(x+2)}{144}$
parallelrisch	$-\frac{35\ln(x-2)x^3-24\ln(x-1)x^3-8\ln(x+1)x^3-3\ln(x+2)x^3-60-70\ln(x-2)x^2+48\ln(x-1)x^2+16\ln(x+1)x^2+6\ln(x+2)x^2-3}{432(x^3-2x^2-}$

input
```
int((x+2)/(x^4-5*x^2+4)^2,x,method=_RETURNVERBOSE)
```

output
```
1/144*ln(x+2)-1/36/(x+1)+1/54*ln(x+1)-1/12/(x-1)+1/18*ln(x-1)-1/36/(x-2)-3
5/432*ln(x-2)
```

3.97.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 103 vs. $2(50) = 100$.

Time = 0.25 (sec) , antiderivative size = 103, normalized size of antiderivative = 1.51

$$\int \frac{2+x}{(4-5x^2+x^4)^2}\,dx = -\frac{60\,x^2-3\,(x^3-2\,x^2-x+2)\log(x+2)-8\,(x^3-2\,x^2-x+2)\log(x+1)-24\,(x^3-2\,x^2-x+2)\log}{432\,(x^3-2\,x^2-x+2)}$$

input
```
integrate((2+x)/(x^4-5*x^2+4)^2,x, algorithm="fricas")
```

output
```
-1/432*(60*x^2 - 3*(x^3 - 2*x^2 - x + 2)*log(x + 2) - 8*(x^3 - 2*x^2 - x +
 2)*log(x + 1) - 24*(x^3 - 2*x^2 - x + 2)*log(x - 1) + 35*(x^3 - 2*x^2 - x
 + 2)*log(x - 2) - 72*x - 60)/(x^3 - 2*x^2 - x + 2)
```

3.97.6 Sympy [A] (verification not implemented)

Time = 0.16 (sec) , antiderivative size = 53, normalized size of antiderivative = 0.78

$$\int \frac{2+x}{(4-5x^2+x^4)^2}\,dx = \frac{-5x^2+6x+5}{36x^3-72x^2-36x+72} - \frac{35\log(x-2)}{432} + \frac{\log(x-1)}{18} + \frac{\log(x+1)}{54} + \frac{\log(x+2)}{144}$$

input
```
integrate((2+x)/(x**4-5*x**2+4)**2,x)
```

output
```
(-5*x**2 + 6*x + 5)/(36*x**3 - 72*x**2 - 36*x + 72) - 35*log(x - 2)/432 +
log(x - 1)/18 + log(x + 1)/54 + log(x + 2)/144
```

3.97.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 52, normalized size of antiderivative = 0.76

$$\int \frac{2+x}{(4-5x^2+x^4)^2}\,dx = -\frac{5\,x^2-6\,x-5}{36\,(x^3-2\,x^2-x+2)} + \frac{1}{144}\log(x+2) + \frac{1}{54}\log(x+1) + \frac{1}{18}\log(x-1) - \frac{35}{432}\log(x-2)$$

input
```
integrate((2+x)/(x^4-5*x^2+4)^2,x, algorithm="maxima")
```

output
```
-1/36*(5*x^2 - 6*x - 5)/(x^3 - 2*x^2 - x + 2) + 1/144*log(x + 2) + 1/54*lo
g(x + 1) + 1/18*log(x - 1) - 35/432*log(x - 2)
```

3.97.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 56, normalized size of antiderivative = 0.82

$$\int \frac{2+x}{(4-5x^2+x^4)^2}\,dx = -\frac{5\,x^2-6\,x-5}{36\,(x+1)(x-1)(x-2)} + \frac{1}{144}\log(|x+2|) + \frac{1}{54}\log(|x+1|) + \frac{1}{18}\log(|x-1|) - \frac{35}{432}\log(|x-2|)$$

input
```
integrate((2+x)/(x^4-5*x^2+4)^2,x, algorithm="giac")
```

output
```
-1/36*(5*x^2 - 6*x - 5)/((x + 1)*(x - 1)*(x - 2)) + 1/144*log(abs(x + 2))
+ 1/54*log(abs(x + 1)) + 1/18*log(abs(x - 1)) - 35/432*log(abs(x - 2))
```

3.97.9 Mupad [B] (verification not implemented)

Time = 0.03 (sec) , antiderivative size = 52, normalized size of antiderivative = 0.76

$$\int \frac{2+x}{(4-5x^2+x^4)^2}\,dx = \frac{\ln(x-1)}{18} + \frac{\ln(x+1)}{54} - \frac{35\ln(x-2)}{432} + \frac{\ln(x+2)}{144} - \frac{-\frac{5x^2}{36}+\frac{x}{6}+\frac{5}{36}}{-x^3+2x^2+x-2}$$

input
```
int((x + 2)/(x^4 - 5*x^2 + 4)^2,x)
```

output
```
log(x - 1)/18 + log(x + 1)/54 - (35*log(x - 2))/432 + log(x + 2)/144 - (x/
6 - (5*x^2)/36 + 5/36)/(x + 2*x^2 - x^3 - 2)
```

3.98 $\int \frac{(2+x)(d+ex)}{\left(4-5x^2+x^4\right)^2} \, dx$

3.98.1 Optimal result

Integrand size = 21, antiderivative size = 105

$$\int \frac{(2+x)(d+ex)}{\left(4-5x^2+x^4\right)^2} \, dx = \frac{d+e}{12(1-x)}+\frac{d+2e}{36(2-x)}-\frac{d-e}{36(1+x)}+\frac{1}{36}(2d+5e)\log(1-x)-\frac{1}{432}(35d+58e)\log(2-x)+\frac{1}{108}(2d+e)\log(1+x)+\frac{1}{144}(d-2e)\log(2+x)$$

output
```
1/12*(d+e)/(1-x)+1/36*(d+2*e)/(2-x)+1/36*(-d+e)/(1+x)+1/36*(2*d+5*e)*ln(1-
x)-1/432*(35*d+58*e)*ln(2-x)+1/108*(2*d+e)*ln(1+x)+1/144*(d-2*e)*ln(2+x)
```

3.98.2 Mathematica [A] (verified)

Time = 0.06 (sec) , antiderivative size = 97, normalized size of antiderivative = 0.92

$$\int \frac{(2+x)(d+ex)}{\left(4-5x^2+x^4\right)^2} \, dx = \frac{1}{432}\left(\frac{12(d(5+6x-5x^2)+2e(5-2x^2))}{2-x-2x^2+x^3}+12(2d+5e)\log(1-x)-(35d+58e)\log(2-x)+4(2d+e)\log(1+x)+3(d-2e)\log(2+x)\right)$$

input
```
Integrate[((2 + x)*(d + e*x))/(4 - 5*x^2 + x^4)^2,x]
```

output
```
((12*(d*(5 + 6*x - 5*x^2) + 2*e*(5 - 2*x^2)))/(2 - x - 2*x^2 + x^3) + 12*(
2*d + 5*e)*Log[1 - x] - (35*d + 58*e)*Log[2 - x] + 4*(2*d + e)*Log[1 + x]
+ 3*(d - 2*e)*Log[2 + x])/432
```

3.98.3 Rubi [A] (verified)

Time = 0.34 (sec) , antiderivative size = 105, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.143$, Rules used = {2019, 2462, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x+2)(d+ex)}{(x^4-5x^2+4)^2}\,dx$$

$$\Big\downarrow \text{2019}$$

$$\int \frac{d+ex}{(x+2)\left(x^3-2x^2-x+2\right)^2}dx$$

$$\Big\downarrow \text{2462}$$

$$\int \left(\frac{-35d-58e}{432(x-2)}+\frac{2d+5e}{36(x-1)}+\frac{2d+e}{108(x+1)}+\frac{d-2e}{144(x+2)}+\frac{d+2e}{36(x-2)^2}+\frac{d+e}{12(x-1)^2}+\frac{d-e}{36(x+1)^2}\right)dx$$

$$\Big\downarrow \text{2009}$$

$$-\frac{d-e}{36(x+1)}+\frac{d+e}{12(1-x)}+\frac{d+2e}{36(2-x)}+\frac{1}{36}(2d+5e)\log(1-x)-\frac{1}{432}(35d+58e)\log(2-x)+\frac{1}{108}(2d+e)\log(x+1)+\frac{1}{144}(d-2e)\log(x+2)$$

input
```
Int[((2 + x)*(d + e*x))/(4 - 5*x^2 + x^4)^2,x]
```

output
```
(d + e)/(12*(1 - x)) + (d + 2*e)/(36*(2 - x)) - (d - e)/(36*(1 + x)) + ((2
*d + 5*e)*Log[1 - x])/36 - ((35*d + 58*e)*Log[2 - x])/432 + ((2*d + e)*Log
[1 + x])/108 + ((d - 2*e)*Log[2 + x])/144
```

3.98.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2462
```
Int[(u_.)*(Px_)^(p_), x_Symbol] :> With[{Qx = Factor[Px]}, Int[ExpandIntegr
and[u*Qx^p, x], x] /;  !SumQ[NonfreeFactors[Qx, x]]] /; PolyQ[Px, x] && GtQ
[Expon[Px, x], 2] &&  !BinomialQ[Px, x] &&  !TrinomialQ[Px, x] && ILtQ[p, 0
] && RationalFunctionQ[u, x]
```

3.98.4 Maple [A] (verified)

Time = 0.08 (sec) , antiderivative size = 92, normalized size of antiderivative = 0.88

method	result
default	$\left(\frac{d}{144}-\frac{e}{72}\right)\ln\left(x+2\right)-\frac{\frac{d}{36}-\frac{e}{36}}{x+1}+\left(\frac{d}{54}+\frac{e}{108}\right)\ln\left(x+1\right)-\frac{\frac{d}{12}+\frac{e}{12}}{x-1}+\left(\frac{d}{18}+\frac{5e}{36}\right)\ln\left(x-1\right)+\left(-\frac{35}{43}\right.$
norman	$\frac{\left(-\frac{5d}{36}-\frac{e}{9}\right)x^3+\left(\frac{17d}{36}+\frac{5e}{18}\right)x+\left(-\frac{d}{9}-\frac{2e}{9}\right)x^2+\frac{5d}{18}+\frac{5e}{9}}{x^4-5x^2+4}+\left(-\frac{35d}{432}-\frac{29e}{216}\right)\ln\left(x-2\right)+\left(\frac{d}{18}+\frac{5e}{36}\right)\ln\left(x-1\right)+\left(\right.$
risch	$\frac{\left(-\frac{5d}{36}-\frac{e}{9}\right)x^2+\frac{dx}{6}+\frac{5d}{36}+\frac{5e}{18}}{x^3-2x^2-x+2}+\frac{\ln(x+2)d}{144}-\frac{\ln(x+2)e}{72}+\frac{\ln(x-1)d}{18}+\frac{5\ln(x-1)e}{36}-\frac{35\ln(2-x)d}{432}-\frac{29\ln(2-x)e}{216}+\ln($
parallelrisch	$-\frac{60d\,x^2-60d-120e-72dx-60\ln(x-1)x^3e-8\ln(x+1)x^3d-4\ln(x+1)x^3e-3\ln(x+2)x^3d+6\ln(x+2)x^3e+70\ln(x-2)d+116\ln}{}$

input
```
int((x+2)*(e*x+d)/(x^4-5*x^2+4)^2,x,method=_RETURNVERBOSE)
```

output
```
(1/144*d-1/72*e)*ln(x+2)-(1/36*d-1/36*e)/(x+1)+(1/54*d+1/108*e)*ln(x+1)-(1
/12*d+1/12*e)/(x-1)+(1/18*d+5/36*e)*ln(x-1)+(-35/432*d-29/216*e)*ln(x-2)-(
1/36*d+1/18*e)/(x-2)
```

3.98.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 211 vs. 2(87) = 174.

Time = 0.28 (sec) , antiderivative size = 211, normalized size of antiderivative = 2.01

$$\int \frac{(2+x)(d+ex)}{(4-5x^2+x^4)^2}\,dx = -\frac{12\,(5\,d+4\,e)x^2-72\,dx-3\,((d-2\,e)x^3-2\,(d-2\,e)x^2-(d-2\,e)x+2\,d-4\,e)\log{(x+2)}-4\,((2\,d+$$

input
```
integrate((2+x)*(e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="fricas")
```

output
```
-1/432*(12*(5*d + 4*e)*x^2 - 72*d*x - 3*((d - 2*e)*x^3 - 2*(d - 2*e)*x^2 -
 (d - 2*e)*x + 2*d - 4*e)*log(x + 2) - 4*((2*d + e)*x^3 - 2*(2*d + e)*x^2
- (2*d + e)*x + 4*d + 2*e)*log(x + 1) - 12*((2*d + 5*e)*x^3 - 2*(2*d + 5*e
)*x^2 - (2*d + 5*e)*x + 4*d + 10*e)*log(x - 1) + ((35*d + 58*e)*x^3 - 2*(3
5*d + 58*e)*x^2 - (35*d + 58*e)*x + 70*d + 116*e)*log(x - 2) - 60*d - 120*
e)/(x^3 - 2*x^2 - x + 2)
```

3.98.6 Sympy [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 1034 vs. 2(82) = 164.

Time = 5.46 (sec) , antiderivative size = 1034, normalized size of antiderivative = 9.85

$$\int \frac{(2+x)(d+ex)}{(4-5x^2+x^4)^2}\,dx = \text{Too large to display}$$

input
```
integrate((2+x)*(e*x+d)/(x**4-5*x**2+4)**2,x)
```

output
```
(d - 2*e)*log(x + (8710660*d**5 + 91884504*d**4*e - 7579779*d**4*(d - 2*e)
/4 + 364910432*d**3*e**2 - 18128055*d**3*e*(d - 2*e) - 83772*d**3*(d - 2*e
)**2 + 686697536*d**2*e**3 - 60296868*d**2*e**2*(d - 2*e) - 597816*d**2*e*
(d - 2*e)**2 + 65907*d**2*(d - 2*e)**3/4 + 614357568*d*e**4 - 85949220*d*e
**3*(d - 2*e) - 1500048*d*e**2*(d - 2*e)**2 + 105840*d*e*(d - 2*e)**3 + 20
8470400*e**5 - 45136356*e**4*(d - 2*e) - 1196064*e**3*(d - 2*e)**2 + 12827
7*e**2*(d - 2*e)**3)/(3374210*d**5 + 38645295*d**4*e + 170558380*d**3*e**2
 + 362061760*d**2*e**3 + 370298160*d*e**4 + 146466320*e**5))/144 + (2*d +
e)*log(x + (8710660*d**5 + 91884504*d**4*e - 2526593*d**4*(2*d + e) + 3649
10432*d**3*e**2 - 24170740*d**3*e*(2*d + e) - 148928*d**3*(2*d + e)**2 + 6
86697536*d**2*e**3 - 80395824*d**2*e**2*(2*d + e) - 1062784*d**2*e*(2*d +
e)**2 + 39056*d**2*(2*d + e)**3 + 614357568*d*e**4 - 114598960*d*e**3*(2*d
 + e) - 2666752*d*e**2*(2*d + e)**2 + 250880*d*e*(2*d + e)**3 + 208470400*
e**5 - 60181808*e**4*(2*d + e) - 2126336*e**3*(2*d + e)**2 + 304064*e**2*(
2*d + e)**3)/(3374210*d**5 + 38645295*d**4*e + 170558380*d**3*e**2 + 36206
1760*d**2*e**3 + 370298160*d*e**4 + 146466320*e**5))/108 + (2*d + 5*e)*log
(x + (8710660*d**5 + 91884504*d**4*e - 7579779*d**4*(2*d + 5*e) + 36491043
2*d**3*e**2 - 72512220*d**3*e*(2*d + 5*e) - 1340352*d**3*(2*d + 5*e)**2 +
686697536*d**2*e**3 - 241187472*d**2*e**2*(2*d + 5*e) - 9565056*d**2*e*(2*
d + 5*e)**2 + 1054512*d**2*(2*d + 5*e)**3 + 614357568*d*e**4 - 34379688...
```

3.98.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 88, normalized size of antiderivative = 0.84

$$\int \frac{(2+x)(d+ex)}{(4-5x^2+x^4)^2}\,dx = \frac{1}{144}\,(d-2\,e)\log\left(x+2\right) + \frac{1}{108}\,(2\,d+e)\log\left(x+1\right) + \frac{1}{36}\,(2\,d+5\,e)\log\left(x-1\right) - \frac{1}{432}\,(35\,d+58\,e)\log\left(x-2\right) - \frac{(5\,d+4\,e)x^2-6\,dx-5\,d-10\,e}{36\,(x^3-2\,x^2-x+2)}$$

input
```
integrate((2+x)*(e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="maxima")
```

output
```
1/144*(d - 2*e)*log(x + 2) + 1/108*(2*d + e)*log(x + 1) + 1/36*(2*d + 5*e)
*log(x - 1) - 1/432*(35*d + 58*e)*log(x - 2) - 1/36*((5*d + 4*e)*x^2 - 6*d
*x - 5*d - 10*e)/(x^3 - 2*x^2 - x + 2)
```

3.98.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 92, normalized size of antiderivative = 0.88

$$\int \frac{(2+x)(d+ex)}{(4-5x^2+x^4)^2}\,dx = \frac{1}{144}\,(d-2\,e)\log\left(|x+2|\right)+\frac{1}{108}\,(2\,d+e)\log\left(|x+1|\right)+\frac{1}{36}\,(2\,d+5\,e)\log\left(|x-1|\right)-\frac{1}{432}\,(35\,d+58\,e)\log\left(|x-2|\right)-\frac{(5\,d+4\,e)x^2-6\,dx-5\,d-10\,e}{36\,(x+1)(x-1)(x-2)}$$

input
```
integrate((2+x)*(e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="giac")
```

output
```
1/144*(d - 2*e)*log(abs(x + 2)) + 1/108*(2*d + e)*log(abs(x + 1)) + 1/36*(
2*d + 5*e)*log(abs(x - 1)) - 1/432*(35*d + 58*e)*log(abs(x - 2)) - 1/36*((
5*d + 4*e)*x^2 - 6*d*x - 5*d - 10*e)/((x + 1)*(x - 1)*(x - 2))
```

3.98.9 Mupad [B] (verification not implemented)

Time = 0.06 (sec) , antiderivative size = 90, normalized size of antiderivative = 0.86

$$\int \frac{(2+x)(d+ex)}{(4-5x^2+x^4)^2}\,dx = \ln(x-1)\left(\frac{d}{18}+\frac{5\,e}{36}\right)-\frac{\left(-\frac{5\,d}{36}-\frac{e}{9}\right)\,x^2+\frac{d\,x}{6}+\frac{5\,d}{36}+\frac{5\,e}{18}}{-x^3+2\,x^2+x-2}+\ln(x+1)\left(\frac{d}{54}+\frac{e}{108}\right)+\ln(x+2)\left(\frac{d}{144}-\frac{e}{72}\right)-\ln(x-2)\left(\frac{35\,d}{432}+\frac{29\,e}{216}\right)$$

input
```
int(((x + 2)*(d + e*x))/(x^4 - 5*x^2 + 4)^2,x)
```

output
```
log(x - 1)*(d/18 + (5*e)/36) - ((5*d)/36 + (5*e)/18 - x^2*((5*d)/36 + e/9)
 + (d*x)/6)/(x + 2*x^2 - x^3 - 2) + log(x + 1)*(d/54 + e/108) + log(x + 2)
*(d/144 - e/72) - log(x - 2)*((35*d)/432 + (29*e)/216)
```

3.99 $\int \frac{(2+x)(d+ex+fx^2)}{(4-5x^2+x^4)^2}\,dx$

3.99.1 Optimal result

Integrand size = 26, antiderivative size = 122

$$\int \frac{(2+x)(d+ex+fx^2)}{(4-5x^2+x^4)^2}\,dx = \frac{d+e+f}{12(1-x)} + \frac{d+2e+4f}{36(2-x)} - \frac{d-e+f}{36(1+x)} + \frac{1}{36}(2d+5e+8f)\log(1-x) - \frac{1}{432}(35d+58e+92f)\log(2-x) + \frac{1}{108}(2d+e-4f)\log(1+x) + \frac{1}{144}(d-2e+4f)\log(2+x)$$

output
```
1/12*(d+e+f)/(1-x)+1/36*(d+2*e+4*f)/(2-x)+1/36*(-d+e-f)/(1+x)+1/36*(2*d+5*
e+8*f)*ln(1-x)-1/432*(35*d+58*e+92*f)*ln(2-x)+1/108*(2*d+e-4*f)*ln(1+x)+1/
144*(d-2*e+4*f)*ln(2+x)
```

3.99.2 Mathematica [A] (verified)

Time = 0.04 (sec) , antiderivative size = 121, normalized size of antiderivative = 0.99

$$\int \frac{(2+x)(d+ex+fx^2)}{(4-5x^2+x^4)^2}\,dx = \frac{1}{432}\left(\frac{12(d(5+6x-5x^2)+e(10-4x^2)+2f(4+3x-4x^2))}{2-x-2x^2+x^3} + 12(2d+5e+8f)\log(1-x) - (35d+58e+92f)\log(2-x) + 4(2d+e-4f)\log(1+x) + 3(d-2e+4f)\log(2+x)\right)$$

input
```
Integrate[((2 + x)*(d + e*x + f*x^2))/(4 - 5*x^2 + x^4)^2,x]
```

output
```
((12*(d*(5 + 6*x - 5*x^2) + e*(10 - 4*x^2) + 2*f*(4 + 3*x - 4*x^2)))/(2 -
x - 2*x^2 + x^3) + 12*(2*d + 5*e + 8*f)*Log[1 - x] - (35*d + 58*e + 92*f)*
Log[2 - x] + 4*(2*d + e - 4*f)*Log[1 + x] + 3*(d - 2*e + 4*f)*Log[2 + x])/
432
```

3.99.3 Rubi [A] (verified)

Time = 0.37 (sec) , antiderivative size = 122, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.115$, Rules used = {2019, 2462, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x+2)\left(d+ex+fx^2\right)}{\left(x^4-5x^2+4\right)^2}dx$$

$$\Big\downarrow \text{2019}$$

$$\int \frac{d+ex+fx^2}{(x+2)\left(x^3-2x^2-x+2\right)^2}dx$$

$$\Big\downarrow \text{2462}$$

$$\int \left(\frac{-35d-58e-92f}{432(x-2)}+\frac{2d+5e+8f}{36(x-1)}+\frac{2d+e-4f}{108(x+1)}+\frac{d-2e+4f}{144(x+2)}+\frac{d+2e+4f}{36(x-2)^2}+\frac{d+e+f}{12(x-1)^2}+\frac{d-e+f}{36(x+1)^2}\right.$$

$$\Big\downarrow \text{2009}$$

$$-\frac{d-e+f}{36(x+1)}+\frac{d+e+f}{12(1-x)}+\frac{d+2e+4f}{36(2-x)}+\frac{1}{36}\log(1-x)(2d+5e+8f)-\frac{1}{432}\log(2-x)(35d+58e+92f)+\frac{1}{108}\log(x+1)(2d+e-4f)+\frac{1}{144}\log(x+2)(d-2e+4f)$$

input
```
Int[((2 + x)*(d + e*x + f*x^2))/(4 - 5*x^2 + x^4)^2,x]
```

output
```
(d + e + f)/(12*(1 - x)) + (d + 2*e + 4*f)/(36*(2 - x)) - (d - e + f)/(36*
(1 + x)) + ((2*d + 5*e + 8*f)*Log[1 - x])/36 - ((35*d + 58*e + 92*f)*Log[2
 - x])/432 + ((2*d + e - 4*f)*Log[1 + x])/108 + ((d - 2*e + 4*f)*Log[2 + x
])/144
```

3.99.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2462
```
Int[(u_.)*(Px_)^(p_), x_Symbol] :> With[{Qx = Factor[Px]}, Int[ExpandIntegr
and[u*Qx^p, x], x] /;  !SumQ[NonfreeFactors[Qx, x]]] /; PolyQ[Px, x] && GtQ
[Expon[Px, x], 2] &&  !BinomialQ[Px, x] &&  !TrinomialQ[Px, x] && ILtQ[p, 0
] && RationalFunctionQ[u, x]
```

3.99.4 Maple [A] (verified)

Time = 0.08 (sec) , antiderivative size = 113, normalized size of antiderivative = 0.93

method	result
default	$\left(\frac{d}{144}-\frac{e}{72}+\frac{f}{36}\right)\ln\left(x+2\right)-\frac{\frac{d}{36}-\frac{e}{36}+\frac{f}{36}}{x+1}+\left(\frac{d}{54}+\frac{e}{108}-\frac{f}{27}\right)\ln\left(x+1\right)-\frac{\frac{d}{12}+\frac{e}{12}+\frac{f}{12}}{x-1}+\left(\frac{d}{18}+\frac{5e}{36}+\right.$
norman	$\frac{\left(-\frac{5d}{36}-\frac{e}{9}-\frac{2f}{9}\right)x^{3}+\left(\frac{17d}{36}+\frac{5e}{18}+\frac{5f}{9}\right)x+\left(-\frac{d}{9}-\frac{2e}{9}-\frac{5f}{18}\right)x^{2}+\frac{5d}{18}+\frac{5e}{9}+\frac{4f}{9}}{x^{4}-5x^{2}+4}+\left(-\frac{35d}{432}-\frac{29e}{216}-\frac{23f}{108}\right)\ln\left(x-2\right)+\left(\frac{d}{18}\right.$
risch	$\frac{\left(-\frac{5d}{36}-\frac{e}{9}-\frac{2f}{9}\right)x^{2}+\left(\frac{d}{6}+\frac{f}{6}\right)x+\frac{5d}{36}+\frac{5e}{18}+\frac{2f}{9}}{x^{3}-2x^{2}-x+2}+\frac{\ln(x+2)d}{144}-\frac{\ln(x+2)e}{72}+\frac{\ln(x+2)f}{36}-\frac{35\ln(2-x)d}{432}-\frac{29\ln(2-x)e}{216}-$
parallelrisch	$-\frac{-96f+60d\,x^{2}-60d-120e+96f\,x^{2}-72dx-60\ln(x-1)x^{3}e-8\ln(x+1)x^{3}d-4\ln(x+1)x^{3}e-3\ln(x+2)x^{3}d+6\ln(x+2)x^{3}e+70}{}$

input
```
int((x+2)*(f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x,method=_RETURNVERBOSE)
```

output
```
(1/144*d-1/72*e+1/36*f)*ln(x+2)-(1/36*d-1/36*e+1/36*f)/(x+1)+(1/54*d+1/108
*e-1/27*f)*ln(x+1)-(1/12*d+1/12*e+1/12*f)/(x-1)+(1/18*d+5/36*e+2/9*f)*ln(x
-1)+(-35/432*d-29/216*e-23/108*f)*ln(x-2)-(1/36*d+1/18*e+1/9*f)/(x-2)
```

3.99.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 267 vs. $2(104) = 208$.

Time = 0.36 (sec) , antiderivative size = 267, normalized size of antiderivative = 2.19

$$\int \frac{(2+x)\left(d+ex+fx^2\right)}{\left(4-5x^2+x^4\right)^2}\,dx = -\frac{12\,(5\,d+4\,e+8\,f)x^2-72\,(d+f)x-3\,((d-2\,e+4\,f)x^3-2\,(d-2\,e+4\,f)x^2-(d-2\,e+4\,f)x+2}{}$$

input
```
integrate((2+x)*(f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="fricas")
```

output
```
-1/432*(12*(5*d + 4*e + 8*f)*x^2 - 72*(d + f)*x - 3*((d - 2*e + 4*f)*x^3 -
 2*(d - 2*e + 4*f)*x^2 - (d - 2*e + 4*f)*x + 2*d - 4*e + 8*f)*log(x + 2) -
 4*((2*d + e - 4*f)*x^3 - 2*(2*d + e - 4*f)*x^2 - (2*d + e - 4*f)*x + 4*d
+ 2*e - 8*f)*log(x + 1) - 12*((2*d + 5*e + 8*f)*x^3 - 2*(2*d + 5*e + 8*f)*
x^2 - (2*d + 5*e + 8*f)*x + 4*d + 10*e + 16*f)*log(x - 1) + ((35*d + 58*e
+ 92*f)*x^3 - 2*(35*d + 58*e + 92*f)*x^2 - (35*d + 58*e + 92*f)*x + 70*d +
 116*e + 184*f)*log(x - 2) - 60*d - 120*e - 96*f)/(x^3 - 2*x^2 - x + 2)
```

3.99.6 Sympy [F(-1)]

Timed out.

$$\int \frac{(2+x)\left(d+ex+fx^2\right)}{\left(4-5x^2+x^4\right)^2}\,dx = \text{Timed out}$$

input
```
integrate((2+x)*(f*x**2+e*x+d)/(x**4-5*x**2+4)**2,x)
```

output
```
Timed out
```

3.99.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 108, normalized size of antiderivative = 0.89

$$\int \frac{(2+x)(d+ex+fx^2)}{(4-5x^2+x^4)^2}\,dx = \frac{1}{144}(d-2e+4f)\log(x+2)+\frac{1}{108}(2d+e-4f)\log(x+1) + \frac{1}{36}(2d+5e+8f)\log(x-1) - \frac{1}{432}(35d+58e+92f)\log(x-2) - \frac{(5d+4e+8f)x^2-6(d+f)x-5d-10e-8f}{36(x^3-2x^2-x+2)}$$

input
```
integrate((2+x)*(f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="maxima")
```

output
```
1/144*(d - 2*e + 4*f)*log(x + 2) + 1/108*(2*d + e - 4*f)*log(x + 1) + 1/36
*(2*d + 5*e + 8*f)*log(x - 1) - 1/432*(35*d + 58*e + 92*f)*log(x - 2) - 1/
36*((5*d + 4*e + 8*f)*x^2 - 6*(d + f)*x - 5*d - 10*e - 8*f)/(x^3 - 2*x^2 -
 x + 2)
```

3.99.8 Giac [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 112, normalized size of antiderivative = 0.92

$$\int \frac{(2+x)(d+ex+fx^2)}{(4-5x^2+x^4)^2}\,dx = \frac{1}{144}(d-2e+4f)\log(|x+2|) + \frac{1}{108}(2d+e-4f)\log(|x+1|) + \frac{1}{36}(2d+5e+8f)\log(|x-1|) - \frac{1}{432}(35d+58e+92f)\log(|x-2|) - \frac{(5d+4e+8f)x^2-6(d+f)x-5d-10e-8f}{36(x+1)(x-1)(x-2)}$$

input
```
integrate((2+x)*(f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="giac")
```

output
```
1/144*(d - 2*e + 4*f)*log(abs(x + 2)) + 1/108*(2*d + e - 4*f)*log(abs(x +
1)) + 1/36*(2*d + 5*e + 8*f)*log(abs(x - 1)) - 1/432*(35*d + 58*e + 92*f)*
log(abs(x - 2)) - 1/36*((5*d + 4*e + 8*f)*x^2 - 6*(d + f)*x - 5*d - 10*e -
 8*f)/((x + 1)*(x - 1)*(x - 2))
```

3.99.9 Mupad [B] (verification not implemented)

Time = 7.90 (sec) , antiderivative size = 113, normalized size of antiderivative = 0.93

$$\int \frac{(2+x)(d+ex+fx^2)}{(4-5x^2+x^4)^2}\,dx = \ln(x-1)\left(\frac{d}{18}+\frac{5\,e}{36}+\frac{2\,f}{9}\right)+\ln(x+1)\left(\frac{d}{54}+\frac{e}{108}-\frac{f}{27}\right) + \ln(x+2)\left(\frac{d}{144}-\frac{e}{72}+\frac{f}{36}\right) - \ln(x-2)\left(\frac{35\,d}{432}+\frac{29\,e}{216}+\frac{23\,f}{108}\right) - \frac{\left(-\frac{5\,d}{36}-\frac{e}{9}-\frac{2\,f}{9}\right)x^2+\left(\frac{d}{6}+\frac{f}{6}\right)x+\frac{5\,d}{36}+\frac{5\,e}{18}+\frac{2\,f}{9}}{-x^3+2\,x^2+x-2}$$

input
```
int(((x + 2)*(d + e*x + f*x^2))/(x^4 - 5*x^2 + 4)^2,x)
```

output
```
log(x - 1)*(d/18 + (5*e)/36 + (2*f)/9) + log(x + 1)*(d/54 + e/108 - f/27)
+ log(x + 2)*(d/144 - e/72 + f/36) - log(x - 2)*((35*d)/432 + (29*e)/216 +
 (23*f)/108) - ((5*d)/36 + (5*e)/18 + (2*f)/9 + x*(d/6 + f/6) - x^2*((5*d)
/36 + e/9 + (2*f)/9))/(x + 2*x^2 - x^3 - 2)
```

3.100 $\int \frac{(2+x)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx$

3.100.1 Optimal result

Integrand size = 31, antiderivative size = 141

$$\int \frac{(2+x)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx = \frac{d+e+f+g}{12(1-x)}+\frac{d+2e+4f+8g}{36(2-x)}-\frac{d-e+f-g}{36(1+x)}+\frac{1}{36}(2d+5e+8f+11g)\log(1-x)-\frac{1}{432}(35d+58e+92f+136g)\log(2-x)+\frac{1}{108}(2d+e-4f+7g)\log(1+x)+\frac{1}{144}(d-2e+4f-8g)\log(2+x)$$

output
```
1/12*(d+e+f+g)/(1-x)+1/36*(d+2*e+4*f+8*g)/(2-x)+1/36*(-d+e-f+g)/(1+x)+1/36
*(2*d+5*e+8*f+11*g)*ln(1-x)-1/432*(35*d+58*e+92*f+136*g)*ln(2-x)+1/108*(2*
d+e-4*f+7*g)*ln(1+x)+1/144*(d-2*e+4*f-8*g)*ln(2+x)
```

3.100.2 Mathematica [A] (verified)

Time = 0.05 (sec) , antiderivative size = 144, normalized size of antiderivative = 1.02

$$\int \frac{(2+x)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx$$
$$= \frac{1}{432}\left(\frac{12(d(5+6x-5x^2)+2(g(8-5x^2)+f(4+3x-4x^2)+e(5-2x^2)))}{2-x-2x^2+x^3} + 12(2d+5e+8f+11g)\log(1-x) - (35d+58e+92f+136g)\log(2-x) + 4(2d+e-4f+7g)\log(1+x) + 3(d-2e+4f-8g)\log(2+x)\right)$$

input
```
Integrate[((2 + x)*(d + e*x + f*x^2 + g*x^3))/(4 - 5*x^2 + x^4)^2,x]
```

output
```
((12*(d*(5 + 6*x - 5*x^2) + 2*(g*(8 - 5*x^2) + f*(4 + 3*x - 4*x^2) + e*(5
- 2*x^2))))/(2 - x - 2*x^2 + x^3) + 12*(2*d + 5*e + 8*f + 11*g)*Log[1 - x]
 - (35*d + 58*e + 92*f + 136*g)*Log[2 - x] + 4*(2*d + e - 4*f + 7*g)*Log[1
 + x] + 3*(d - 2*e + 4*f - 8*g)*Log[2 + x])/432
```

3.100.3 Rubi [A] (verified)

Time = 0.40 (sec) , antiderivative size = 141, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}} = 0.097$, Rules used = {2019, 2462, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x+2)(d+ex+fx^2+gx^3)}{(x^4-5x^2+4)^2}\,dx$$
$$\Big\downarrow \text{2019}$$
$$\int \frac{d+ex+fx^2+gx^3}{(x+2)(x^3-2x^2-x+2)^2}dx$$
$$\Big\downarrow \text{2462}$$
$$\int \left(\frac{-35d-58e-92f-136g}{432(x-2)} + \frac{2d+5e+8f+11g}{36(x-1)} + \frac{2d+e-4f+7g}{108(x+1)} + \frac{d-2e+4f-8g}{144(x+2)} + \frac{d+2e+4f+8}{36(x-2)^2}\right.$$
$$\Big\downarrow \text{2009}$$

$$-\frac{d-e+f-g}{36(x+1)}+\frac{d+e+f+g}{12(1-x)}+\frac{d+2e+4f+8g}{36(2-x)}+\frac{1}{36}\log(1-x)(2d+5e+8f+11g)-\frac{1}{432}\log(2-x)(35d+58e+92f+136g)+\frac{1}{108}\log(x+1)(2d+e-4f+7g)+\frac{1}{144}\log(x+2)(d-2e+4f-8g)$$

input
```
Int[((2 + x)*(d + e*x + f*x^2 + g*x^3))/(4 - 5*x^2 + x^4)^2,x]
```

output
```
(d + e + f + g)/(12*(1 - x)) + (d + 2*e + 4*f + 8*g)/(36*(2 - x)) - (d - e
 + f - g)/(36*(1 + x)) + ((2*d + 5*e + 8*f + 11*g)*Log[1 - x])/36 - ((35*d
 + 58*e + 92*f + 136*g)*Log[2 - x])/432 + ((2*d + e - 4*f + 7*g)*Log[1 + x
])/108 + ((d - 2*e + 4*f - 8*g)*Log[2 + x])/144
```

3.100.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2462
```
Int[(u_.)*(Px_)^(p_), x_Symbol] :> With[{Qx = Factor[Px]}, Int[ExpandIntegr
and[u*Qx^p, x], x] /;  !SumQ[NonfreeFactors[Qx, x]]] /; PolyQ[Px, x] && GtQ
[Expon[Px, x], 2] &&  !BinomialQ[Px, x] &&  !TrinomialQ[Px, x] && ILtQ[p, 0
] && RationalFunctionQ[u, x]
```

3.100.4 Maple [A] (verified)

Time = 0.09 (sec) , antiderivative size = 134, normalized size of antiderivative = 0.95

method	result
default	$\left(\frac{d}{144}-\frac{e}{72}+\frac{f}{36}-\frac{g}{18}\right)\ln\left(x+2\right)-\frac{\frac{d}{36}-\frac{e}{36}+\frac{f}{36}-\frac{g}{36}}{x+1}+\left(\frac{d}{54}+\frac{e}{108}-\frac{f}{27}+\frac{7g}{108}\right)\ln\left(x+1\right)-$
norman	$\frac{\left(-\frac{5d}{36}-\frac{e}{9}-\frac{2f}{9}-\frac{5g}{18}\right)x^3+\left(\frac{17d}{36}+\frac{5e}{18}+\frac{5f}{9}+\frac{4g}{9}\right)x+\left(-\frac{d}{9}-\frac{2e}{9}-\frac{5f}{18}-\frac{5g}{9}\right)x^2+\frac{5d}{18}+\frac{5e}{9}+\frac{4f}{9}+\frac{8g}{9}}{x^4-5x^2+4}+\left(-\frac{35d}{432}-\frac{29e}{216}-\frac{23f}{108}-\right.$
risch	$\frac{\left(-\frac{5d}{36}-\frac{e}{9}-\frac{2f}{9}-\frac{5g}{18}\right)x^2+\left(\frac{d}{6}+\frac{f}{6}\right)x+\frac{5d}{36}+\frac{5e}{18}+\frac{2f}{9}+\frac{4g}{9}}{x^3-2x^2-x+2}+\frac{\ln(x+2)d}{144}-\frac{\ln(x+2)e}{72}+\frac{\ln(x+2)f}{36}-\frac{\ln(x+2)g}{18}+\frac{\ln(-x-1)d}{54}$
parallelrisch	$-$ $-96f-192g+60d\,x^2-60d-120e+96f\,x^2+120g\,x^2-72dx-60\ln(x-1)x^3e-8\ln(x+1)x^3d-4\ln(x+1)x^3e-3\ln(x+2)x^3d+6$

input
```
int((x+2)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x,method=_RETURNVERBOSE)
```

output
```
(1/144*d-1/72*e+1/36*f-1/18*g)*ln(x+2)-(1/36*d-1/36*e+1/36*f-1/36*g)/(x+1)
+(1/54*d+1/108*e-1/27*f+7/108*g)*ln(x+1)-(1/12*d+1/12*e+1/12*f+1/12*g)/(x-
1)+(1/18*d+5/36*e+2/9*f+11/36*g)*ln(x-1)+(-35/432*d-29/216*e-23/108*f-17/5
4*g)*ln(x-2)-(1/36*d+1/18*e+1/9*f+2/9*g)/(x-2)
```

3.100.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 321 vs. $2(123) = 246$.

Time = 0.78 (sec) , antiderivative size = 321, normalized size of antiderivative = 2.28

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3\right)}{\left(4-5x^2+x^4\right)^2}\,dx = -\frac{12\,(5\,d+4\,e+8\,f+10\,g)x^2-72\,(d+f)x-3\,((d-2\,e+4\,f-8\,g)x^3-2\,(d-2\,e+4\,f-8\,g)x^2-(d$$

input
```
integrate((2+x)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="fricas")
```

output
```
-1/432*(12*(5*d + 4*e + 8*f + 10*g)*x^2 - 72*(d + f)*x - 3*((d - 2*e + 4*f
 - 8*g)*x^3 - 2*(d - 2*e + 4*f - 8*g)*x^2 - (d - 2*e + 4*f - 8*g)*x + 2*d
- 4*e + 8*f - 16*g)*log(x + 2) - 4*((2*d + e - 4*f + 7*g)*x^3 - 2*(2*d + e
 - 4*f + 7*g)*x^2 - (2*d + e - 4*f + 7*g)*x + 4*d + 2*e - 8*f + 14*g)*log(
x + 1) - 12*((2*d + 5*e + 8*f + 11*g)*x^3 - 2*(2*d + 5*e + 8*f + 11*g)*x^2
 - (2*d + 5*e + 8*f + 11*g)*x + 4*d + 10*e + 16*f + 22*g)*log(x - 1) + ((3
5*d + 58*e + 92*f + 136*g)*x^3 - 2*(35*d + 58*e + 92*f + 136*g)*x^2 - (35*
d + 58*e + 92*f + 136*g)*x + 70*d + 116*e + 184*f + 272*g)*log(x - 2) - 60
*d - 120*e - 96*f - 192*g)/(x^3 - 2*x^2 - x + 2)
```

3.100.6 Sympy [F(-1)]

Timed out.

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3\right)}{\left(4-5x^2+x^4\right)^2}\,dx = \text{Timed out}$$

input
```
integrate((2+x)*(g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4)**2,x)
```

output
```
Timed out
```

3.100.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 126, normalized size of antiderivative = 0.89

$$\int \frac{(2+x)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx$$
$$= \frac{1}{144}(d-2e+4f-8g)\log(x+2)+\frac{1}{108}(2d+e-4f+7g)\log(x+1) + \frac{1}{36}(2d+5e+8f+11g)\log(x-1)-\frac{1}{432}(35d+58e+92f+136g)\log(x-2) - \frac{(5d+4e+8f+10g)x^2-6(d+f)x-5d-10e-8f-16g}{36(x^3-2x^2-x+2)}$$

input
```
integrate((2+x)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="maxima")
```

output
```
1/144*(d - 2*e + 4*f - 8*g)*log(x + 2) + 1/108*(2*d + e - 4*f + 7*g)*log(x
 + 1) + 1/36*(2*d + 5*e + 8*f + 11*g)*log(x - 1) - 1/432*(35*d + 58*e + 92
*f + 136*g)*log(x - 2) - 1/36*((5*d + 4*e + 8*f + 10*g)*x^2 - 6*(d + f)*x
- 5*d - 10*e - 8*f - 16*g)/(x^3 - 2*x^2 - x + 2)
```

3.100.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 130, normalized size of antiderivative = 0.92

$$\int \frac{(2+x)(d+ex+fx^2+gx^3)}{(4-5x^2+x^4)^2}\,dx$$
$$= \frac{1}{144}(d-2e+4f-8g)\log(|x+2|)+\frac{1}{108}(2d+e-4f+7g)\log(|x+1|) + \frac{1}{36}(2d+5e+8f+11g)\log(|x-1|)-\frac{1}{432}(35d+58e+92f+136g)\log(|x-2|) - \frac{(5d+4e+8f+10g)x^2-6(d+f)x-5d-10e-8f-16g}{36(x+1)(x-1)(x-2)}$$

input
```
integrate((2+x)*(g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="giac")
```

output
```
1/144*(d - 2*e + 4*f - 8*g)*log(abs(x + 2)) + 1/108*(2*d + e - 4*f + 7*g)*
log(abs(x + 1)) + 1/36*(2*d + 5*e + 8*f + 11*g)*log(abs(x - 1)) - 1/432*(3
5*d + 58*e + 92*f + 136*g)*log(abs(x - 2)) - 1/36*((5*d + 4*e + 8*f + 10*g
)*x^2 - 6*(d + f)*x - 5*d - 10*e - 8*f - 16*g)/((x + 1)*(x - 1)*(x - 2))
```

3.100.9 Mupad [B] (verification not implemented)

Time = 8.21 (sec) , antiderivative size = 131, normalized size of antiderivative = 0.93

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3\right)}{\left(4-5x^2+x^4\right)^2}\,dx = \ln(x-1)\left(\frac{d}{18}+\frac{5e}{36}+\frac{2f}{9}+\frac{11g}{36}\right)+\ln(x+2)\left(\frac{d}{144}-\frac{e}{72}+\frac{f}{36}-\frac{g}{18}\right) + \ln(x+1)\left(\frac{d}{54}+\frac{e}{108}-\frac{f}{27}+\frac{7g}{108}\right)-\ln(x-2)\left(\frac{35d}{432}+\frac{29e}{216}+\frac{23f}{108}+\frac{17g}{54}\right) - \frac{\left(-\frac{5d}{36}-\frac{e}{9}-\frac{2f}{9}-\frac{5g}{18}\right)x^2+\left(\frac{d}{6}+\frac{f}{6}\right)x+\frac{5d}{36}+\frac{5e}{18}+\frac{2f}{9}+\frac{4g}{9}}{-x^3+2x^2+x-2}$$

input
```
int(((x + 2)*(d + e*x + f*x^2 + g*x^3))/(x^4 - 5*x^2 + 4)^2,x)
```

output
```
log(x - 1)*(d/18 + (5*e)/36 + (2*f)/9 + (11*g)/36) + log(x + 2)*(d/144 - e
/72 + f/36 - g/18) + log(x + 1)*(d/54 + e/108 - f/27 + (7*g)/108) - log(x
- 2)*((35*d)/432 + (29*e)/216 + (23*f)/108 + (17*g)/54) - ((5*d)/36 + (5*e
)/18 + (2*f)/9 + (4*g)/9 - x^2*((5*d)/36 + e/9 + (2*f)/9 + (5*g)/18) + x*(
d/6 + f/6))/(x + 2*x^2 - x^3 - 2)
```

3.101 $\int \frac{(2+x)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx$

3.101.1 Optimal result

Integrand size = 36, antiderivative size = 158

$$\int \frac{(2+x)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx = \frac{d+e+f+g+h}{12(1-x)} + \frac{d+2e+4f+8g+16h}{36(2-x)} - \frac{d-e+f-g+h}{36(1+x)} + \frac{1}{36}(2d+5e+8f+11g+14h)\log(1-x) - \frac{1}{432}(35d+58e+92f+136g+176h)\log(2-x) + \frac{1}{108}(2d+e-4f+7g-10h)\log(1+x) + \frac{1}{144}(d-2e+4f-8g+16h)\log(2+x)$$

output
```
1/12*(d+e+f+g+h)/(1-x)+1/36*(d+2*e+4*f+8*g+16*h)/(2-x)+1/36*(-d+e-f+g-h)/(
1+x)+1/36*(2*d+5*e+8*f+11*g+14*h)*ln(1-x)-1/432*(35*d+58*e+92*f+136*g+176*
h)*ln(2-x)+1/108*(2*d+e-4*f+7*g-10*h)*ln(1+x)+1/144*(d-2*e+4*f-8*g+16*h)*l
n(2+x)
```

3.101.2 Mathematica [A] (verified)

Time = 0.06 (sec) , antiderivative size = 169, normalized size of antiderivative = 1.07

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4\right)}{\left(4-5x^2+x^4\right)^2}\,dx$$
$$=\frac{1}{432}\left(\frac{12(d(5+6x-5x^2)+2(8g+10h+3hx-5gx^2-10hx^2+f(4+3x-4x^2)+e(5-2x^2)))}{2-x-2x^2+x^3}\right.$$
$$+12(2d+5e+8f+11g+14h)\log(1-x)-(35d+58e+92f+136g+176h)\log(2-x)$$
$$\left.+4(2d+e-4f+7g-10h)\log(1+x)+3(d-2e+4f-8g+16h)\log(2+x)\right)$$

input
```
Integrate[((2 + x)*(d + e*x + f*x^2 + g*x^3 + h*x^4))/(4 - 5*x^2 + x^4)^2,
x]
```

output
```
((12*(d*(5 + 6*x - 5*x^2) + 2*(8*g + 10*h + 3*h*x - 5*g*x^2 - 10*h*x^2 + f
*(4 + 3*x - 4*x^2) + e*(5 - 2*x^2))))/(2 - x - 2*x^2 + x^3) + 12*(2*d + 5*
e + 8*f + 11*g + 14*h)*Log[1 - x] - (35*d + 58*e + 92*f + 136*g + 176*h)*L
og[2 - x] + 4*(2*d + e - 4*f + 7*g - 10*h)*Log[1 + x] + 3*(d - 2*e + 4*f -
 8*g + 16*h)*Log[2 + x])/432
```

3.101.3 Rubi [A] (verified)

Time = 0.45 (sec) , antiderivative size = 158, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.083, Rules used = {2019, 2462, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x+2)\left(d+ex+fx^2+gx^3+hx^4\right)}{\left(x^4-5x^2+4\right)^2}dx$$
$$\Big\downarrow \text{2019}$$
$$\int \frac{d+ex+fx^2+gx^3+hx^4}{(x+2)\left(x^3-2x^2-x+2\right)^2}dx$$
$$\Big\downarrow \text{2462}$$
$$\int \left(\frac{-35d-58e-92f-136g-176h}{432(x-2)}+\frac{2d+5e+8f+11g+14h}{36(x-1)}+\frac{2d+e-4f+7g-10h}{108(x+1)}+\frac{d-2e+4f-8}{144(x+}\right.$$

$$\downarrow 2009$$

$$-\frac{d-e+f-g+h}{36(x+1)}+\frac{d+e+f+g+h}{12(1-x)}+\frac{d+2e+4f+8g+16h}{36(2-x)}+\frac{1}{36}\log(1-x)(2d+5e+8f+11g+14h)-\frac{1}{432}\log(2-x)(35d+58e+92f+136g+176h)+\frac{1}{108}\log(x+1)(2d+e-4f+7g-10h)+\frac{1}{144}\log(x+2)(d-2e+4f-8g+16h)$$

input
```
Int[((2 + x)*(d + e*x + f*x^2 + g*x^3 + h*x^4))/(4 - 5*x^2 + x^4)^2,x]
```

output
```
(d + e + f + g + h)/(12*(1 - x)) + (d + 2*e + 4*f + 8*g + 16*h)/(36*(2 - x
)) - (d - e + f - g + h)/(36*(1 + x)) + ((2*d + 5*e + 8*f + 11*g + 14*h)*L
og[1 - x])/36 - ((35*d + 58*e + 92*f + 136*g + 176*h)*Log[2 - x])/432 + ((
2*d + e - 4*f + 7*g - 10*h)*Log[1 + x])/108 + ((d - 2*e + 4*f - 8*g + 16*h
)*Log[2 + x])/144
```

3.101.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2462
```
Int[(u_.)*(Px_)^(p_), x_Symbol] :> With[{Qx = Factor[Px]}, Int[ExpandIntegr
and[u*Qx^p, x], x] /;  !SumQ[NonfreeFactors[Qx, x]]] /; PolyQ[Px, x] && GtQ
[Expon[Px, x], 2] &&  !BinomialQ[Px, x] &&  !TrinomialQ[Px, x] && ILtQ[p, 0
] && RationalFunctionQ[u, x]
```

3.101.4 Maple [A] (verified)

Time = 0.11 (sec) , antiderivative size = 155, normalized size of antiderivative = 0.98

method	result
default	$\left(\frac{d}{144}-\frac{e}{72}+\frac{f}{36}-\frac{g}{18}+\frac{h}{9}\right)\ln\left(x+2\right)-\frac{\frac{d}{36}-\frac{e}{36}+\frac{f}{36}-\frac{g}{36}+\frac{h}{36}}{x+1}+\left(\frac{d}{54}+\frac{e}{108}-\frac{f}{27}+\frac{7g}{108}-\frac{5h}{54}\right)\ln\left(x+\right.$
norman	$\frac{\left(-\frac{5d}{36}-\frac{e}{9}-\frac{2f}{9}-\frac{5g}{18}-\frac{5h}{9}\right)x^{3}+\left(\frac{17d}{36}+\frac{5e}{18}+\frac{5f}{9}+\frac{4g}{9}+\frac{8h}{9}\right)x+\left(-\frac{d}{9}-\frac{2e}{9}-\frac{5f}{18}-\frac{5g}{9}-\frac{17h}{18}\right)x^{2}+\frac{5d}{18}+\frac{5e}{9}+\frac{10h}{9}+\frac{4f}{9}+\frac{8g}{9}}{x^{4}-5x^{2}+4}+\left(-\frac{35d}{432}-\right.$
risch	$\frac{\left(-\frac{5d}{36}-\frac{e}{9}-\frac{2f}{9}-\frac{5g}{18}-\frac{5h}{9}\right)x^{2}+\left(\frac{h}{6}+\frac{f}{6}+\frac{d}{6}\right)x+\frac{5d}{36}+\frac{5e}{18}+\frac{2f}{9}+\frac{4g}{9}+\frac{5h}{9}}{x^{3}-2x^{2}-x+2}+\frac{\ln(x+2)d}{144}-\frac{\ln(x+2)e}{72}+\frac{\ln(x+2)f}{36}-\frac{\ln(x+2)g}{18}+$
parallelrisch	$-\frac{-96f-192g+60d\,x^{2}-60d-240h-120e+96f\,x^{2}-176\ln(x-2)xh+168\ln(x-1)xh-40\ln(x+1)xh+48\ln(x+2)xh+120g\,x^{2}-}{}$

input
```
int((x+2)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x,method=_RETURNVERBOS
E)
```

output
```
(1/144*d-1/72*e+1/36*f-1/18*g+1/9*h)*ln(x+2)-(1/36*d-1/36*e+1/36*f-1/36*g+
1/36*h)/(x+1)+(1/54*d+1/108*e-1/27*f+7/108*g-5/54*h)*ln(x+1)-(1/12*d+1/12*
e+1/12*f+1/12*g+1/12*h)/(x-1)+(1/18*d+5/36*e+2/9*f+11/36*g+7/18*h)*ln(x-1)
+(-35/432*d-29/216*e-23/108*f-17/54*g-11/27*h)*ln(x-2)-(1/36*d+1/18*e+1/9*
f+2/9*g+4/9*h)/(x-2)
```

3.101.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 376 vs. 2(140) = 280.

Time = 3.63 (sec) , antiderivative size = 376, normalized size of antiderivative = 2.38

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4\right)}{\left(4-5x^2+x^4\right)^2}\,dx =$$
$$-\frac{12\,(5\,d+4\,e+8\,f+10\,g+20\,h)x^2-72\,(d+f+h)x-3\,((d-2\,e+4\,f-8\,g+16\,h)x^3-2\,(d-2\,e-}{}$$

input
```
integrate((2+x)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="fr
icas")
```

output
```
-1/432*(12*(5*d + 4*e + 8*f + 10*g + 20*h)*x^2 - 72*(d + f + h)*x - 3*((d
- 2*e + 4*f - 8*g + 16*h)*x^3 - 2*(d - 2*e + 4*f - 8*g + 16*h)*x^2 - (d -
2*e + 4*f - 8*g + 16*h)*x + 2*d - 4*e + 8*f - 16*g + 32*h)*log(x + 2) - 4*
((2*d + e - 4*f + 7*g - 10*h)*x^3 - 2*(2*d + e - 4*f + 7*g - 10*h)*x^2 - (
2*d + e - 4*f + 7*g - 10*h)*x + 4*d + 2*e - 8*f + 14*g - 20*h)*log(x + 1)
- 12*((2*d + 5*e + 8*f + 11*g + 14*h)*x^3 - 2*(2*d + 5*e + 8*f + 11*g + 14
*h)*x^2 - (2*d + 5*e + 8*f + 11*g + 14*h)*x + 4*d + 10*e + 16*f + 22*g + 2
8*h)*log(x - 1) + ((35*d + 58*e + 92*f + 136*g + 176*h)*x^3 - 2*(35*d + 58
*e + 92*f + 136*g + 176*h)*x^2 - (35*d + 58*e + 92*f + 136*g + 176*h)*x +
70*d + 116*e + 184*f + 272*g + 352*h)*log(x - 2) - 60*d - 120*e - 96*f - 1
92*g - 240*h)/(x^3 - 2*x^2 - x + 2)
```

3.101.6 Sympy [F(-1)]

Timed out.

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4\right)}{\left(4-5x^2+x^4\right)^2}\,dx=\text{Timed out}$$

input
```
integrate((2+x)*(h*x**4+g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4)**2,x)
```

output
```
Timed out
```

3.101.7 Maxima [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 145, normalized size of antiderivative = 0.92

$$\begin{aligned}
&\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4\right)}{\left(4-5x^2+x^4\right)^2}\,dx \\
&= \frac{1}{144}\left(d-2\,e+4\,f-8\,g+16\,h\right)\log\left(x+2\right)+\frac{1}{108}\left(2\,d+e-4\,f+7\,g-10\,h\right)\log\left(x+1\right) \\
&\quad+\frac{1}{36}\left(2\,d+5\,e+8\,f+11\,g+14\,h\right)\log\left(x-1\right) \\
&\quad-\frac{1}{432}\left(35\,d+58\,e+92\,f+136\,g+176\,h\right)\log\left(x-2\right) \\
&\quad-\frac{(5\,d+4\,e+8\,f+10\,g+20\,h)x^2-6\,(d+f+h)x-5\,d-10\,e-8\,f-16\,g-20\,h}{36\,(x^3-2\,x^2-x+2)}
\end{aligned}$$

input
```
integrate((2+x)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="ma
xima")
```

output
```
1/144*(d - 2*e + 4*f - 8*g + 16*h)*log(x + 2) + 1/108*(2*d + e - 4*f + 7*g
 - 10*h)*log(x + 1) + 1/36*(2*d + 5*e + 8*f + 11*g + 14*h)*log(x - 1) - 1/
432*(35*d + 58*e + 92*f + 136*g + 176*h)*log(x - 2) - 1/36*((5*d + 4*e + 8
*f + 10*g + 20*h)*x^2 - 6*(d + f + h)*x - 5*d - 10*e - 8*f - 16*g - 20*h)/
(x^3 - 2*x^2 - x + 2)
```

3.101.8 Giac [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 149, normalized size of antiderivative = 0.94

$$\int \frac{(2+x)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx$$
$$= \frac{1}{144}(d-2e+4f-8g+16h)\log(|x+2|)$$
$$+ \frac{1}{108}(2d+e-4f+7g-10h)\log(|x+1|)$$
$$+ \frac{1}{36}(2d+5e+8f+11g+14h)\log(|x-1|)$$
$$- \frac{1}{432}(35d+58e+92f+136g+176h)\log(|x-2|)$$
$$- \frac{(5d+4e+8f+10g+20h)x^2-6(d+f+h)x-5d-10e-8f-16g-20h}{36(x+1)(x-1)(x-2)}$$

input
```
integrate((2+x)*(h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorithm="gi
ac")
```

output
```
1/144*(d - 2*e + 4*f - 8*g + 16*h)*log(abs(x + 2)) + 1/108*(2*d + e - 4*f
+ 7*g - 10*h)*log(abs(x + 1)) + 1/36*(2*d + 5*e + 8*f + 11*g + 14*h)*log(a
bs(x - 1)) - 1/432*(35*d + 58*e + 92*f + 136*g + 176*h)*log(abs(x - 2)) -
1/36*((5*d + 4*e + 8*f + 10*g + 20*h)*x^2 - 6*(d + f + h)*x - 5*d - 10*e -
 8*f - 16*g - 20*h)/((x + 1)*(x - 1)*(x - 2))
```

3.101.9 Mupad [B] (verification not implemented)

Time = 8.43 (sec) , antiderivative size = 152, normalized size of antiderivative = 0.96

$$\int \frac{(2+x)(d+ex+fx^2+gx^3+hx^4)}{(4-5x^2+x^4)^2}\,dx$$
$$= \ln(x-1)\left(\frac{d}{18}+\frac{5e}{36}+\frac{2f}{9}+\frac{11g}{36}+\frac{7h}{18}\right) - \frac{\left(-\frac{5d}{36}-\frac{e}{9}-\frac{2f}{9}-\frac{5g}{18}-\frac{5h}{9}\right)x^2+\left(\frac{d}{6}+\frac{f}{6}+\frac{h}{6}\right)x+\frac{5d}{36}+\frac{5e}{18}+\frac{2f}{9}+\frac{4g}{9}+\frac{5h}{9}}{-x^3+2x^2+x-2} + \ln(x+2)\left(\frac{d}{144}-\frac{e}{72}+\frac{f}{36}-\frac{g}{18}+\frac{h}{9}\right)+\ln(x+1)\left(\frac{d}{54}+\frac{e}{108}-\frac{f}{27}+\frac{7g}{108}-\frac{5h}{54}\right) - \ln(x-2)\left(\frac{35d}{432}+\frac{29e}{216}+\frac{23f}{108}+\frac{17g}{54}+\frac{11h}{27}\right)$$

input
```
int(((x + 2)*(d + e*x + f*x^2 + g*x^3 + h*x^4))/(x^4 - 5*x^2 + 4)^2,x)
```

output
```
log(x - 1)*(d/18 + (5*e)/36 + (2*f)/9 + (11*g)/36 + (7*h)/18) - ((5*d)/36
+ (5*e)/18 + (2*f)/9 + (4*g)/9 + (5*h)/9 - x^2*((5*d)/36 + e/9 + (2*f)/9 +
 (5*g)/18 + (5*h)/9) + x*(d/6 + f/6 + h/6))/(x + 2*x^2 - x^3 - 2) + log(x
+ 2)*(d/144 - e/72 + f/36 - g/18 + h/9) + log(x + 1)*(d/54 + e/108 - f/27
+ (7*g)/108 - (5*h)/54) - log(x - 2)*((35*d)/432 + (29*e)/216 + (23*f)/108
 + (17*g)/54 + (11*h)/27)
```

3.102 $\int \frac{(2+x)(d+ex+fx^2+gx^3+hx^4+ix^5)}{(4-5x^2+x^4)^2}\,dx$

3.102.1 Optimal result

Integrand size = 41, antiderivative size = 177

$$\begin{aligned}
&\int \frac{(2+x)(d+ex+fx^2+gx^3+hx^4+ix^5)}{(4-5x^2+x^4)^2}\,dx \\
&= \frac{d+e+f+g+h+i}{12(1-x)}+\frac{d+2e+4f+8g+16h+32i}{36(2-x)} \\
&\quad -\frac{d-e+f-g+h-i}{36(1+x)}+\frac{1}{36}(2d+5e+8f+11g+14h+17i)\log(1-x) \\
&\quad -\frac{1}{432}(35d+58e+92f+136g+176h+160i)\log(2-x) \\
&\quad +\frac{1}{108}(2d+e-4f+7g-10h+13i)\log(1+x)+\frac{1}{144}(d-2e+4f-8g+16h-32i)\log(2+x)
\end{aligned}$$

output

```
1/12*(d+e+f+g+h+i)/(1-x)+1/36*(d+2*e+4*f+8*g+16*h+32*i)/(2-x)+1/36*(-d+e-f
+g-h+i)/(1+x)+1/36*(2*d+5*e+8*f+11*g+14*h+17*i)*ln(1-x)-1/432*(35*d+58*e+9
2*f+136*g+176*h+160*i)*ln(2-x)+1/108*(2*d+e-4*f+7*g-10*h+13*i)*ln(1+x)+1/1
44*(d-2*e+4*f-8*g+16*h-32*i)*ln(2+x)
```

3.102.2 Mathematica [A] (verified)

Time = 0.07 (sec) , antiderivative size = 195, normalized size of antiderivative = 1.10

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)}{\left(4-5x^2+x^4\right)^2}\,dx$$
$$=\frac{5d+10e+8f+16g+20h+40i+6dx+6fx+6hx-5dx^2-4ex^2-8fx^2-10gx^2-20hx^2-34ix^2}{36\left(2-x-2x^2+x^3\right)}$$
$$+\frac{1}{36}(2d+5e+8f+11g+14h+17i)\log(1-x)$$
$$+\frac{1}{432}(-35d-58e-92f-136g-176h-160i)\log(2-x)$$
$$+\frac{1}{108}(2d+e-4f+7g-10h+13i)\log(1+x)+\frac{1}{144}(d-2e+4f-8g+16h-32i)\log(2+x)$$

input
```
Integrate[((2 + x)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5))/(4 - 5*x^2 +
 x^4)^2,x]
```

output
```
(5*d + 10*e + 8*f + 16*g + 20*h + 40*i + 6*d*x + 6*f*x + 6*h*x - 5*d*x^2 -
 4*e*x^2 - 8*f*x^2 - 10*g*x^2 - 20*h*x^2 - 34*i*x^2)/(36*(2 - x - 2*x^2 +
x^3)) + ((2*d + 5*e + 8*f + 11*g + 14*h + 17*i)*Log[1 - x])/36 + ((-35*d -
 58*e - 92*f - 136*g - 176*h - 160*i)*Log[2 - x])/432 + ((2*d + e - 4*f +
7*g - 10*h + 13*i)*Log[1 + x])/108 + ((d - 2*e + 4*f - 8*g + 16*h - 32*i)*
Log[2 + x])/144
```

3.102.3 Rubi [A] (verified)

Time = 0.50 (sec) , antiderivative size = 177, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.073, Rules used = {2019, 2462, 2009}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{(x+2)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)}{\left(x^4-5x^2+4\right)^2}\,dx$$
$$\Big\downarrow \; 2019$$
$$\int \frac{d+ex+fx^2+gx^3+hx^4+ix^5}{(x+2)\left(x^3-2x^2-x+2\right)^2}dx$$

$$\Big\downarrow \text{2462}$$

$$\int \left(\frac{-35d-58e-92f-136g-176h-160i}{432(x-2)}+\frac{2d+5e+8f+11g+14h+17i}{36(x-1)}+\frac{2d+e-4f+7g-10h+13}{108(x+1)}\right.$$

$$\Big\downarrow \text{2009}$$

$$-\frac{d-e+f-g+h-i}{36(x+1)}+\frac{d+e+f+g+h+i}{12(1-x)}+\frac{d+2e+4f+8g+16h+32i}{36(2-x)}+\frac{1}{36}\log(1-x)(2d+5e+8f+11g+14h+17i)-\frac{1}{432}\log(2-x)(35d+58e+92f+136g+176h+160i)+\frac{1}{108}\log(x+1)(2d+e-4f+7g-10h+13i)+\frac{1}{144}\log(x+2)(d-2e+4f-8g+16h-32i)$$

input
```
Int[((2 + x)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5))/(4 - 5*x^2 + x^4)^
2,x]
```

output
```
(d + e + f + g + h + i)/(12*(1 - x)) + (d + 2*e + 4*f + 8*g + 16*h + 32*i)
/(36*(2 - x)) - (d - e + f - g + h - i)/(36*(1 + x)) + ((2*d + 5*e + 8*f +
 11*g + 14*h + 17*i)*Log[1 - x])/36 - ((35*d + 58*e + 92*f + 136*g + 176*h
 + 160*i)*Log[2 - x])/432 + ((2*d + e - 4*f + 7*g - 10*h + 13*i)*Log[1 + x
])/108 + ((d - 2*e + 4*f - 8*g + 16*h - 32*i)*Log[2 + x])/144
```

3.102.3.1 Defintions of rubi rules used

rule 2009
```
Int[u_, x_Symbol] :> Simp[IntSum[u, x], x] /; SumQ[u]
```

rule 2019
```
Int[(u_.)*(Px_)^(p_.)*(Qx_)^(q_.), x_Symbol] :> Int[u*PolynomialQuotient[Px
, Qx, x]^p*Qx^(p + q), x] /; FreeQ[q, x] && PolyQ[Px, x] && PolyQ[Qx, x] &&
 EqQ[PolynomialRemainder[Px, Qx, x], 0] && IntegerQ[p] && LtQ[p*q, 0]
```

rule 2462
```
Int[(u_.)*(Px_)^(p_), x_Symbol] :> With[{Qx = Factor[Px]}, Int[ExpandIntegr
and[u*Qx^p, x], x] /;  !SumQ[NonfreeFactors[Qx, x]]] /; PolyQ[Px, x] && GtQ
[Expon[Px, x], 2] &&  !BinomialQ[Px, x] &&  !TrinomialQ[Px, x] && ILtQ[p, 0
] && RationalFunctionQ[u, x]
```

3.102.4 Maple [A] (verified)

Time = 0.13 (sec) , antiderivative size = 176, normalized size of antiderivative = 0.99

method	result
default	$\left(\frac{d}{144}-\frac{e}{72}+\frac{f}{36}-\frac{g}{18}+\frac{h}{9}-\frac{2i}{9}\right)\ln\left(x+2\right)-\frac{\frac{d}{36}-\frac{e}{36}+\frac{f}{36}-\frac{g}{36}+\frac{h}{36}-\frac{i}{36}}{x+1}+\left(\frac{d}{54}+\frac{e}{108}-\frac{f}{27}+\frac{7g}{108}-\frac{5h}{54}\right.$
norman	$\frac{\left(-\frac{5d}{36}-\frac{e}{9}-\frac{2f}{9}-\frac{5g}{18}-\frac{5h}{9}-\frac{17i}{18}\right)x^{3}+\left(\frac{17d}{36}+\frac{5e}{18}+\frac{5f}{9}+\frac{4g}{9}+\frac{8h}{9}+\frac{10i}{9}\right)x+\left(-\frac{d}{9}-\frac{2e}{9}-\frac{5f}{18}-\frac{5g}{9}-\frac{17h}{18}-\frac{17i}{9}\right)x^{2}+\frac{5d}{18}+\frac{5e}{9}+\frac{10h}{9}+\frac{20i}{9}+}{x^{4}-5x^{2}+4}$
risch	$\frac{7\ln(-x-1)g}{108}+\frac{\ln(x-1)d}{18}+\frac{5\ln(x-1)e}{36}+\frac{\ln(x+2)f}{36}-\frac{5\ln(-x-1)h}{54}-\frac{23\ln(2-x)f}{108}-\frac{35\ln(2-x)d}{432}-\frac{29\ln(2-x)e}{216}$
parallelrisch	Expression too large to display

input
```
int((x+2)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x,method=_RETURN
VERBOSE)
```

output
```
(1/144*d-1/72*e+1/36*f-1/18*g+1/9*h-2/9*i)*ln(x+2)-(1/36*d-1/36*e+1/36*f-1
/36*g+1/36*h-1/36*i)/(x+1)+(1/54*d+1/108*e-1/27*f+7/108*g-5/54*h+13/108*i)
*ln(x+1)-(1/12*d+1/12*e+1/12*f+1/12*g+1/12*h+1/12*i)/(x-1)+(1/18*d+5/36*e+
2/9*f+11/36*g+7/18*h+17/36*i)*ln(x-1)+(-35/432*d-29/216*e-23/108*f-17/54*g
-11/27*h-10/27*i)*ln(x-2)-(1/36*d+1/18*e+1/9*f+2/9*g+4/9*h+8/9*i)/(x-2)
```

3.102.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 430 vs. 2(159) = 318.

Time = 20.84 (sec) , antiderivative size = 430, normalized size of antiderivative = 2.43

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)}{\left(4-5x^2+x^4\right)^2}\,dx =$$
$$-\frac{12\,(5\,d+4\,e+8\,f+10\,g+20\,h+34\,i)x^2-72\,(d+f+h)x-3\,((d-2\,e+4\,f-8\,g+16\,h-32\,i)x^3}{}$$

input
```
integrate((2+x)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorit
hm="fricas")
```

output
```
-1/432*(12*(5*d + 4*e + 8*f + 10*g + 20*h + 34*i)*x^2 - 72*(d + f + h)*x -
 3*((d - 2*e + 4*f - 8*g + 16*h - 32*i)*x^3 - 2*(d - 2*e + 4*f - 8*g + 16*
h - 32*i)*x^2 - (d - 2*e + 4*f - 8*g + 16*h - 32*i)*x + 2*d - 4*e + 8*f -
16*g + 32*h - 64*i)*log(x + 2) - 4*((2*d + e - 4*f + 7*g - 10*h + 13*i)*x^
3 - 2*(2*d + e - 4*f + 7*g - 10*h + 13*i)*x^2 - (2*d + e - 4*f + 7*g - 10*
h + 13*i)*x + 4*d + 2*e - 8*f + 14*g - 20*h + 26*i)*log(x + 1) - 12*((2*d
+ 5*e + 8*f + 11*g + 14*h + 17*i)*x^3 - 2*(2*d + 5*e + 8*f + 11*g + 14*h +
 17*i)*x^2 - (2*d + 5*e + 8*f + 11*g + 14*h + 17*i)*x + 4*d + 10*e + 16*f
+ 22*g + 28*h + 34*i)*log(x - 1) + ((35*d + 58*e + 92*f + 136*g + 176*h +
160*i)*x^3 - 2*(35*d + 58*e + 92*f + 136*g + 176*h + 160*i)*x^2 - (35*d +
58*e + 92*f + 136*g + 176*h + 160*i)*x + 70*d + 116*e + 184*f + 272*g + 35
2*h + 320*i)*log(x - 2) - 60*d - 120*e - 96*f - 192*g - 240*h - 480*i)/(x^
3 - 2*x^2 - x + 2)
```

3.102.6 Sympy [F(-1)]

Timed out.

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)}{\left(4-5x^2+x^4\right)^2}\,dx = \text{Timed out}$$

input
```
integrate((2+x)*(i*x**5+h*x**4+g*x**3+f*x**2+e*x+d)/(x**4-5*x**2+4)**2,x)
```

output
```
Timed out
```

3.102.7 Maxima [A] (verification not implemented)

Time = 0.20 (sec) , antiderivative size = 163, normalized size of antiderivative = 0.92

$$\begin{aligned}
&\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)}{\left(4-5x^2+x^4\right)^2}\,dx \\
&= \frac{1}{144}\left(d-2e+4f-8g+16h-32i\right)\log\left(x+2\right) \\
&\quad + \frac{1}{108}\left(2d+e-4f+7g-10h+13i\right)\log\left(x+1\right) \\
&\quad + \frac{1}{36}\left(2d+5e+8f+11g+14h+17i\right)\log\left(x-1\right) \\
&\quad - \frac{1}{432}\left(35d+58e+92f+136g+176h+160i\right)\log\left(x-2\right) \\
&\quad - \frac{(5d+4e+8f+10g+20h+34i)x^2-6(d+f+h)x-5d-10e-8f-16g-20h-40i}{36\left(x^3-2x^2-x+2\right)}
\end{aligned}$$

input
```
integrate((2+x)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorit
hm="maxima")
```

output
```
1/144*(d - 2*e + 4*f - 8*g + 16*h - 32*i)*log(x + 2) + 1/108*(2*d + e - 4*
f + 7*g - 10*h + 13*i)*log(x + 1) + 1/36*(2*d + 5*e + 8*f + 11*g + 14*h +
17*i)*log(x - 1) - 1/432*(35*d + 58*e + 92*f + 136*g + 176*h + 160*i)*log(
x - 2) - 1/36*((5*d + 4*e + 8*f + 10*g + 20*h + 34*i)*x^2 - 6*(d + f + h)*
x - 5*d - 10*e - 8*f - 16*g - 20*h - 40*i)/(x^3 - 2*x^2 - x + 2)
```

3.102.8 Giac [A] (verification not implemented)

Time = 0.30 (sec) , antiderivative size = 167, normalized size of antiderivative = 0.94

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)}{\left(4-5x^2+x^4\right)^2}\,dx$$
$$= \frac{1}{144}\left(d-2\,e+4\,f-8\,g+16\,h-32\,i\right)\log\left(|x+2|\right) + \frac{1}{108}\left(2\,d+e-4\,f+7\,g-10\,h+13\,i\right)\log\left(|x+1|\right) + \frac{1}{36}\left(2\,d+5\,e+8\,f+11\,g+14\,h+17\,i\right)\log\left(|x-1|\right) - \frac{1}{432}\left(35\,d+58\,e+92\,f+136\,g+176\,h+160\,i\right)\log\left(|x-2|\right) - \frac{(5\,d+4\,e+8\,f+10\,g+20\,h+34\,i)x^2-6\,(d+f+h)x-5\,d-10\,e-8\,f-16\,g-20\,h-40\,i}{36\,(x+1)(x-1)(x-2)}$$

input
```
integrate((2+x)*(i*x^5+h*x^4+g*x^3+f*x^2+e*x+d)/(x^4-5*x^2+4)^2,x, algorit
hm="giac")
```

output
```
1/144*(d - 2*e + 4*f - 8*g + 16*h - 32*i)*log(abs(x + 2)) + 1/108*(2*d + e
 - 4*f + 7*g - 10*h + 13*i)*log(abs(x + 1)) + 1/36*(2*d + 5*e + 8*f + 11*g
 + 14*h + 17*i)*log(abs(x - 1)) - 1/432*(35*d + 58*e + 92*f + 136*g + 176*
h + 160*i)*log(abs(x - 2)) - 1/36*((5*d + 4*e + 8*f + 10*g + 20*h + 34*i)*
x^2 - 6*(d + f + h)*x - 5*d - 10*e - 8*f - 16*g - 20*h - 40*i)/((x + 1)*(x
 - 1)*(x - 2))
```

3.102.9 Mupad [B] (verification not implemented)

Time = 8.69 (sec) , antiderivative size = 170, normalized size of antiderivative = 0.96

$$\int \frac{(2+x)\left(d+ex+fx^2+gx^3+hx^4+ix^5\right)}{\left(4-5x^2+x^4\right)^2}\,dx$$
$$=\ln(x-1)\left(\frac{d}{18}+\frac{5e}{36}+\frac{2f}{9}+\frac{11g}{36}+\frac{7h}{18}+\frac{17i}{36}\right)+\ln(x+2)\left(\frac{d}{144}-\frac{e}{72}+\frac{f}{36}-\frac{g}{18}+\frac{h}{9}-\frac{2i}{9}\right)$$
$$+\ln(x+1)\left(\frac{d}{54}+\frac{e}{108}-\frac{f}{27}+\frac{7g}{108}-\frac{5h}{54}+\frac{13i}{108}\right)$$
$$-\ln(x-2)\left(\frac{35d}{432}+\frac{29e}{216}+\frac{23f}{108}+\frac{17g}{54}+\frac{11h}{27}+\frac{10i}{27}\right)$$
$$-\frac{\left(-\frac{5d}{36}-\frac{e}{9}-\frac{2f}{9}-\frac{5g}{18}-\frac{5h}{9}-\frac{17i}{18}\right)x^2+\left(\frac{d}{6}+\frac{f}{6}+\frac{h}{6}\right)x+\frac{5d}{36}+\frac{5e}{18}+\frac{2f}{9}+\frac{4g}{9}+\frac{5h}{9}+\frac{10i}{9}}{-x^3+2x^2+x-2}$$

input
```
int(((x + 2)*(d + e*x + f*x^2 + g*x^3 + h*x^4 + i*x^5))/(x^4 - 5*x^2 + 4)^
2,x)
```

output
```
log(x - 1)*(d/18 + (5*e)/36 + (2*f)/9 + (11*g)/36 + (7*h)/18 + (17*i)/36)
+ log(x + 2)*(d/144 - e/72 + f/36 - g/18 + h/9 - (2*i)/9) + log(x + 1)*(d/
54 + e/108 - f/27 + (7*g)/108 - (5*h)/54 + (13*i)/108) - log(x - 2)*((35*d
)/432 + (29*e)/216 + (23*f)/108 + (17*g)/54 + (11*h)/27 + (10*i)/27) - ((5
*d)/36 + (5*e)/18 + (2*f)/9 + (4*g)/9 + (5*h)/9 + (10*i)/9 - x^2*((5*d)/36
 + e/9 + (2*f)/9 + (5*g)/18 + (5*h)/9 + (17*i)/18) + x*(d/6 + f/6 + h/6))/
(x + 2*x^2 - x^3 - 2)
```

3.103 $\int (d+ex+fx^2+gx^3)(a+bx^2+cx^4)^{3/2}\,dx$

3.103.1 Optimal result

Integrand size = 32, antiderivative size = 717

$$\int (d+ex+fx^2+gx^3)\left(a+bx^2+cx^4\right)^{3/2}\,dx =$$
$$-\frac{(18b^3cd-144abc^2d-8b^4f+57ab^2cf-84a^2c^2f)\,x\sqrt{a+bx^2+cx^4}}{315c^{5/2}\left(\sqrt{a}+\sqrt{c}x^2\right)}$$
$$-\frac{3(b^2-4ac)(2ce-bg)(b+2cx^2)\sqrt{a+bx^2+cx^4}}{256c^3}$$
$$+\frac{x(9b^2cd+90ac^2d-4b^3f+9abcf+3c(9bcd-4b^2f+14acf)x^2)\sqrt{a+bx^2+cx^4}}{315c^2}$$
$$+\frac{(2ce-bg)(b+2cx^2)\left(a+bx^2+cx^4\right)^{3/2}}{32c^2}+\frac{x(3(3cd+bf)+7cfx^2)\left(a+bx^2+cx^4\right)^{3/2}}{63c}$$
$$+\frac{g\left(a+bx^2+cx^4\right)^{5/2}}{10c}+\frac{3(b^2-4ac)^2(2ce-bg)\operatorname{arctanh}\left(\frac{b+2cx^2}{2\sqrt{c}\sqrt{a+bx^2+cx^4}}\right)}{512c^{7/2}}$$
$$+\frac{\sqrt[4]{a}(18b^3cd-144abc^2d-8b^4f+57ab^2cf-84a^2c^2f)\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}E\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right)|\frac{1}{4}(2\right.}{315c^{11/4}\sqrt{a+bx^2+cx^4}}$$
$$-\frac{\sqrt[4]{a}\left(18b^3cd-144abc^2d-8b^4f+57ab^2cf-84a^2c^2f+\sqrt{a}\sqrt{c}(9b^2cd-180ac^2d-4b^3f+24abcf)\right)\left(\sqrt{a}+\right.}{630c^{11/4}\sqrt{a+bx^2+cx^4}}$$

output
```
1/32*(-b*g+2*c*e)*(2*c*x^2+b)*(c*x^4+b*x^2+a)^(3/2)/c^2+1/63*x*(7*c*f*x^2+
3*b*f+9*c*d)*(c*x^4+b*x^2+a)^(3/2)/c+1/10*g*(c*x^4+b*x^2+a)^(5/2)/c+3/512*
(-4*a*c+b^2)^2*(-b*g+2*c*e)*arctanh(1/2*(2*c*x^2+b)/c^(1/2)/(c*x^4+b*x^2+a
)^(1/2))/c^(7/2)-3/256*(-4*a*c+b^2)*(-b*g+2*c*e)*(2*c*x^2+b)*(c*x^4+b*x^2+
a)^(1/2)/c^3+1/315*x*(9*b^2*c*d+90*a*c^2*d-4*b^3*f+9*a*b*c*f+3*c*(14*a*c*f
-4*b^2*f+9*b*c*d)*x^2)*(c*x^4+b*x^2+a)^(1/2)/c^2-1/315*(-84*a^2*c^2*f+57*a
*b^2*c*f-144*a*b*c^2*d-8*b^4*f+18*b^3*c*d)*x*(c*x^4+b*x^2+a)^(1/2)/c^(5/2)
/(a^(1/2)+x^2*c^(1/2))+1/315*a^(1/4)*(-84*a^2*c^2*f+57*a*b^2*c*f-144*a*b*c
^2*d-8*b^4*f+18*b^3*c*d)*(cos(2*arctan(c^(1/4)*x/a^(1/4)))^2)^(1/2)/cos(2*
arctan(c^(1/4)*x/a^(1/4)))*EllipticE(sin(2*arctan(c^(1/4)*x/a^(1/4))),1/2*
(2-b/a^(1/2)/c^(1/2))^(1/2))*(a^(1/2)+x^2*c^(1/2))*((c*x^4+b*x^2+a)/(a^(1/
2)+x^2*c^(1/2))^2)^(1/2)/c^(11/4)/(c*x^4+b*x^2+a)^(1/2)-1/630*a^(1/4)*(cos
(2*arctan(c^(1/4)*x/a^(1/4)))^2)^(1/2)/cos(2*arctan(c^(1/4)*x/a^(1/4)))*El
lipticF(sin(2*arctan(c^(1/4)*x/a^(1/4))),1/2*(2-b/a^(1/2)/c^(1/2))^(1/2))*
(a^(1/2)+x^2*c^(1/2))*(18*b^3*c*d-144*a*b*c^2*d-8*b^4*f+57*a*b^2*c*f-84*a^
2*c^2*f+(24*a*b*c*f-180*a*c^2*d-4*b^3*f+9*b^2*c*d)*a^(1/2)*c^(1/2))*((c*x^
4+b*x^2+a)/(a^(1/2)+x^2*c^(1/2))^2)^(1/2)/c^(11/4)/(c*x^4+b*x^2+a)^(1/2)
```

3.103.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 11.69 (sec) , antiderivative size = 2588, normalized size of antiderivative = 3.61

$$\int \left(d+ex+fx^2+gx^3\right)\left(a+bx^2+cx^4\right)^{3/2}\,dx=\text{Result too large to show}$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3)*(a + b*x^2 + c*x^4)^(3/2),x]
```

output
```
(-2*Sqrt[c]*Sqrt[c/(b + Sqrt[b^2 - 4*a*c])]*(a + b*x^2 + c*x^4)*(-945*b^4*
g + 2*b^3*c*(945*e + x*(512*f + 315*g*x)) - 12*b^2*c*(-525*a*g + c*x*(192*
d + 105*e*x + 64*f*x^2 + 42*g*x^3)) - 8*b*c^2*(3*a*(525*e + 256*f*x + 147*
g*x^2) + 2*c*x^3*(1152*d + 945*e*x + 800*f*x^2 + 693*g*x^3)) - 16*c^2*(504
*a^2*g + 2*c^2*x^5*(360*d + 7*x*(45*e + 40*f*x + 36*g*x^2)) + a*c*x*(2160*
d + 7*x*(225*e + 16*x*(11*f + 9*g*x))))) + (2304*I)*Sqrt[2]*b^3*c^(3/2)*(b
 - Sqrt[b^2 - 4*a*c])*d*Sqrt[(b + Sqrt[b^2 - 4*a*c] + 2*c*x^2)/(b + Sqrt[b
^2 - 4*a*c])]*Sqrt[1 + (2*c*x^2)/(b - Sqrt[b^2 - 4*a*c])]*(EllipticE[I*Arc
Sinh[Sqrt[2]*Sqrt[c/(b + Sqrt[b^2 - 4*a*c])]*x], (b + Sqrt[b^2 - 4*a*c])/(
b - Sqrt[b^2 - 4*a*c])] - EllipticF[I*ArcSinh[Sqrt[2]*Sqrt[c/(b + Sqrt[b^2
 - 4*a*c])]*x], (b + Sqrt[b^2 - 4*a*c])/(b - Sqrt[b^2 - 4*a*c])]) + (18432
*I)*Sqrt[2]*a*b*c^(5/2)*(-b + Sqrt[b^2 - 4*a*c])*d*Sqrt[(b + Sqrt[b^2 - 4*
a*c] + 2*c*x^2)/(b + Sqrt[b^2 - 4*a*c])]*Sqrt[1 + (2*c*x^2)/(b - Sqrt[b^2
- 4*a*c])]*(EllipticE[I*ArcSinh[Sqrt[2]*Sqrt[c/(b + Sqrt[b^2 - 4*a*c])]*x]
, (b + Sqrt[b^2 - 4*a*c])/(b - Sqrt[b^2 - 4*a*c])] - EllipticF[I*ArcSinh[S
qrt[2]*Sqrt[c/(b + Sqrt[b^2 - 4*a*c])]*x], (b + Sqrt[b^2 - 4*a*c])/(b - Sq
rt[b^2 - 4*a*c])]) + (7296*I)*Sqrt[2]*a*b^2*c^(3/2)*(b - Sqrt[b^2 - 4*a*c]
)*f*Sqrt[(b + Sqrt[b^2 - 4*a*c] + 2*c*x^2)/(b + Sqrt[b^2 - 4*a*c])]*Sqrt[1
 + (2*c*x^2)/(b - Sqrt[b^2 - 4*a*c])]*(EllipticE[I*ArcSinh[Sqrt[2]*Sqrt[c/
(b + Sqrt[b^2 - 4*a*c])]*x], (b + Sqrt[b^2 - 4*a*c])/(b - Sqrt[b^2 - 4*...
```

3.103.3 Rubi [A] (verified)

Time = 1.25 (sec) , antiderivative size = 693, normalized size of antiderivative = 0.97, number of steps used = 15, number of rules used = 14, $\frac{\text{number of rules}}{\text{integrand size}} = 0.438$, Rules used = {2202, 1490, 1490, 25, 1511, 27, 1416, 1509, 1576, 1160, 1087, 1087, 1092, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \left(a+bx^2+cx^4\right)^{3/2} \left(d+ex+fx^2+gx^3\right) \, dx$$

$$\Big\downarrow \text{2202}$$

$$\int \left(fx^2+d\right) \left(cx^4+bx^2+a\right)^{3/2}dx+\int x\left(gx^2+e\right) \left(cx^4+bx^2+a\right)^{3/2}dx$$

$$\Big\downarrow \text{1490}$$

$$\frac{\int \left(\left(-4fb^2+9cdb+14acf\right)x^2+a(18cd-bf)\right)\sqrt{cx^4+bx^2+a}dx}{21c}+$$
$$\int x\left(gx^2+e\right) \left(cx^4+bx^2+a\right)^{3/2}dx+\frac{x\left(a+bx^2+cx^4\right)^{3/2}\left(3(bf+3cd)+7cfx^2\right)}{63c}$$

$$\Big\downarrow \; 1490$$

$$\frac{\frac{\int -\frac{\left(-8fb^4+18cdb^3+57acfb^2-144ac^2db-84a^2c^2f\right)x^2+a\left(-4fb^3+9cdb^2+24acfb-180ac^2d\right)}{\sqrt{cx^4+bx^2+a}}dx}{15c}+\frac{x\sqrt{a+bx^2+cx^4}\left(3cx^2\left(14acf-4b^2f+9bcd\right)+9abcf+90a\cdots\right.}{15c}}{21c}$$

$$\int x\left(gx^2+e\right)\left(cx^4+bx^2+a\right)^{3/2}dx+\frac{x\left(a+bx^2+cx^4\right)^{3/2}\left(3(bf+3cd)+7cfx^2\right)}{63c}$$

$$\Big\downarrow \; 25$$

$$\frac{\frac{x\sqrt{a+bx^2+cx^4}\left(3cx^2\left(14acf-4b^2f+9bcd\right)+9abcf+90ac^2d-4b^3f+9b^2cd\right)}{15c}-\frac{\int \frac{\left(-8fb^4+18cdb^3+57acfb^2-144ac^2db-84a^2c^2f\right)x^2+a\left(-4fb^3+9cdb^2+24\cdots\right.}{\sqrt{cx^4+bx^2+a}}}{15c}}{21c}$$

$$\int x\left(gx^2+e\right)\left(cx^4+bx^2+a\right)^{3/2}dx+\frac{x\left(a+bx^2+cx^4\right)^{3/2}\left(3(bf+3cd)+7cfx^2\right)}{63c}$$

$$\Big\downarrow \; 1511$$

$$\frac{\frac{x\sqrt{a+bx^2+cx^4}\left(3cx^2\left(14acf-4b^2f+9bcd\right)+9abcf+90ac^2d-4b^3f+9b^2cd\right)}{15c}-\frac{\frac{\sqrt{a}\left(-84a^2c^2f+57ab^2cf+\sqrt{a}\sqrt{c}\left(24abcf-180ac^2d-4b^3f+9b^2cd\right)-144abc^2\cdots\right.}{\sqrt{c}}}{\cdots}}{21c}$$

$$\int x\left(gx^2+e\right)\left(cx^4+bx^2+a\right)^{3/2}dx+\frac{x\left(a+bx^2+cx^4\right)^{3/2}\left(3(bf+3cd)+7cfx^2\right)}{63c}$$

$$\Big\downarrow \; 27$$

$$\frac{\frac{x\sqrt{a+bx^2+cx^4}\left(3cx^2\left(14acf-4b^2f+9bcd\right)+9abcf+90ac^2d-4b^3f+9b^2cd\right)}{15c}-\frac{\frac{\sqrt{a}\left(-84a^2c^2f+57ab^2cf+\sqrt{a}\sqrt{c}\left(24abcf-180ac^2d-4b^3f+9b^2cd\right)-144abc^2\cdots\right.}{\sqrt{c}}}{\cdots}}{21c}$$

$$\int x\left(gx^2+e\right)\left(cx^4+bx^2+a\right)^{3/2}dx+\frac{x\left(a+bx^2+cx^4\right)^{3/2}\left(3(bf+3cd)+7cfx^2\right)}{63c}$$

$$\Big\downarrow \; 1416$$

$$\frac{x\sqrt{a+bx^2+cx^4}\left(3cx^2\left(14acf-4b^2f+9bcd\right)+9abcf+90ac^2d-4b^3f+9b^2cd\right)}{15c}-\frac{\sqrt[4]{a}\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}\left(-84a^2c^2f+57ab^2cf+\sqrt{a}\sqrt{c}\left(24abcf-1\cdots\right.\right.}{\cdots}$$

$$\int x\left(gx^2+e\right)\left(cx^4+bx^2+a\right)^{3/2}dx+\frac{x\left(a+bx^2+cx^4\right)^{3/2}\left(3(bf+3cd)+7cfx^2\right)}{63c}$$

$$\Big\downarrow \; 1509$$

3.103. $\int \left(d+ex+fx^2+gx^3\right)\left(a+bx^2+cx^4\right)^{3/2}\,dx$

$$\int x\left(gx^2+e\right)\left(cx^4+bx^2+a\right)^{3/2}dx+$$

$$\frac{\frac{x\sqrt{a+bx^2+cx^4}\left(3cx^2\left(14acf-4b^2f+9bcd\right)+9abcf+90ac^2d-4b^3f+9b^2cd\right)}{15c}-\frac{\sqrt[4]{a}\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}\left(-84a^2c^2f+57ab^2cf+\sqrt{a}\sqrt{c}\left(24abcf-1\cdots\right.\right.}{\cdots}}{\cdots}$$

$$\frac{x\left(a+bx^2+cx^4\right)^{3/2}\left(3(bf+3cd)+7cfx^2\right)}{63c}$$

$$\Big\downarrow \text{ 1576}$$

$$\frac{1}{2}\int\left(gx^2+e\right)\left(cx^4+bx^2+a\right)^{3/2}dx^2+$$

$$\frac{\frac{x\sqrt{a+bx^2+cx^4}\left(3cx^2\left(14acf-4b^2f+9bcd\right)+9abcf+90ac^2d-4b^3f+9b^2cd\right)}{15c}-\frac{\sqrt[4]{a}\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}\left(-84a^2c^2f+57ab^2cf+\sqrt{a}\sqrt{c}\left(24abcf-1\cdots\right.\right.}{\cdots}}{\cdots}$$

$$\frac{x\left(a+bx^2+cx^4\right)^{3/2}\left(3(bf+3cd)+7cfx^2\right)}{63c}$$

$$\Big\downarrow \text{ 1160}$$

$$\frac{1}{2}\left(\frac{(2ce-bg)\int\left(cx^4+bx^2+a\right)^{3/2}dx^2}{2c}+\frac{g\left(a+bx^2+cx^4\right)^{5/2}}{5c}\right)+$$

$$\frac{\frac{x\sqrt{a+bx^2+cx^4}\left(3cx^2\left(14acf-4b^2f+9bcd\right)+9abcf+90ac^2d-4b^3f+9b^2cd\right)}{15c}-\frac{\sqrt[4]{a}\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}\left(-84a^2c^2f+57ab^2cf+\sqrt{a}\sqrt{c}\left(24abcf-1\cdots\right.\right.}{\cdots}}{\cdots}$$

$$\frac{x\left(a+bx^2+cx^4\right)^{3/2}\left(3(bf+3cd)+7cfx^2\right)}{63c}$$

$$\Big\downarrow \text{ 1087}$$

3.103. $\int\left(d+ex+fx^2+gx^3\right)\left(a+bx^2+cx^4\right)^{3/2}dx$

$$\frac{1}{2}\left(\frac{(2ce-bg)\left(\frac{(b+2cx^2)(a+bx^2+cx^4)^{3/2}}{8c}-\frac{3(b^2-4ac)\int\sqrt{cx^4+bx^2+a}dx^2}{16c}\right)}{2c}+\frac{g(a+bx^2+cx^4)^{5/2}}{5c}\right)+$$

$$\frac{\frac{x\sqrt{a+bx^2+cx^4}(3cx^2(14acf-4b^2f+9bcd)+9abcf+90ac^2d-4b^3f+9b^2cd)}{15c}-\frac{\sqrt[4]{a}(\sqrt{a}+\sqrt{c}x^2)\sqrt{\frac{a+bx^2+cx^4}{(\sqrt{a}+\sqrt{c}x^2)^2}}(-84a^2c^2f+57ab^2cf+\sqrt{a}\sqrt{c}(24abcf-1\ldots}{\ldots}}{\ldots}$$

$$\frac{x(a+bx^2+cx^4)^{3/2}(3(bf+3cd)+7cfx^2)}{63c}$$

$\big\downarrow$ 1087

$$\frac{1}{2}\left(\frac{(2ce-bg)\left(\frac{(b+2cx^2)(a+bx^2+cx^4)^{3/2}}{8c}-\frac{3(b^2-4ac)\left(\frac{(b+2cx^2)\sqrt{a+bx^2+cx^4}}{4c}-\frac{(b^2-4ac)\int\frac{1}{\sqrt{cx^4+bx^2+a}}dx^2}{8c}\right)}{16c}\right)}{2c}+\frac{g(a+bx^2+cx\ldots}{5c}\right.$$

$$\frac{\frac{x\sqrt{a+bx^2+cx^4}(3cx^2(14acf-4b^2f+9bcd)+9abcf+90ac^2d-4b^3f+9b^2cd)}{15c}-\frac{\sqrt[4]{a}(\sqrt{a}+\sqrt{c}x^2)\sqrt{\frac{a+bx^2+cx^4}{(\sqrt{a}+\sqrt{c}x^2)^2}}(-84a^2c^2f+57ab^2cf+\sqrt{a}\sqrt{c}(24abcf-1\ldots}{\ldots}}{\ldots}$$

$$\frac{x(a+bx^2+cx^4)^{3/2}(3(bf+3cd)+7cfx^2)}{63c}$$

$\big\downarrow$ 1092

3.103. $\int(d+ex+fx^2+gx^3)(a+bx^2+cx^4)^{3/2}\,dx$

$$\frac{1}{2}\left(\frac{(2ce-bg)\left(\frac{(b+2cx^2)(a+bx^2+cx^4)^{3/2}}{8c}-\frac{3(b^2-4ac)\left(\frac{(b+2cx^2)\sqrt{a+bx^2+cx^4}}{4c}-\frac{(b^2-4ac)\int\frac{1}{4c-x^4}d\frac{2cx^2+b}{\sqrt{cx^4+bx^2+a}}}{4c}\right)}{16c}\right)}{2c}+\frac{g(a+bx^2+\ldots}{5c\ldots}\right.$$

$$\frac{\frac{x\sqrt{a+bx^2+cx^4}(3cx^2(14acf-4b^2f+9bcd)+9abcf+90ac^2d-4b^3f+9b^2cd)}{15c}-\frac{\sqrt[4]{a}(\sqrt{a}+\sqrt{c}x^2)\sqrt{\frac{a+bx^2+cx^4}{(\sqrt{a}+\sqrt{c}x^2)^2}}\left(-84a^2c^2f+57ab^2cf+\sqrt{a}\sqrt{c}(24abcf-1\ldots\right.}{\ldots}}{\ldots}$$

$$\frac{x(a+bx^2+cx^4)^{3/2}(3(bf+3cd)+7cfx^2)}{63c}$$

$\big\downarrow$ 219

$$\frac{\frac{x\sqrt{a+bx^2+cx^4}(3cx^2(14acf-4b^2f+9bcd)+9abcf+90ac^2d-4b^3f+9b^2cd)}{15c}-\frac{\sqrt[4]{a}(\sqrt{a}+\sqrt{c}x^2)\sqrt{\frac{a+bx^2+cx^4}{(\sqrt{a}+\sqrt{c}x^2)^2}}\left(-84a^2c^2f+57ab^2cf+\sqrt{a}\sqrt{c}(24abcf-1\ldots\right.}{\ldots}}{\ldots}$$

$$\frac{1}{2}\left(\frac{(2ce-bg)\left(\frac{(b+2cx^2)(a+bx^2+cx^4)^{3/2}}{8c}-\frac{3(b^2-4ac)\left(\frac{(b+2cx^2)\sqrt{a+bx^2+cx^4}}{4c}-\frac{(b^2-4ac)\text{arctanh}\left(\frac{b+2cx^2}{2\sqrt{c}\sqrt{a+bx^2+cx^4}}\right)}{8c^{3/2}}\right)}{16c}\right)}{2c}+\frac{g(a+\ldots}{\ldots}\right.$$

$$\frac{x(a+bx^2+cx^4)^{3/2}(3(bf+3cd)+7cfx^2)}{63c}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3)*(a + b*x^2 + c*x^4)^(3/2),x]
```

3.103. $\int (d+ex+fx^2+gx^3)(a+bx^2+cx^4)^{3/2}\,dx$

output
```
(x*(3*(3*c*d + b*f) + 7*c*f*x^2)*(a + b*x^2 + c*x^4)^(3/2))/(63*c) + ((g*(
a + b*x^2 + c*x^4)^(5/2))/(5*c) + ((2*c*e - b*g)*(((b + 2*c*x^2)*(a + b*x^
2 + c*x^4)^(3/2))/(8*c) - (3*(b^2 - 4*a*c)*(((b + 2*c*x^2)*Sqrt[a + b*x^2
+ c*x^4])/(4*c) - ((b^2 - 4*a*c)*ArcTanh[(b + 2*c*x^2)/(2*Sqrt[c]*Sqrt[a +
 b*x^2 + c*x^4])])/(8*c^(3/2))))/(16*c)))/(2*c))/2 + ((x*(9*b^2*c*d + 90*a
*c^2*d - 4*b^3*f + 9*a*b*c*f + 3*c*(9*b*c*d - 4*b^2*f + 14*a*c*f)*x^2)*Sqr
t[a + b*x^2 + c*x^4])/(15*c) - (-(((18*b^3*c*d - 144*a*b*c^2*d - 8*b^4*f +
 57*a*b^2*c*f - 84*a^2*c^2*f)*(-((x*Sqrt[a + b*x^2 + c*x^4])/(Sqrt[a] + Sq
rt[c]*x^2)) + (a^(1/4)*(Sqrt[a] + Sqrt[c]*x^2)*Sqrt[(a + b*x^2 + c*x^4)/(S
qrt[a] + Sqrt[c]*x^2)^2]*EllipticE[2*ArcTan[(c^(1/4)*x)/a^(1/4)], (2 - b/(
Sqrt[a]*Sqrt[c]))/4])/(c^(1/4)*Sqrt[a + b*x^2 + c*x^4])))/Sqrt[c]) + (a^(1
/4)*(18*b^3*c*d - 144*a*b*c^2*d - 8*b^4*f + 57*a*b^2*c*f - 84*a^2*c^2*f +
Sqrt[a]*Sqrt[c]*(9*b^2*c*d - 180*a*c^2*d - 4*b^3*f + 24*a*b*c*f))*(Sqrt[a]
 + Sqrt[c]*x^2)*Sqrt[(a + b*x^2 + c*x^4)/(Sqrt[a] + Sqrt[c]*x^2)^2]*Ellipt
icF[2*ArcTan[(c^(1/4)*x)/a^(1/4)], (2 - b/(Sqrt[a]*Sqrt[c]))/4])/(2*c^(3/4
)*Sqrt[a + b*x^2 + c*x^4]))/(15*c))/(21*c)
```

3.103.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1087
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^p/(2*c*(2*p + 1))), x] - Simp[p*((b^2 - 4*a*c)/(2*c*(2*
p + 1)))   Int[(a + b*x + c*x^2)^(p - 1), x], x] /; FreeQ[{a, b, c}, x] &&
GtQ[p, 0] && (IntegerQ[4*p] || IntegerQ[3*p])
```

rule 1092
```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Simp[2   Subst[I
nt[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a
, b, c}, x]
```

rule 1160
```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[e*((a + b*x + c*x^2)^(p + 1)/(2*c*(p + 1))), x] + Simp[(2*c*d - b
*e)/(2*c)   Int[(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, p}, x]
 && NeQ[p, -1]
```

rule 1416
```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4], x_Symbol] :> With[{q = Rt[c
/a, 4]}, Simp[(1 + q^2*x^2)*(Sqrt[(a + b*x^2 + c*x^4)/(a*(1 + q^2*x^2)^2)]/
(2*q*Sqrt[a + b*x^2 + c*x^4]))*EllipticF[2*ArcTan[q*x], 1/2 - b*(q^2/(4*c))
], x]] /; FreeQ[{a, b, c}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[c/a]
```

rule 1490
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(2*b*e*p + c*d*(4*p + 3) + c*e*(4*p + 1)*x^2)*((a + b*x^2 + c
*x^4)^p/(c*(4*p + 1)*(4*p + 3))), x] + Simp[2*(p/(c*(4*p + 1)*(4*p + 3)))
 Int[Simp[2*a*c*d*(4*p + 3) - a*b*e + (2*a*c*e*(4*p + 1) + b*c*d*(4*p + 3)
- b^2*e*(2*p + 1))*x^2, x]*(a + b*x^2 + c*x^4)^(p - 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 GtQ[p, 0] && FractionQ[p] && IntegerQ[2*p]
```

rule 1509
```
Int[((d_) + (e_.)*(x_)^2)/Sqrt[(a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4], x_Symbo
l] :> With[{q = Rt[c/a, 4]}, Simp[(-d)*x*(Sqrt[a + b*x^2 + c*x^4]/(a*(1 + q
^2*x^2))), x] + Simp[d*(1 + q^2*x^2)*(Sqrt[(a + b*x^2 + c*x^4)/(a*(1 + q^2*
x^2)^2)]/(q*Sqrt[a + b*x^2 + c*x^4]))*EllipticE[2*ArcTan[q*x], 1/2 - b*(q^2
/(4*c))], x] /; EqQ[e + d*q^2, 0]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2
- 4*a*c, 0] && PosQ[c/a]
```

rule 1511
```
Int[((d_) + (e_.)*(x_)^2)/Sqrt[(a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4], x_Symbo
l] :> With[{q = Rt[c/a, 2]}, Simp[(e + d*q)/q   Int[1/Sqrt[a + b*x^2 + c*x^
4], x], x] - Simp[e/q   Int[(1 - q*x^2)/Sqrt[a + b*x^2 + c*x^4], x], x] /;
NeQ[e + d*q, 0]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && Pos
Q[c/a]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

3.103. $\int (d+ex+fx^2+gx^3)\left(a+bx^2+cx^4\right)^{3/2}\,dx$

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.103.4 Maple [A] (verified)

Time = 5.79 (sec) , antiderivative size = 1209, normalized size of antiderivative = 1.69

method	result	size
risch	Expression too large to display	1209
elliptic	Expression too large to display	1376
default	Expression too large to display	1580

input
```
int((g*x^3+f*x^2+e*x+d)*(c*x^4+b*x^2+a)^(3/2),x,method=_RETURNVERBOSE)
```

output
```
1/80640/c^3*(8064*c^4*g*x^8+8960*c^4*f*x^7+11088*b*c^3*g*x^6+10080*c^4*e*x
^6+12800*b*c^3*f*x^5+11520*c^4*d*x^5+16128*a*c^3*g*x^4+504*b^2*c^2*g*x^4+1
5120*b*c^3*e*x^4+19712*a*c^3*f*x^3+768*b^2*c^2*f*x^3+18432*b*c^3*d*x^3+352
8*a*b*c^2*g*x^2+25200*a*c^3*e*x^2-630*b^3*c*g*x^2+1260*b^2*c^2*e*x^2+6144*
a*b*c^2*f*x+34560*a*c^3*d*x-1024*b^3*c*f*x+2304*b^2*c^2*d*x+8064*a^2*c^2*g
-6300*a*b^2*c*g+12600*a*b*c^2*e+945*b^4*g-1890*b^3*c*e)*(c*x^4+b*x^2+a)^(1
/2)-1/80640/c^3*(-11520*a^2*c^3*d*2^(1/2)/((-b+(-4*a*c+b^2)^(1/2))/a)^(1/2
)*(4-2*(-b+(-4*a*c+b^2)^(1/2))/a*x^2)^(1/2)*(4+2*(b+(-4*a*c+b^2)^(1/2))/a*
x^2)^(1/2)/(c*x^4+b*x^2+a)^(1/2)*EllipticF(1/2*x*2^(1/2)*((-b+(-4*a*c+b^2)
^(1/2))/a)^(1/2),1/2*(-4+2*b*(b+(-4*a*c+b^2)^(1/2))/a/c)^(1/2))+1536*a^2*b
*c^2*f*2^(1/2)/((-b+(-4*a*c+b^2)^(1/2))/a)^(1/2)*(4-2*(-b+(-4*a*c+b^2)^(1/
2))/a*x^2)^(1/2)*(4+2*(b+(-4*a*c+b^2)^(1/2))/a*x^2)^(1/2)/(c*x^4+b*x^2+a)^
(1/2)*EllipticF(1/2*x*2^(1/2)*((-b+(-4*a*c+b^2)^(1/2))/a)^(1/2),1/2*(-4+2*
b*(b+(-4*a*c+b^2)^(1/2))/a/c)^(1/2))-256*a*b^3*c*f*2^(1/2)/((-b+(-4*a*c+b^
2)^(1/2))/a)^(1/2)*(4-2*(-b+(-4*a*c+b^2)^(1/2))/a*x^2)^(1/2)*(4+2*(b+(-4*a
*c+b^2)^(1/2))/a*x^2)^(1/2)/(c*x^4+b*x^2+a)^(1/2)*EllipticF(1/2*x*2^(1/2)*
((-b+(-4*a*c+b^2)^(1/2))/a)^(1/2),1/2*(-4+2*b*(b+(-4*a*c+b^2)^(1/2))/a/c)^
(1/2))+576*a*b^2*c^2*d*2^(1/2)/((-b+(-4*a*c+b^2)^(1/2))/a)^(1/2)*(4-2*(-b+
(-4*a*c+b^2)^(1/2))/a*x^2)^(1/2)*(4+2*(b+(-4*a*c+b^2)^(1/2))/a*x^2)^(1/2)/
(c*x^4+b*x^2+a)^(1/2)*EllipticF(1/2*x*2^(1/2)*((-b+(-4*a*c+b^2)^(1/2))/...
```

3.103.5 Fricas [A] (verification not implemented)

Time = 0.31 (sec) , antiderivative size = 911, normalized size of antiderivative = 1.27

$$\int \left(d+ex+fx^2+gx^3\right)\left(a+bx^2+cx^4\right)^{3/2}\,dx=$$

$$-\frac{512\sqrt{\frac{1}{2}}\left(\left(18\,(b^3c^2-8\,abc^3)d-(8\,b^4c-57\,ab^2c^2+84\,a^2c^3)f\right)x\sqrt{\frac{b^2-4\,ac}{c^2}}-(18\,(b^4c-8\,ab^2c^2)d-(8\,b^5-5\cdots}{\cdots}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)*(c*x^4+b*x^2+a)^(3/2),x, algorithm="fricas")
```

output
```
-1/322560*(512*sqrt(1/2)*((18*(b^3*c^2 - 8*a*b*c^3)*d - (8*b^4*c - 57*a*b^
2*c^2 + 84*a^2*c^3)*f)*x*sqrt((b^2 - 4*a*c)/c^2) - (18*(b^4*c - 8*a*b^2*c^
2)*d - (8*b^5 - 57*a*b^3*c + 84*a^2*b*c^2)*f)*x)*sqrt(c)*sqrt((c*sqrt((b^2
 - 4*a*c)/c^2) - b)/c)*elliptic_e(arcsin(sqrt(1/2)*sqrt((c*sqrt((b^2 - 4*a
*c)/c^2) - b)/c)/x), 1/2*(b*c*sqrt((b^2 - 4*a*c)/c^2) + b^2 - 2*a*c)/(a*c)
) - 512*sqrt(1/2)*((9*(2*b^3*c^2 + 20*a*c^4 - (16*a*b + b^2)*c^3)*d - (8*b
^4*c + 12*(7*a^2 + 2*a*b)*c^3 - (57*a*b^2 + 4*b^3)*c^2)*f)*x*sqrt((b^2 - 4
*a*c)/c^2) - (9*(2*b^4*c - 20*a*b*c^3 - (16*a*b^2 - b^3)*c^2)*d - (8*b^5 +
 12*(7*a^2*b - 2*a*b^2)*c^2 - (57*a*b^3 - 4*b^4)*c)*f)*x)*sqrt(c)*sqrt((c*
sqrt((b^2 - 4*a*c)/c^2) - b)/c)*elliptic_f(arcsin(sqrt(1/2)*sqrt((c*sqrt((
b^2 - 4*a*c)/c^2) - b)/c)/x), 1/2*(b*c*sqrt((b^2 - 4*a*c)/c^2) + b^2 - 2*a
*c)/(a*c)) + 945*(2*(b^4*c - 8*a*b^2*c^2 + 16*a^2*c^3)*e - (b^5 - 8*a*b^3*
c + 16*a^2*b*c^2)*g)*sqrt(c)*x*log(8*c^2*x^4 + 8*b*c*x^2 + b^2 - 4*sqrt(c*
x^4 + b*x^2 + a)*(2*c*x^2 + b)*sqrt(c) + 4*a*c) - 4*(8064*c^5*g*x^9 + 8960
*c^5*f*x^8 + 1008*(10*c^5*e + 11*b*c^4*g)*x^7 + 1280*(9*c^5*d + 10*b*c^4*f
)*x^6 + 504*(30*b*c^4*e + (b^2*c^3 + 32*a*c^4)*g)*x^5 + 256*(72*b*c^4*d +
(3*b^2*c^3 + 77*a*c^4)*f)*x^4 + 126*(10*(b^2*c^3 + 20*a*c^4)*e - (5*b^3*c^
2 - 28*a*b*c^3)*g)*x^3 + 256*(9*(b^2*c^3 + 15*a*c^4)*d - 4*(b^3*c^2 - 6*a*
b*c^3)*f)*x^2 - 4608*(b^3*c^2 - 8*a*b*c^3)*d + 256*(8*b^4*c - 57*a*b^2*c^2
 + 84*a^2*c^3)*f - 63*(10*(3*b^3*c^2 - 20*a*b*c^3)*e - (15*b^4*c - 100*...
```

3.103.6 Sympy [F]

$$\int (d+ex+fx^2+gx^3) (a+bx^2+cx^4)^{3/2} dx = \int (a+bx^2+cx^4)^{\frac{3}{2}} (d+ex+fx^2+gx^3) dx$$

input
```
integrate((g*x**3+f*x**2+e*x+d)*(c*x**4+b*x**2+a)**(3/2),x)
```

output
```
Integral((a + b*x**2 + c*x**4)**(3/2)*(d + e*x + f*x**2 + g*x**3), x)
```

3.103.7 Maxima [F]

$$\int (d+ex+fx^2+gx^3) (a+bx^2+cx^4)^{3/2} dx = \int (cx^4+bx^2+a)^{\frac{3}{2}}(gx^3+fx^2+ex+d) dx$$

input
```
integrate((g*x^3+f*x^2+e*x+d)*(c*x^4+b*x^2+a)^(3/2),x, algorithm="maxima")
```

output
```
integrate((c*x^4 + b*x^2 + a)^(3/2)*(g*x^3 + f*x^2 + e*x + d), x)
```

3.103.8 Giac [F]

$$\int (d+ex+fx^2+gx^3) (a+bx^2+cx^4)^{3/2} dx = \int (cx^4+bx^2+a)^{\frac{3}{2}}(gx^3+fx^2+ex+d) dx$$

input
```
integrate((g*x^3+f*x^2+e*x+d)*(c*x^4+b*x^2+a)^(3/2),x, algorithm="giac")
```

output
```
integrate((c*x^4 + b*x^2 + a)^(3/2)*(g*x^3 + f*x^2 + e*x + d), x)
```

3.103.9 Mupad [F(-1)]

Timed out.

$$\int \left(d+ex+fx^2+gx^3\right) \left(a+bx^2+cx^4\right)^{3/2} \,dx=\int \left(c\,x^4+b\,x^2+a\right)^{3/2}\,\left(g\,x^3+f\,x^2+e\,x+d\right) \,dx$$

input
```
int((a + b*x^2 + c*x^4)^(3/2)*(d + e*x + f*x^2 + g*x^3),x)
```

output
```
int((a + b*x^2 + c*x^4)^(3/2)*(d + e*x + f*x^2 + g*x^3), x)
```

3.104 $\int (d+ex+fx^2+gx^3)\sqrt{a+bx^2+cx^4}\,dx$

3.104.1 Optimal result

Integrand size = 32, antiderivative size = 505

$$\int (d+ex+fx^2+gx^3)\sqrt{a+bx^2+cx^4}\,dx = \frac{(5bcd-2b^2f+6acf)\,x\sqrt{a+bx^2+cx^4}}{15c^{3/2}\left(\sqrt{a}+\sqrt{c}x^2\right)} + \frac{(2ce-bg)\,(b+2cx^2)\sqrt{a+bx^2+cx^4}}{16c^2} + \frac{x(5cd+bf+3cfx^2)\sqrt{a+bx^2+cx^4}}{15c} + \frac{g(a+bx^2+cx^4)^{3/2}}{6c} - \frac{(b^2-4ac)\,(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{2\sqrt{c}\sqrt{a+bx^2+cx^4}}\right)}{32c^{5/2}} - \frac{\sqrt[4]{a}(5bcd-2b^2f+6acf)\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}E\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right)|\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{15c^{7/4}\sqrt{a+bx^2+cx^4}} + \frac{\sqrt[4]{a}\left(b+2\sqrt{a}\sqrt{c}\right)\left(5cd-2bf+3\sqrt{a}\sqrt{c}f\right)\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}\,\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2\right.\right.}{30c^{7/4}\sqrt{a+bx^2+cx^4}}$$

output
```
1/6*g*(c*x^4+b*x^2+a)^(3/2)/c-1/32*(-4*a*c+b^2)*(-b*g+2*c*e)*arctanh(1/2*(
2*c*x^2+b)/c^(1/2)/(c*x^4+b*x^2+a)^(1/2))/c^(5/2)+1/16*(-b*g+2*c*e)*(2*c*x
^2+b)*(c*x^4+b*x^2+a)^(1/2)/c^2+1/15*x*(3*c*f*x^2+b*f+5*c*d)*(c*x^4+b*x^2+
a)^(1/2)/c+1/15*(6*a*c*f-2*b^2*f+5*b*c*d)*x*(c*x^4+b*x^2+a)^(1/2)/c^(3/2)/
(a^(1/2)+x^2*c^(1/2))-1/15*a^(1/4)*(6*a*c*f-2*b^2*f+5*b*c*d)*(cos(2*arctan
(c^(1/4)*x/a^(1/4)))^2)^(1/2)/cos(2*arctan(c^(1/4)*x/a^(1/4)))*EllipticE(s
in(2*arctan(c^(1/4)*x/a^(1/4))),1/2*(2-b/a^(1/2)/c^(1/2))^(1/2))*(a^(1/2)+
x^2*c^(1/2))*((c*x^4+b*x^2+a)/(a^(1/2)+x^2*c^(1/2))^2)^(1/2)/c^(7/4)/(c*x^
4+b*x^2+a)^(1/2)+1/30*a^(1/4)*(cos(2*arctan(c^(1/4)*x/a^(1/4)))^2)^(1/2)/c
os(2*arctan(c^(1/4)*x/a^(1/4)))*EllipticF(sin(2*arctan(c^(1/4)*x/a^(1/4)))
,1/2*(2-b/a^(1/2)/c^(1/2))^(1/2))*(a^(1/2)+x^2*c^(1/2))*(b+2*a^(1/2)*c^(1/
2))*(5*c*d-2*b*f+3*f*a^(1/2)*c^(1/2))*((c*x^4+b*x^2+a)/(a^(1/2)+x^2*c^(1/2
))^2)^(1/2)/c^(7/4)/(c*x^4+b*x^2+a)^(1/2)
```

3.104.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 12.54 (sec) , antiderivative size = 661, normalized size of antiderivative = 1.31

$$\int \left(d+ex+fx^2+gx^3\right)\sqrt{a+bx^2+cx^4}\,dx$$

$$= \frac{2\sqrt{c}(a+bx^2+cx^4)\left(-15b^2g+2bc(15e+x(8f+5gx))+4c(10ag+cx(20d+x(15e+2x(6f+5gx))))\right)}{}$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3)*Sqrt[a + b*x^2 + c*x^4],x]
```

output
```
(2*Sqrt[c]*(a + b*x^2 + c*x^4)*(-15*b^2*g + 2*b*c*(15*e + x*(8*f + 5*g*x))
 + 4*c*(10*a*g + c*x*(20*d + x*(15*e + 2*x*(6*f + 5*g*x))))) + ((-8*I)*Sqr
t[2]*Sqrt[c]*(-b + Sqrt[b^2 - 4*a*c])*(-5*b*c*d + 2*b^2*f - 6*a*c*f)*Sqrt[
(b - Sqrt[b^2 - 4*a*c] + 2*c*x^2)/(b - Sqrt[b^2 - 4*a*c])]*Sqrt[(b + Sqrt[
b^2 - 4*a*c] + 2*c*x^2)/(b + Sqrt[b^2 - 4*a*c])]*EllipticE[I*ArcSinh[Sqrt[
2]*Sqrt[c/(b + Sqrt[b^2 - 4*a*c])]*x], (b + Sqrt[b^2 - 4*a*c])/(b - Sqrt[b
^2 - 4*a*c])] + (8*I)*Sqrt[2]*Sqrt[c]*(-2*b^3*f + b*c*(-5*Sqrt[b^2 - 4*a*c
]*d + 8*a*f) + b^2*(5*c*d + 2*Sqrt[b^2 - 4*a*c]*f) - 2*a*c*(10*c*d + 3*Sqr
t[b^2 - 4*a*c]*f))*Sqrt[(b - Sqrt[b^2 - 4*a*c] + 2*c*x^2)/(b - Sqrt[b^2 -
4*a*c])]*Sqrt[(b + Sqrt[b^2 - 4*a*c] + 2*c*x^2)/(b + Sqrt[b^2 - 4*a*c])]*E
llipticF[I*ArcSinh[Sqrt[2]*Sqrt[c/(b + Sqrt[b^2 - 4*a*c])]*x], (b + Sqrt[b
^2 - 4*a*c])/(b - Sqrt[b^2 - 4*a*c])] - 15*(b^2 - 4*a*c)*Sqrt[c/(b + Sqrt[
b^2 - 4*a*c])]*(-2*c*e + b*g)*Sqrt[a + b*x^2 + c*x^4]*Log[b + 2*c*x^2 - 2*
Sqrt[c]*Sqrt[a + b*x^2 + c*x^4]])/Sqrt[c/(b + Sqrt[b^2 - 4*a*c])])/(480*c^
(5/2)*Sqrt[a + b*x^2 + c*x^4])
```

3.104.3 Rubi [A] (verified)

Time = 0.79 (sec) , antiderivative size = 498, normalized size of antiderivative = 0.99, number of steps used = 12, number of rules used = 11, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.344, Rules used = {2202, 1490, 1511, 27, 1416, 1509, 1576, 1160, 1087, 1092, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \sqrt{a+bx^2+cx^4}\left(d+ex+fx^2+gx^3\right)dx$$

$$\Big\downarrow \text{2202}$$

$$\int \left(fx^2+d\right)\sqrt{cx^4+bx^2+a}dx+\int x\left(gx^2+e\right)\sqrt{cx^4+bx^2+a}dx$$

$$\Big\downarrow \text{1490}$$

$$\frac{\int \frac{\left(-2fb^2+5cdb+6acf\right)x^2+a(10cd-bf)}{\sqrt{cx^4+bx^2+a}}dx}{15c}+\int x\left(gx^2+e\right)\sqrt{cx^4+bx^2+a}dx+\frac{x\sqrt{a+bx^2+cx^4}\left(bf+5cd+3cfx^2\right)}{15c}$$

$$\Big\downarrow \text{1511}$$

$$\frac{\frac{\sqrt{a}(2\sqrt{a}\sqrt{c}+b)(3\sqrt{a}\sqrt{c}f-2bf+5cd)\int\frac{1}{\sqrt{cx^4+bx^2+a}}dx}{\sqrt{c}}-\frac{\sqrt{a}(6acf-2b^2f+5bcd)\int\frac{\sqrt{a}-\sqrt{c}x^2}{\sqrt{a}\sqrt{cx^4+bx^2+a}}dx}{\sqrt{c}}}{15c}+$$

$$\int x\left(gx^2+e\right)\sqrt{cx^4+bx^2+a}dx+\frac{x\sqrt{a+bx^2+cx^4}\left(bf+5cd+3cfx^2\right)}{15c}$$

$$\Big\downarrow \; 27$$

$$\frac{\frac{\sqrt{a}(2\sqrt{a}\sqrt{c}+b)(3\sqrt{a}\sqrt{c}f-2bf+5cd)\int\frac{1}{\sqrt{cx^4+bx^2+a}}dx}{\sqrt{c}}-\frac{(6acf-2b^2f+5bcd)\int\frac{\sqrt{a}-\sqrt{c}x^2}{\sqrt{cx^4+bx^2+a}}dx}{\sqrt{c}}}{15c}+$$

$$\int x\left(gx^2+e\right)\sqrt{cx^4+bx^2+a}dx+\frac{x\sqrt{a+bx^2+cx^4}\left(bf+5cd+3cfx^2\right)}{15c}$$

$$\Big\downarrow \; 1416$$

$$\frac{\frac{\sqrt[4]{a}(2\sqrt{a}\sqrt{c}+b)(\sqrt{a}+\sqrt{c}x^2)\sqrt{\frac{a+bx^2+cx^4}{(\sqrt{a}+\sqrt{c}x^2)^2}}(3\sqrt{a}\sqrt{c}f-2bf+5cd)\,\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2c^{3/4}\sqrt{a+bx^2+cx^4}}-\frac{(6acf-2b^2f+5bcd)\int\frac{\sqrt{a}-\sqrt{c}x^2}{\sqrt{cx^4+b\dots}}}{\sqrt{c}}}{15c}$$

$$\int x\left(gx^2+e\right)\sqrt{cx^4+bx^2+a}dx+\frac{x\sqrt{a+bx^2+cx^4}\left(bf+5cd+3cfx^2\right)}{15c}$$

$$\Big\downarrow \; 1509$$

$$\int x\left(gx^2+e\right)\sqrt{cx^4+bx^2+a}dx+$$

$$\frac{\frac{\sqrt[4]{a}(2\sqrt{a}\sqrt{c}+b)(\sqrt{a}+\sqrt{c}x^2)\sqrt{\frac{a+bx^2+cx^4}{(\sqrt{a}+\sqrt{c}x^2)^2}}(3\sqrt{a}\sqrt{c}f-2bf+5cd)\,\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2c^{3/4}\sqrt{a+bx^2+cx^4}}-\frac{(6acf-2b^2f+5bcd)\left(\frac{\sqrt[4]{a}(\dots}{\dots}\right.}{\dots}}{15c}$$

$$\frac{x\sqrt{a+bx^2+cx^4}\left(bf+5cd+3cfx^2\right)}{15c}$$

$$\Big\downarrow \; 1576$$

$$\frac{1}{2}\int\left(gx^2+e\right)\sqrt{cx^4+bx^2+a}dx^2+$$

$$\frac{\frac{\sqrt[4]{a}(2\sqrt{a}\sqrt{c}+b)(\sqrt{a}+\sqrt{c}x^2)\sqrt{\frac{a+bx^2+cx^4}{(\sqrt{a}+\sqrt{c}x^2)^2}}(3\sqrt{a}\sqrt{c}f-2bf+5cd)\,\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2c^{3/4}\sqrt{a+bx^2+cx^4}}-\frac{(6acf-2b^2f+5bcd)\left(\frac{\sqrt[4]{a}(\dots}{\dots}\right.}{\dots}}{15c}$$

$$\frac{x\sqrt{a+bx^2+cx^4}\left(bf+5cd+3cfx^2\right)}{15c}$$

$$\Big\downarrow \; 1160$$

3.104. $\int\left(d+ex+fx^2+gx^3\right)\sqrt{a+bx^2+cx^4}\,dx$

$$\frac{1}{2}\left(\frac{(2ce-bg)\int \sqrt{cx^4+bx^2+a}dx^2}{2c}+\frac{g\left(a+bx^2+cx^4\right)^{3/2}}{3c}\right)+$$

$$\frac{\frac{\sqrt[4]{a}(2\sqrt{a}\sqrt{c}+b)\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}(3\sqrt{a}\sqrt{c}f-2bf+5cd)\,\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2c^{3/4}\sqrt{a+bx^2+cx^4}}-\frac{(6acf-2b^2f+5bcd)\left(\frac{\sqrt[4]{a}(\sqrt{}\right.}{}}{15c}$$

$$\frac{x\sqrt{a+bx^2+cx^4}\left(bf+5cd+3cfx^2\right)}{15c}$$

$$\Big\downarrow \text{1087}$$

$$\frac{1}{2}\left(\frac{(2ce-bg)\left(\frac{(b+2cx^2)\sqrt{a+bx^2+cx^4}}{4c}-\frac{(b^2-4ac)\int \frac{1}{\sqrt{cx^4+bx^2+a}}dx^2}{8c}\right)}{2c}+\frac{g\left(a+bx^2+cx^4\right)^{3/2}}{3c}\right)+$$

$$\frac{\frac{\sqrt[4]{a}(2\sqrt{a}\sqrt{c}+b)\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}(3\sqrt{a}\sqrt{c}f-2bf+5cd)\,\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2c^{3/4}\sqrt{a+bx^2+cx^4}}-\frac{(6acf-2b^2f+5bcd)\left(\frac{\sqrt[4]{a}(\sqrt{}\right.}{}}{15c}$$

$$\frac{x\sqrt{a+bx^2+cx^4}\left(bf+5cd+3cfx^2\right)}{15c}$$

$$\Big\downarrow \text{1092}$$

$$\frac{1}{2}\left(\frac{(2ce-bg)\left(\frac{(b+2cx^2)\sqrt{a+bx^2+cx^4}}{4c}-\frac{(b^2-4ac)\int \frac{1}{4c-x^4}d\frac{2cx^2+b}{\sqrt{cx^4+bx^2+a}}}{4c}\right)}{2c}+\frac{g\left(a+bx^2+cx^4\right)^{3/2}}{3c}\right)+$$

$$\frac{\frac{\sqrt[4]{a}(2\sqrt{a}\sqrt{c}+b)\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}(3\sqrt{a}\sqrt{c}f-2bf+5cd)\,\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2c^{3/4}\sqrt{a+bx^2+cx^4}}-\frac{(6acf-2b^2f+5bcd)\left(\frac{\sqrt[4]{a}(\sqrt{}\right.}{}}{15c}$$

$$\frac{x\sqrt{a+bx^2+cx^4}\left(bf+5cd+3cfx^2\right)}{15c}$$

$$\Big\downarrow \text{219}$$

3.104. $\int \left(d+ex+fx^2+gx^3\right)\sqrt{a+bx^2+cx^4}\,dx$

$$\frac{\frac{\sqrt[4]{a}(2\sqrt{a}\sqrt{c}+b)(\sqrt{a}+\sqrt{c}x^2)\sqrt{\frac{a+bx^2+cx^4}{(\sqrt{a}+\sqrt{c}x^2)^2}}(3\sqrt{a}\sqrt{c}f-2bf+5cd)\,\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2c^{3/4}\sqrt{a+bx^2+cx^4}} - \frac{(6acf-2b^2f+5bcd)\left(\frac{\sqrt[4]{a}(\sqrt{}\cdots}{}\right.}{}}{15c}$$

$$\frac{1}{2}\left(\frac{(2ce-bg)\left(\frac{(b+2cx^2)\sqrt{a+bx^2+cx^4}}{4c}-\frac{(b^2-4ac)\,\text{arctanh}\left(\frac{b+2cx^2}{2\sqrt{c}\sqrt{a+bx^2+cx^4}}\right)}{8c^{3/2}}\right)}{2c}+\frac{g(a+bx^2+cx^4)^{3/2}}{3c}\right)+$$

$$\frac{x\sqrt{a+bx^2+cx^4}(bf+5cd+3cfx^2)}{15c}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3)*Sqrt[a + b*x^2 + c*x^4],x]
```

output
```
(x*(5*c*d + b*f + 3*c*f*x^2)*Sqrt[a + b*x^2 + c*x^4])/(15*c) + ((g*(a + b*
x^2 + c*x^4)^(3/2))/(3*c) + ((2*c*e - b*g)*(((b + 2*c*x^2)*Sqrt[a + b*x^2
+ c*x^4])/(4*c) - ((b^2 - 4*a*c)*ArcTanh[(b + 2*c*x^2)/(2*Sqrt[c]*Sqrt[a +
 b*x^2 + c*x^4])])/(8*c^(3/2))))/(2*c))/2 + (-(((5*b*c*d - 2*b^2*f + 6*a*c
*f)*(-((x*Sqrt[a + b*x^2 + c*x^4])/(Sqrt[a] + Sqrt[c]*x^2)) + (a^(1/4)*(Sq
rt[a] + Sqrt[c]*x^2)*Sqrt[(a + b*x^2 + c*x^4)/(Sqrt[a] + Sqrt[c]*x^2)^2]*E
llipticE[2*ArcTan[(c^(1/4)*x)/a^(1/4)], (2 - b/(Sqrt[a]*Sqrt[c]))/4])/(c^(
1/4)*Sqrt[a + b*x^2 + c*x^4])))/Sqrt[c]) + (a^(1/4)*(b + 2*Sqrt[a]*Sqrt[c]
)*(5*c*d - 2*b*f + 3*Sqrt[a]*Sqrt[c]*f)*(Sqrt[a] + Sqrt[c]*x^2)*Sqrt[(a +
b*x^2 + c*x^4)/(Sqrt[a] + Sqrt[c]*x^2)^2]*EllipticF[2*ArcTan[(c^(1/4)*x)/a
^(1/4)], (2 - b/(Sqrt[a]*Sqrt[c]))/4])/(2*c^(3/4)*Sqrt[a + b*x^2 + c*x^4])
)/(15*c)
```

3.104.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1087
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^p/(2*c*(2*p + 1))), x] - Simp[p*((b^2 - 4*a*c)/(2*c*(2*
p + 1)))   Int[(a + b*x + c*x^2)^(p - 1), x], x] /; FreeQ[{a, b, c}, x] &&
GtQ[p, 0] && (IntegerQ[4*p] || IntegerQ[3*p])
```

rule 1092
```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Simp[2   Subst[I
nt[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a
, b, c}, x]
```

rule 1160
```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[e*((a + b*x + c*x^2)^(p + 1)/(2*c*(p + 1))), x] + Simp[(2*c*d - b
*e)/(2*c)   Int[(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, p}, x]
 && NeQ[p, -1]
```

rule 1416
```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4], x_Symbol] :> With[{q = Rt[c
/a, 4]}, Simp[(1 + q^2*x^2)*(Sqrt[(a + b*x^2 + c*x^4)/(a*(1 + q^2*x^2)^2)]/
(2*q*Sqrt[a + b*x^2 + c*x^4]))*EllipticF[2*ArcTan[q*x], 1/2 - b*(q^2/(4*c))
], x]] /; FreeQ[{a, b, c}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[c/a]
```

rule 1490
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(2*b*e*p + c*d*(4*p + 3) + c*e*(4*p + 1)*x^2)*((a + b*x^2 + c
*x^4)^p/(c*(4*p + 1)*(4*p + 3))), x] + Simp[2*(p/(c*(4*p + 1)*(4*p + 3)))
 Int[Simp[2*a*c*d*(4*p + 3) - a*b*e + (2*a*c*e*(4*p + 1) + b*c*d*(4*p + 3)
- b^2*e*(2*p + 1))*x^2, x]*(a + b*x^2 + c*x^4)^(p - 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 GtQ[p, 0] && FractionQ[p] && IntegerQ[2*p]
```

rule 1509
```
Int[((d_) + (e_.)*(x_)^2)/Sqrt[(a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4], x_Symbo
l] :> With[{q = Rt[c/a, 4]}, Simp[(-d)*x*(Sqrt[a + b*x^2 + c*x^4]/(a*(1 + q
^2*x^2))), x] + Simp[d*(1 + q^2*x^2)*(Sqrt[(a + b*x^2 + c*x^4)/(a*(1 + q^2*
x^2)^2)]/(q*Sqrt[a + b*x^2 + c*x^4]))*EllipticE[2*ArcTan[q*x], 1/2 - b*(q^2
/(4*c))], x] /; EqQ[e + d*q^2, 0]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2
- 4*a*c, 0] && PosQ[c/a]
```

3.104. $\int (d + ex + fx^2 + gx^3) \sqrt{a + bx^2 + cx^4}\, dx$

rule 1511
```
Int[((d_) + (e_.)*(x_)^2)/Sqrt[(a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4], x_Symbo
l] :> With[{q = Rt[c/a, 2]}, Simp[(e + d*q)/q   Int[1/Sqrt[a + b*x^2 + c*x^
4], x], x] - Simp[e/q   Int[(1 - q*x^2)/Sqrt[a + b*x^2 + c*x^4], x], x] /;
NeQ[e + d*q, 0]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && Pos
Q[c/a]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.104.4 Maple [A] (verified)

Time = 5.34 (sec) , antiderivative size = 620, normalized size of antiderivative = 1.23

method	result
elliptic	$\frac{g\,x^4\sqrt{c\,x^4+b\,x^2+a}}{6}+\frac{f\,x^3\sqrt{c\,x^4+b\,x^2+a}}{5}+\frac{\left(\frac{bg}{6}+ec\right)x^2\sqrt{c\,x^4+b\,x^2+a}}{4c}+\frac{\left(\frac{bf}{5}+cd\right)x\sqrt{c\,x^4+b\,x^2+a}}{3c}+\frac{\left(\frac{ag}{3}+be-\frac{3\left(\frac{bg}{6}+ec\right)b}{4c}\right)}{2c}$
risch	$\frac{\left(40g\,x^4c^2+48f\,x^3c^2+10bcg\,x^2+60c^2e\,x^2+16bfxc+80c^2dx+40acg-15b^2g+30ebc\right)\sqrt{c\,x^4+b\,x^2+a}}{240c^2}-\frac{40a\,c^2d\sqrt{2}\sqrt{4-\frac{2\left(-b+\sqrt{-4ac+}\right.}{a}}}{}$
default	$d\left(\frac{x\sqrt{c\,x^4+b\,x^2+a}}{3}+\frac{a\sqrt{2}\sqrt{4-\frac{2\left(-b+\sqrt{-4ac+b^2}\right)x^2}{a}}\sqrt{4+\frac{2\left(b+\sqrt{-4ac+b^2}\right)x^2}{a}}\,F\left(\frac{x\sqrt{2}\sqrt{\frac{-b+\sqrt{-4ac+b^2}}{a}}}{2},\frac{\sqrt{-4+\frac{2b\left(b+\sqrt{-4ac+b^2}\right)}{ac}}}{2}\right)}{6\sqrt{\frac{-b+\sqrt{-4ac+b^2}}{a}}\sqrt{c\,x^4+b\,x^2+a}}\right.$

input
```
int((g*x^3+f*x^2+e*x+d)*(c*x^4+b*x^2+a)^(1/2),x,method=_RETURNVERBOSE)
```

output
```
1/6*g*x^4*(c*x^4+b*x^2+a)^(1/2)+1/5*f*x^3*(c*x^4+b*x^2+a)^(1/2)+1/4*(1/6*b
*g+e*c)/c*x^2*(c*x^4+b*x^2+a)^(1/2)+1/3*(1/5*b*f+c*d)/c*x*(c*x^4+b*x^2+a)^
(1/2)+1/2*(1/3*a*g+b*e-3/4*(1/6*b*g+e*c)/c*b)/c*(c*x^4+b*x^2+a)^(1/2)+1/4*
(d*a-1/3*(1/5*b*f+c*d)/c*a)*2^(1/2)/((-b+(-4*a*c+b^2)^(1/2))/a)^(1/2)*(4-2
*(-b+(-4*a*c+b^2)^(1/2))/a*x^2)^(1/2)*(4+2*(b+(-4*a*c+b^2)^(1/2))/a*x^2)^(
1/2)/(c*x^4+b*x^2+a)^(1/2)*EllipticF(1/2*x*2^(1/2)*((-b+(-4*a*c+b^2)^(1/2)
)/a)^(1/2),1/2*(-4+2*b*(b+(-4*a*c+b^2)^(1/2))/a/c)^(1/2))+1/2*(a*e-1/2*(1/
6*b*g+e*c)/c*a-1/2*(1/3*a*g+b*e-3/4*(1/6*b*g+e*c)/c*b)/c*b)*ln((2*c*x^2+b)
/c^(1/2)+2*(c*x^4+b*x^2+a)^(1/2))/c^(1/2)-1/2*(2/5*a*f+b*d-2/3*(1/5*b*f+c*
d)/c*b)*a*2^(1/2)/((-b+(-4*a*c+b^2)^(1/2))/a)^(1/2)*(4-2*(-b+(-4*a*c+b^2)^
(1/2))/a*x^2)^(1/2)*(4+2*(b+(-4*a*c+b^2)^(1/2))/a*x^2)^(1/2)/(c*x^4+b*x^2+
a)^(1/2)/(b+(-4*a*c+b^2)^(1/2))*(EllipticF(1/2*x*2^(1/2)*((-b+(-4*a*c+b^2)
^(1/2))/a)^(1/2),1/2*(-4+2*b*(b+(-4*a*c+b^2)^(1/2))/a/c)^(1/2))-EllipticE(
1/2*x*2^(1/2)*((-b+(-4*a*c+b^2)^(1/2))/a)^(1/2),1/2*(-4+2*b*(b+(-4*a*c+b^2
)^(1/2))/a/c)^(1/2)))
```

3.104.5 Fricas [A] (verification not implemented)

Time = 0.26 (sec) , antiderivative size = 574, normalized size of antiderivative = 1.14

$$\int \left(d+ex+fx^2+gx^3\right)\sqrt{a+bx^2+cx^4}\,dx$$

$$= \frac{32\sqrt{\frac{1}{2}}\left(\left(5\,bc^2d-2\,(b^2c-3\,ac^2)f\right)x\sqrt{\frac{b^2-4\,ac}{c^2}}-\left(5\,b^2cd-2\,(b^3-3\,abc)f\right)x\right)\sqrt{c}\sqrt{\frac{c\sqrt{\frac{b^2-4\,ac}{c^2}}-b}{c}}E\left(\arcsin\left(\sqrt{\cdots}\right.\right.}{\cdots}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)*(c*x^4+b*x^2+a)^(1/2),x, algorithm="fricas")
```

output
```
1/960*(32*sqrt(1/2)*((5*b*c^2*d - 2*(b^2*c - 3*a*c^2)*f)*x*sqrt((b^2 - 4*a
*c)/c^2) - (5*b^2*c*d - 2*(b^3 - 3*a*b*c)*f)*x)*sqrt(c)*sqrt((c*sqrt((b^2
- 4*a*c)/c^2) - b)/c)*elliptic_e(arcsin(sqrt(1/2)*sqrt((c*sqrt((b^2 - 4*a*
c)/c^2) - b)/c)/x), 1/2*(b*c*sqrt((b^2 - 4*a*c)/c^2) + b^2 - 2*a*c)/(a*c))
 - 32*sqrt(1/2)*((5*(b*c^2 - 2*c^3)*d - (2*b^2*c - (6*a + b)*c^2)*f)*x*sqr
t((b^2 - 4*a*c)/c^2) - (5*(b^2*c + 2*b*c^2)*d - (2*b^3 - (6*a*b - b^2)*c)*
f)*x)*sqrt(c)*sqrt((c*sqrt((b^2 - 4*a*c)/c^2) - b)/c)*elliptic_f(arcsin(sq
rt(1/2)*sqrt((c*sqrt((b^2 - 4*a*c)/c^2) - b)/c)/x), 1/2*(b*c*sqrt((b^2 - 4
*a*c)/c^2) + b^2 - 2*a*c)/(a*c)) + 15*(2*(b^2*c - 4*a*c^2)*e - (b^3 - 4*a*
b*c)*g)*sqrt(c)*x*log(8*c^2*x^4 + 8*b*c*x^2 + b^2 - 4*sqrt(c*x^4 + b*x^2 +
 a)*(2*c*x^2 + b)*sqrt(c) + 4*a*c) + 4*(40*c^3*g*x^5 + 48*c^3*f*x^4 + 80*b
*c^2*d + 10*(6*c^3*e + b*c^2*g)*x^3 + 16*(5*c^3*d + b*c^2*f)*x^2 - 32*(b^2
*c - 3*a*c^2)*f + 5*(6*b*c^2*e - (3*b^2*c - 8*a*c^2)*g)*x)*sqrt(c*x^4 + b*
x^2 + a))/(c^3*x)
```

3.104.6 Sympy [F]

$$\int (d+ex+fx^2+gx^3)\sqrt{a+bx^2+cx^4}\,dx = \int \sqrt{a+bx^2+cx^4}\,(d+ex+fx^2+gx^3)\;dx$$

input
```
integrate((g*x**3+f*x**2+e*x+d)*(c*x**4+b*x**2+a)**(1/2),x)
```

output
```
Integral(sqrt(a + b*x**2 + c*x**4)*(d + e*x + f*x**2 + g*x**3), x)
```

3.104.7 Maxima [F]

$$\int (d+ex+fx^2+gx^3)\sqrt{a+bx^2+cx^4}\,dx = \int \sqrt{cx^4+bx^2+a}\,(gx^3+fx^2+ex+d)\,dx$$

input
```
integrate((g*x^3+f*x^2+e*x+d)*(c*x^4+b*x^2+a)^(1/2),x, algorithm="maxima")
```

output
```
integrate(sqrt(c*x^4 + b*x^2 + a)*(g*x^3 + f*x^2 + e*x + d), x)
```

3.104.8 Giac [F]

$$\int \left(d+ex+fx^2+gx^3\right)\sqrt{a+bx^2+cx^4}\,dx=\int \sqrt{cx^4+bx^2+a}\left(gx^3+fx^2+ex+d\right)dx$$

input
```
integrate((g*x^3+f*x^2+e*x+d)*(c*x^4+b*x^2+a)^(1/2),x, algorithm="giac")
```

output
```
integrate(sqrt(c*x^4 + b*x^2 + a)*(g*x^3 + f*x^2 + e*x + d), x)
```

3.104.9 Mupad [F(-1)]

Timed out.

$$\int \left(d+ex+fx^2+gx^3\right)\sqrt{a+bx^2+cx^4}\,dx=\int \sqrt{c\,x^4+b\,x^2+a}\,\left(g\,x^3+f\,x^2+e\,x+d\right)\,dx$$

input
```
int((a + b*x^2 + c*x^4)^(1/2)*(d + e*x + f*x^2 + g*x^3),x)
```

output
```
int((a + b*x^2 + c*x^4)^(1/2)*(d + e*x + f*x^2 + g*x^3), x)
```

3.105 $\int \frac{d+ex+fx^2+gx^3}{\sqrt{a+bx^2+cx^4}}\,dx$

3.105.1 Optimal result

Integrand size = 32, antiderivative size = 359

$$\int \frac{d+ex+fx^2+gx^3}{\sqrt{a+bx^2+cx^4}}\,dx = \frac{g\sqrt{a+bx^2+cx^4}}{2c}+\frac{fx\sqrt{a+bx^2+cx^4}}{\sqrt{c}\left(\sqrt{a}+\sqrt{c}x^2\right)}+\frac{(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{2\sqrt{c}\sqrt{a+bx^2+cx^4}}\right)}{4c^{3/2}} - \frac{\sqrt[4]{a}f\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}E\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right)|\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{c^{3/4}\sqrt{a+bx^2+cx^4}} + \frac{\sqrt[4]{a}\left(\frac{\sqrt{c}d}{\sqrt{a}}+f\right)\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}\,\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2c^{3/4}\sqrt{a+bx^2+cx^4}}$$

output
```
1/4*(-b*g+2*c*e)*arctanh(1/2*(2*c*x^2+b)/c^(1/2)/(c*x^4+b*x^2+a)^(1/2))/c^
(3/2)+1/2*g*(c*x^4+b*x^2+a)^(1/2)/c+f*x*(c*x^4+b*x^2+a)^(1/2)/c^(1/2)/(a^(
1/2)+x^2*c^(1/2))-a^(1/4)*f*(cos(2*arctan(c^(1/4)*x/a^(1/4)))^2)^(1/2)/cos
(2*arctan(c^(1/4)*x/a^(1/4)))*EllipticE(sin(2*arctan(c^(1/4)*x/a^(1/4))),1
/2*(2-b/a^(1/2)/c^(1/2))^(1/2))*(a^(1/2)+x^2*c^(1/2))*((c*x^4+b*x^2+a)/(a^
(1/2)+x^2*c^(1/2))^2)^(1/2)/c^(3/4)/(c*x^4+b*x^2+a)^(1/2)+1/2*a^(1/4)*(cos
(2*arctan(c^(1/4)*x/a^(1/4)))^2)^(1/2)/cos(2*arctan(c^(1/4)*x/a^(1/4)))*El
lipticF(sin(2*arctan(c^(1/4)*x/a^(1/4))),1/2*(2-b/a^(1/2)/c^(1/2))^(1/2))*
(a^(1/2)+x^2*c^(1/2))*(f+d*c^(1/2)/a^(1/2))*((c*x^4+b*x^2+a)/(a^(1/2)+x^2*
c^(1/2))^2)^(1/2)/c^(3/4)/(c*x^4+b*x^2+a)^(1/2)
```

3.105.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 10.87 (sec) , antiderivative size = 525, normalized size of antiderivative = 1.46

$$\int \frac{d+ex+fx^2+gx^3}{\sqrt{a+bx^2+cx^4}}\,dx$$
$$= \frac{i\sqrt{2}\sqrt{c}\left(-b+\sqrt{b^2-4ac}\right) f\sqrt{\frac{b-\sqrt{b^2-4ac}+2cx^2}{b-\sqrt{b^2-4ac}}}\sqrt{\frac{b+\sqrt{b^2-4ac}+2cx^2}{b+\sqrt{b^2-4ac}}}E\left(i\text{arcsinh}\left(\sqrt{2}\sqrt{\frac{c}{b+\sqrt{b^2-4ac}}}x\right)|\frac{b+\sqrt{b^2-4ac}}{b-\sqrt{b^2-4ac}}\right)}{}$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3)/Sqrt[a + b*x^2 + c*x^4],x]
```

output
```
(I*Sqrt[2]*Sqrt[c]*(-b + Sqrt[b^2 - 4*a*c])*f*Sqrt[(b - Sqrt[b^2 - 4*a*c]
+ 2*c*x^2)/(b - Sqrt[b^2 - 4*a*c])]*Sqrt[(b + Sqrt[b^2 - 4*a*c] + 2*c*x^2)
/(b + Sqrt[b^2 - 4*a*c])]*EllipticE[I*ArcSinh[Sqrt[2]*Sqrt[c/(b + Sqrt[b^2
 - 4*a*c])]*x], (b + Sqrt[b^2 - 4*a*c])/(b - Sqrt[b^2 - 4*a*c])] - I*Sqrt[
2]*Sqrt[c]*(2*c*d + (-b + Sqrt[b^2 - 4*a*c])*f)*Sqrt[(b - Sqrt[b^2 - 4*a*c
] + 2*c*x^2)/(b - Sqrt[b^2 - 4*a*c])]*Sqrt[(b + Sqrt[b^2 - 4*a*c] + 2*c*x^
2)/(b + Sqrt[b^2 - 4*a*c])]*EllipticF[I*ArcSinh[Sqrt[2]*Sqrt[c/(b + Sqrt[b
^2 - 4*a*c])]*x], (b + Sqrt[b^2 - 4*a*c])/(b - Sqrt[b^2 - 4*a*c])] + Sqrt[
c/(b + Sqrt[b^2 - 4*a*c])]*(2*Sqrt[c]*g*(a + b*x^2 + c*x^4) + (-2*c*e + b*
g)*Sqrt[a + b*x^2 + c*x^4]*Log[b + 2*c*x^2 - 2*Sqrt[c]*Sqrt[a + b*x^2 + c*
x^4]]))/(4*c^(3/2)*Sqrt[c/(b + Sqrt[b^2 - 4*a*c])]*Sqrt[a + b*x^2 + c*x^4]
)
```

3.105.3 Rubi [A] (verified)

Time = 0.55 (sec) , antiderivative size = 363, normalized size of antiderivative = 1.01, number of steps used = 10, number of rules used = 9, $\frac{\text{number of rules}}{\text{integrand size}} = 0.281$, Rules used = {2202, 1511, 27, 1416, 1509, 1576, 1160, 1092, 219}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3}{\sqrt{a+bx^2+cx^4}}\,dx$$
$$\Big\downarrow \text{2202}$$

$$\int \frac{fx^2+d}{\sqrt{cx^4+bx^2+a}}dx+\int \frac{x(gx^2+e)}{\sqrt{cx^4+bx^2+a}}dx$$

$$\Big\downarrow \text{1511}$$

$$\left(\frac{\sqrt{a}f}{\sqrt{c}}+d\right)\int \frac{1}{\sqrt{cx^4+bx^2+a}}dx+\int \frac{x(gx^2+e)}{\sqrt{cx^4+bx^2+a}}dx-\frac{\sqrt{a}f\int \frac{\sqrt{a}-\sqrt{c}x^2}{\sqrt{a}\sqrt{cx^4+bx^2+a}}dx}{\sqrt{c}}$$

$$\Big\downarrow \text{27}$$

$$\left(\frac{\sqrt{a}f}{\sqrt{c}}+d\right)\int \frac{1}{\sqrt{cx^4+bx^2+a}}dx+\int \frac{x(gx^2+e)}{\sqrt{cx^4+bx^2+a}}dx-\frac{f\int \frac{\sqrt{a}-\sqrt{c}x^2}{\sqrt{cx^4+bx^2+a}}dx}{\sqrt{c}}$$

$$\Big\downarrow \text{1416}$$

$$\int \frac{x(gx^2+e)}{\sqrt{cx^4+bx^2+a}}dx-\frac{f\int \frac{\sqrt{a}-\sqrt{c}x^2}{\sqrt{cx^4+bx^2+a}}dx}{\sqrt{c}}+\frac{\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}\left(\frac{\sqrt{a}f}{\sqrt{c}}+d\right)\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2\sqrt[4]{a}\sqrt[4]{c}\sqrt{a+bx^2+cx^4}}$$

$$\Big\downarrow \text{1509}$$

$$\int \frac{x(gx^2+e)}{\sqrt{cx^4+bx^2+a}}dx+\frac{\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}\left(\frac{\sqrt{a}f}{\sqrt{c}}+d\right)\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2\sqrt[4]{a}\sqrt[4]{c}\sqrt{a+bx^2+cx^4}}-\frac{f\left(\frac{\sqrt[4]{a}\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}E\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right)|\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{\sqrt[4]{c}\sqrt{a+bx^2+cx^4}}-\frac{x\sqrt{a+bx^2+cx^4}}{\sqrt{a}+\sqrt{c}x^2}\right)}{\sqrt{c}}$$

$$\Big\downarrow \text{1576}$$

$$\frac{1}{2}\int \frac{gx^2+e}{\sqrt{cx^4+bx^2+a}}dx^2+\frac{\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}\left(\frac{\sqrt{a}f}{\sqrt{c}}+d\right)\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2\sqrt[4]{a}\sqrt[4]{c}\sqrt{a+bx^2+cx^4}}-\frac{f\left(\frac{\sqrt[4]{a}\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}E\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right)|\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{\sqrt[4]{c}\sqrt{a+bx^2+cx^4}}-\frac{x\sqrt{a+bx^2+cx^4}}{\sqrt{a}+\sqrt{c}x^2}\right)}{\sqrt{c}}$$

$$\Big\downarrow \text{1160}$$

3.105. $\int \frac{d+ex+fx^2+gx^3}{\sqrt{a+bx^2+cx^4}}\,dx$

$$\frac{1}{2}\left(\frac{(2ce-bg)\int \frac{1}{\sqrt{cx^4+bx^2+a}}dx^2}{2c}+\frac{g\sqrt{a+bx^2+cx^4}}{c}\right)+\frac{\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}\left(\frac{\sqrt{a}f}{\sqrt{c}}+d\right)\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2\sqrt[4]{a}\sqrt[4]{c}\sqrt{a+bx^2+cx^4}}-\frac{f\left(\frac{\sqrt[4]{a}\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}E\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right)|\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{\sqrt[4]{c}\sqrt{a+bx^2+cx^4}}-\frac{x\sqrt{a+bx^2+cx^4}}{\sqrt{a}+\sqrt{c}x^2}\right)}{\sqrt{c}}$$

$$\Big\downarrow \text{1092}$$

$$\frac{1}{2}\left(\frac{(2ce-bg)\int \frac{1}{4c-x^4}d\frac{2cx^2+b}{\sqrt{cx^4+bx^2+a}}}{c}+\frac{g\sqrt{a+bx^2+cx^4}}{c}\right)+\frac{\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}\left(\frac{\sqrt{a}f}{\sqrt{c}}+d\right)\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2\sqrt[4]{a}\sqrt[4]{c}\sqrt{a+bx^2+cx^4}}-\frac{f\left(\frac{\sqrt[4]{a}\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}E\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right)|\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{\sqrt[4]{c}\sqrt{a+bx^2+cx^4}}-\frac{x\sqrt{a+bx^2+cx^4}}{\sqrt{a}+\sqrt{c}x^2}\right)}{\sqrt{c}}$$

$$\Big\downarrow \text{219}$$

$$\frac{\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}\left(\frac{\sqrt{a}f}{\sqrt{c}}+d\right)\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2\sqrt[4]{a}\sqrt[4]{c}\sqrt{a+bx^2+cx^4}}-\frac{f\left(\frac{\sqrt[4]{a}\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}E\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right)|\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{\sqrt[4]{c}\sqrt{a+bx^2+cx^4}}-\frac{x\sqrt{a+bx^2+cx^4}}{\sqrt{a}+\sqrt{c}x^2}\right)}{\sqrt{c}}+\frac{1}{2}\left(\frac{(2ce-bg)\text{arctanh}\left(\frac{b+2cx^2}{2\sqrt{c}\sqrt{a+bx^2+cx^4}}\right)}{2c^{3/2}}+\frac{g\sqrt{a+bx^2+cx^4}}{c}\right)$$

input
```
Int[(d + e*x + f*x^2 + g*x^3)/Sqrt[a + b*x^2 + c*x^4],x]
```

3.105. $\int \frac{d+ex+fx^2+gx^3}{\sqrt{a+bx^2+cx^4}}\,dx$

output
```
((g*Sqrt[a + b*x^2 + c*x^4])/c + ((2*c*e - b*g)*ArcTanh[(b + 2*c*x^2)/(2*S
qrt[c]*Sqrt[a + b*x^2 + c*x^4])])/(2*c^(3/2)))/2 - (f*(-((x*Sqrt[a + b*x^2
 + c*x^4])/(Sqrt[a] + Sqrt[c]*x^2)) + (a^(1/4)*(Sqrt[a] + Sqrt[c]*x^2)*Sqr
t[(a + b*x^2 + c*x^4)/(Sqrt[a] + Sqrt[c]*x^2)^2]*EllipticE[2*ArcTan[(c^(1/
4)*x)/a^(1/4)], (2 - b/(Sqrt[a]*Sqrt[c]))/4])/(c^(1/4)*Sqrt[a + b*x^2 + c*
x^4])))/Sqrt[c] + ((d + (Sqrt[a]*f)/Sqrt[c])*(Sqrt[a] + Sqrt[c]*x^2)*Sqrt[
(a + b*x^2 + c*x^4)/(Sqrt[a] + Sqrt[c]*x^2)^2]*EllipticF[2*ArcTan[(c^(1/4)
*x)/a^(1/4)], (2 - b/(Sqrt[a]*Sqrt[c]))/4])/(2*a^(1/4)*c^(1/4)*Sqrt[a + b*
x^2 + c*x^4])
```

3.105.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 219
```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

rule 1092
```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Simp[2   Subst[I
nt[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a
, b, c}, x]
```

rule 1160
```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[e*((a + b*x + c*x^2)^(p + 1)/(2*c*(p + 1))), x] + Simp[(2*c*d - b
*e)/(2*c)   Int[(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, p}, x]
 && NeQ[p, -1]
```

rule 1416
```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4], x_Symbol] :> With[{q = Rt[c
/a, 4]}, Simp[(1 + q^2*x^2)*(Sqrt[(a + b*x^2 + c*x^4)/(a*(1 + q^2*x^2)^2)]/
(2*q*Sqrt[a + b*x^2 + c*x^4]))*EllipticF[2*ArcTan[q*x], 1/2 - b*(q^2/(4*c))
], x]] /; FreeQ[{a, b, c}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[c/a]
```

rule 1509
```
Int[((d_) + (e_.)*(x_)^2)/Sqrt[(a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4], x_Symbo
l] :> With[{q = Rt[c/a, 4]}, Simp[(-d)*x*(Sqrt[a + b*x^2 + c*x^4]/(a*(1 + q
^2*x^2))), x] + Simp[d*(1 + q^2*x^2)*(Sqrt[(a + b*x^2 + c*x^4)/(a*(1 + q^2*
x^2)^2)]/(q*Sqrt[a + b*x^2 + c*x^4]))*EllipticE[2*ArcTan[q*x], 1/2 - b*(q^2
/(4*c))], x] /; EqQ[e + d*q^2, 0]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2
- 4*a*c, 0] && PosQ[c/a]
```

rule 1511
```
Int[((d_) + (e_.)*(x_)^2)/Sqrt[(a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4], x_Symbo
l] :> With[{q = Rt[c/a, 2]}, Simp[(e + d*q)/q   Int[1/Sqrt[a + b*x^2 + c*x^
4], x], x] - Simp[e/q   Int[(1 - q*x^2)/Sqrt[a + b*x^2 + c*x^4], x], x] /;
NeQ[e + d*q, 0]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && Pos
Q[c/a]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.105.4 Maple [A] (verified)

Time = 4.08 (sec) , antiderivative size = 426, normalized size of antiderivative = 1.19

method	result
elliptic	$\frac{g\sqrt{c\,x^4+b\,x^2+a}}{2c}+\frac{d\sqrt{2}\,\sqrt{4-\frac{2\left(-b+\sqrt{-4ac+b^2}\right)x^2}{a}}\,\sqrt{4+\frac{2\left(b+\sqrt{-4ac+b^2}\right)x^2}{a}}\,F\left(\frac{x\sqrt{2}\,\sqrt{\frac{-b+\sqrt{-4ac+b^2}}{a}}}{2},\frac{\sqrt{-4+\frac{2b\left(b+\sqrt{-4ac+b^2}\right)}{ac}}}{2}\right)}{4\sqrt{\frac{-b+\sqrt{-4ac+b^2}}{a}}\,\sqrt{c\,x^4+b\,x^2+a}}+$
risch	$\frac{g\sqrt{c\,x^4+b\,x^2+a}}{2c}-\frac{-\frac{cd\sqrt{2}\,\sqrt{4-\frac{2\left(-b+\sqrt{-4ac+b^2}\right)x^2}{a}}\,\sqrt{4+\frac{2\left(b+\sqrt{-4ac+b^2}\right)x^2}{a}}\,F\left(\frac{x\sqrt{2}\,\sqrt{\frac{-b+\sqrt{-4ac+b^2}}{a}}}{2},\frac{\sqrt{-4+\frac{2b\left(b+\sqrt{-4ac+b^2}\right)}{ac}}}{2}\right)}{2\sqrt{\frac{-b+\sqrt{-4ac+b^2}}{a}}\,\sqrt{c\,x^4+b\,x^2+a}}+-}{}$
default	$\frac{d\sqrt{2}\,\sqrt{4-\frac{2\left(-b+\sqrt{-4ac+b^2}\right)x^2}{a}}\,\sqrt{4+\frac{2\left(b+\sqrt{-4ac+b^2}\right)x^2}{a}}\,F\left(\frac{x\sqrt{2}\,\sqrt{\frac{-b+\sqrt{-4ac+b^2}}{a}}}{2},\frac{\sqrt{-4+\frac{2b\left(b+\sqrt{-4ac+b^2}\right)}{ac}}}{2}\right)}{4\sqrt{\frac{-b+\sqrt{-4ac+b^2}}{a}}\,\sqrt{c\,x^4+b\,x^2+a}}+g\left(\frac{\sqrt{c\,x^4+b\,x^2+a}}{2c}\right.$

input
```
int((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^(1/2),x,method=_RETURNVERBOSE)
```

output
```
1/2*g*(c*x^4+b*x^2+a)^(1/2)/c+1/4*d*2^(1/2)/((-b+(-4*a*c+b^2)^(1/2))/a)^(1
/2)*(4-2*(-b+(-4*a*c+b^2)^(1/2))/a*x^2)^(1/2)*(4+2*(b+(-4*a*c+b^2)^(1/2))/
a*x^2)^(1/2)/(c*x^4+b*x^2+a)^(1/2)*EllipticF(1/2*x*2^(1/2)*((-b+(-4*a*c+b^
2)^(1/2))/a)^(1/2),1/2*(-4+2*b*(b+(-4*a*c+b^2)^(1/2))/a/c)^(1/2))+1/2*(e-1
/2*g/c*b)*ln((2*c*x^2+b)/c^(1/2)+2*(c*x^4+b*x^2+a)^(1/2))/c^(1/2)-1/2*f*a*
2^(1/2)/((-b+(-4*a*c+b^2)^(1/2))/a)^(1/2)*(4-2*(-b+(-4*a*c+b^2)^(1/2))/a*x
^2)^(1/2)*(4+2*(b+(-4*a*c+b^2)^(1/2))/a*x^2)^(1/2)/(c*x^4+b*x^2+a)^(1/2)/(
b+(-4*a*c+b^2)^(1/2))*(EllipticF(1/2*x*2^(1/2)*((-b+(-4*a*c+b^2)^(1/2))/a)
^(1/2),1/2*(-4+2*b*(b+(-4*a*c+b^2)^(1/2))/a/c)^(1/2))-EllipticE(1/2*x*2^(1
/2)*((-b+(-4*a*c+b^2)^(1/2))/a)^(1/2),1/2*(-4+2*b*(b+(-4*a*c+b^2)^(1/2))/a
/c)^(1/2)))
```

3.105.5 Fricas [A] (verification not implemented)

Time = 0.19 (sec) , antiderivative size = 376, normalized size of antiderivative = 1.05

$$\int \frac{d+ex+fx^2+gx^3}{\sqrt{a+bx^2+cx^4}}\,dx$$
$$= \frac{4\sqrt{\frac{1}{2}}\left(acfx\sqrt{\frac{b^2-4\,ac}{c^2}}-abfx\right)\sqrt{c}\sqrt{\frac{c\sqrt{\frac{b^2-4\,ac}{c^2}}-b}{c}}E(\arcsin\left(\frac{\sqrt{\frac{1}{2}}\sqrt{\frac{c\sqrt{\frac{b^2-4\,ac}{c^2}}-b}{c}}}{x}\right) \mid \frac{bc\sqrt{\frac{b^2-4\,ac}{c^2}}+b^2-2\,ac}{2\,ac})+4\sqrt{\frac{1}{2}}\left(\left(\right.\right.}{}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^(1/2),x, algorithm="fricas")
```

output
```
1/8*(4*sqrt(1/2)*(a*c*f*x*sqrt((b^2 - 4*a*c)/c^2) - a*b*f*x)*sqrt(c)*sqrt(
(c*sqrt((b^2 - 4*a*c)/c^2) - b)/c)*elliptic_e(arcsin(sqrt(1/2)*sqrt((c*sqr
t((b^2 - 4*a*c)/c^2) - b)/c)/x), 1/2*(b*c*sqrt((b^2 - 4*a*c)/c^2) + b^2 -
2*a*c)/(a*c)) + 4*sqrt(1/2)*((c^2*d - a*c*f)*x*sqrt((b^2 - 4*a*c)/c^2) + (
b*c*d + a*b*f)*x)*sqrt(c)*sqrt((c*sqrt((b^2 - 4*a*c)/c^2) - b)/c)*elliptic
_f(arcsin(sqrt(1/2)*sqrt((c*sqrt((b^2 - 4*a*c)/c^2) - b)/c)/x), 1/2*(b*c*s
qrt((b^2 - 4*a*c)/c^2) + b^2 - 2*a*c)/(a*c)) - (2*a*c*e - a*b*g)*sqrt(c)*x
*log(8*c^2*x^4 + 8*b*c*x^2 + b^2 - 4*sqrt(c*x^4 + b*x^2 + a)*(2*c*x^2 + b)
*sqrt(c) + 4*a*c) + 4*sqrt(c*x^4 + b*x^2 + a)*(a*c*g*x + 2*a*c*f))/(a*c^2*
x)
```

3.105.6 Sympy [F]

$$\int \frac{d+ex+fx^2+gx^3}{\sqrt{a+bx^2+cx^4}}\,dx=\int \frac{d+ex+fx^2+gx^3}{\sqrt{a+bx^2+cx^4}}\,dx$$

input
```
integrate((g*x**3+f*x**2+e*x+d)/(c*x**4+b*x**2+a)**(1/2),x)
```

output
```
Integral((d + e*x + f*x**2 + g*x**3)/sqrt(a + b*x**2 + c*x**4), x)
```

3.105.7 Maxima [F]

$$\int \frac{d+ex+fx^2+gx^3}{\sqrt{a+bx^2+cx^4}}\,dx=\int \frac{gx^3+fx^2+ex+d}{\sqrt{cx^4+bx^2+a}}\,dx$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^(1/2),x, algorithm="maxima")
```

output
```
integrate((g*x^3 + f*x^2 + e*x + d)/sqrt(c*x^4 + b*x^2 + a), x)
```

3.105.8 Giac [F]

$$\int \frac{d+ex+fx^2+gx^3}{\sqrt{a+bx^2+cx^4}}\,dx=\int \frac{gx^3+fx^2+ex+d}{\sqrt{cx^4+bx^2+a}}\,dx$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^(1/2),x, algorithm="giac")
```

output
```
integrate((g*x^3 + f*x^2 + e*x + d)/sqrt(c*x^4 + b*x^2 + a), x)
```

3.105.9 Mupad [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3}{\sqrt{a+bx^2+cx^4}}\,dx=\int \frac{g\,x^3+f\,x^2+e\,x+d}{\sqrt{c\,x^4+b\,x^2+a}}\,dx$$

input
```
int((d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4)^(1/2),x)
```

output
```
int((d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4)^(1/2), x)
```

3.106 $\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{3/2}} dx$

3.106.1 Optimal result

Integrand size = 32, antiderivative size = 447

$$\begin{aligned}\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{3/2}}\,dx = {} & \frac{x(b^2d-2acd-abf+c(bd-2af)x^2)}{a(b^2-4ac)\sqrt{a+bx^2+cx^4}} \\ & - \frac{be-2ag+(2ce-bg)x^2}{(b^2-4ac)\sqrt{a+bx^2+cx^4}} - \frac{\sqrt{c}(bd-2af)x\sqrt{a+bx^2+cx^4}}{a(b^2-4ac)(\sqrt{a}+\sqrt{c}x^2)} \\ & + \frac{\sqrt[4]{c}(bd-2af)(\sqrt{a}+\sqrt{c}x^2)\sqrt{\frac{a+bx^2+cx^4}{(\sqrt{a}+\sqrt{c}x^2)^2}}E\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right)|\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{a^{3/4}(b^2-4ac)\sqrt{a+bx^2+cx^4}} \\ & - \frac{(\sqrt{c}d-\sqrt{a}f)(\sqrt{a}+\sqrt{c}x^2)\sqrt{\frac{a+bx^2+cx^4}{(\sqrt{a}+\sqrt{c}x^2)^2}}\,\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2a^{3/4}(b-2\sqrt{a}\sqrt{c})\sqrt[4]{c}\sqrt{a+bx^2+cx^4}}\end{aligned}$$

output
```
x*(b^2*d-2*a*c*d-a*b*f+c*(-2*a*f+b*d)*x^2)/a/(-4*a*c+b^2)/(c*x^4+b*x^2+a)^
(1/2)+(-b*e+2*a*g-(-b*g+2*c*e)*x^2)/(-4*a*c+b^2)/(c*x^4+b*x^2+a)^(1/2)-(-2
*a*f+b*d)*x*c^(1/2)*(c*x^4+b*x^2+a)^(1/2)/a/(-4*a*c+b^2)/(a^(1/2)+x^2*c^(1
/2))+c^(1/4)*(-2*a*f+b*d)*(cos(2*arctan(c^(1/4)*x/a^(1/4)))^2)^(1/2)/cos(2
*arctan(c^(1/4)*x/a^(1/4)))*EllipticE(sin(2*arctan(c^(1/4)*x/a^(1/4))),1/2
*(2-b/a^(1/2)/c^(1/2))^(1/2))*(a^(1/2)+x^2*c^(1/2))*((c*x^4+b*x^2+a)/(a^(1
/2)+x^2*c^(1/2))^2)^(1/2)/a^(3/4)/(-4*a*c+b^2)/(c*x^4+b*x^2+a)^(1/2)-1/2*(
cos(2*arctan(c^(1/4)*x/a^(1/4)))^2)^(1/2)/cos(2*arctan(c^(1/4)*x/a^(1/4)))
*EllipticF(sin(2*arctan(c^(1/4)*x/a^(1/4))),1/2*(2-b/a^(1/2)/c^(1/2))^(1/2
))*(-f*a^(1/2)+d*c^(1/2))*(a^(1/2)+x^2*c^(1/2))*((c*x^4+b*x^2+a)/(a^(1/2)+
x^2*c^(1/2))^2)^(1/2)/a^(3/4)/c^(1/4)/(b-2*a^(1/2)*c^(1/2))/(c*x^4+b*x^2+a
)^(1/2)
```

3.106.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 11.16 (sec) , antiderivative size = 513, normalized size of antiderivative = 1.15

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{3/2}}\,dx =$$

$$-\frac{4\sqrt{\frac{c}{b+\sqrt{b^2-4ac}}}(-2a^2g-bdx(b+cx^2)+2acx(d+x(e+fx))+ab(e+x(f-gx)))+i\left(-b+\sqrt{b^2-4ac}\right)(b}{}$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4)^(3/2),x]
```

output
```
-1/4*(4*Sqrt[c/(b + Sqrt[b^2 - 4*a*c])]*(-2*a^2*g - b*d*x*(b + c*x^2) + 2*
a*c*x*(d + x*(e + f*x)) + a*b*(e + x*(f - g*x))) + I*(-b + Sqrt[b^2 - 4*a*
c])*(b*d - 2*a*f)*Sqrt[(b + Sqrt[b^2 - 4*a*c] + 2*c*x^2)/(b + Sqrt[b^2 - 4
*a*c])]*Sqrt[(2*b - 2*Sqrt[b^2 - 4*a*c] + 4*c*x^2)/(b - Sqrt[b^2 - 4*a*c])
]*EllipticE[I*ArcSinh[Sqrt[2]*Sqrt[c/(b + Sqrt[b^2 - 4*a*c])]*x], (b + Sqr
t[b^2 - 4*a*c])/(b - Sqrt[b^2 - 4*a*c])] - I*(-(b^2*d) + 4*a*c*d + b*Sqrt[
b^2 - 4*a*c]*d - 2*a*Sqrt[b^2 - 4*a*c]*f)*Sqrt[(b + Sqrt[b^2 - 4*a*c] + 2*
c*x^2)/(b + Sqrt[b^2 - 4*a*c])]*Sqrt[(2*b - 2*Sqrt[b^2 - 4*a*c] + 4*c*x^2)
/(b - Sqrt[b^2 - 4*a*c])]*EllipticF[I*ArcSinh[Sqrt[2]*Sqrt[c/(b + Sqrt[b^2
 - 4*a*c])]*x], (b + Sqrt[b^2 - 4*a*c])/(b - Sqrt[b^2 - 4*a*c])])/(a*(b^2
- 4*a*c)*Sqrt[c/(b + Sqrt[b^2 - 4*a*c])]*Sqrt[a + b*x^2 + c*x^4])
```

3.106.3 Rubi [A] (verified)

Time = 0.68 (sec) , antiderivative size = 433, normalized size of antiderivative = 0.97, number of steps used = 9, number of rules used = 8, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.250, Rules used = {2202, 1492, 1511, 27, 1416, 1509, 1576, 1158}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{3/2}}\,dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^{3/2}}dx+\int \frac{x(gx^2+e)}{(cx^4+bx^2+a)^{3/2}}dx$$

$$\Big\downarrow \text{1492}$$

$$-\frac{\int \frac{c(bd-2af)x^2+a(2cd-bf)}{\sqrt{cx^4+bx^2+a}}dx}{a(b^2-4ac)}+\int \frac{x(gx^2+e)}{(cx^4+bx^2+a)^{3/2}}dx+\frac{x(cx^2(bd-2af)-abf-2acd+b^2d)}{a(b^2-4ac)\sqrt{a+bx^2+cx^4}}$$

$$\Big\downarrow \text{1511}$$

$$-\frac{\sqrt{a}\left(\sqrt{c}(bd-2af)+\sqrt{a}(2cd-bf)\right)\int \frac{1}{\sqrt{cx^4+bx^2+a}}dx-\sqrt{a}\sqrt{c}(bd-2af)\int \frac{\sqrt{a}-\sqrt{c}x^2}{\sqrt{a}\sqrt{cx^4+bx^2+a}}dx}{a(b^2-4ac)}+\int \frac{x(gx^2+e)}{(cx^4+bx^2+a)^{3/2}}dx+\frac{x(cx^2(bd-2af)-abf-2acd+b^2d)}{a(b^2-4ac)\sqrt{a+bx^2+cx^4}}$$

$$\Big\downarrow \text{27}$$

$$-\frac{\sqrt{a}\left(\sqrt{c}(bd-2af)+\sqrt{a}(2cd-bf)\right)\int \frac{1}{\sqrt{cx^4+bx^2+a}}dx-\sqrt{c}(bd-2af)\int \frac{\sqrt{a}-\sqrt{c}x^2}{\sqrt{cx^4+bx^2+a}}dx}{a(b^2-4ac)}+\int \frac{x(gx^2+e)}{(cx^4+bx^2+a)^{3/2}}dx+\frac{x(cx^2(bd-2af)-abf-2acd+b^2d)}{a(b^2-4ac)\sqrt{a+bx^2+cx^4}}$$

$$\Big\downarrow \text{1416}$$

$$-\frac{\frac{\sqrt[4]{a}(\sqrt{a}+\sqrt{c}x^2)\sqrt{\frac{a+bx^2+cx^4}{(\sqrt{a}+\sqrt{c}x^2)^2}}(\sqrt{c}(bd-2af)+\sqrt{a}(2cd-bf))\,\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2\sqrt[4]{c}\sqrt{a+bx^2+cx^4}}-\sqrt{c}(bd-2af)\int \frac{\sqrt{a}-\sqrt{c}x}{\sqrt{cx^4+bx^2}} \ldots}{a(b^2-4ac)}+\int \frac{x(gx^2+e)}{(cx^4+bx^2+a)^{3/2}}dx+\frac{x(cx^2(bd-2af)-abf-2acd+b^2d)}{a(b^2-4ac)\sqrt{a+bx^2+cx^4}}$$

$$\Big\downarrow \text{1509}$$

$$\int \frac{x(gx^2+e)}{(cx^4+bx^2+a)^{3/2}}dx - \frac{\frac{\sqrt[4]{a}(\sqrt{a}+\sqrt{c}x^2)\sqrt{\frac{a+bx^2+cx^4}{(\sqrt{a}+\sqrt{c}x^2)^2}}(\sqrt{c}(bd-2af)+\sqrt{a}(2cd-bf))\,\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2\sqrt[4]{c}\sqrt{a+bx^2+cx^4}} - \sqrt{c}(bd-2af)\left(\frac{\sqrt[4]{a}(\sqrt{a}+\sqrt{c}\ldots}{\ldots}\right.}{a(b^2-4ac)}$$

$$\frac{x(cx^2(bd-2af)-abf-2acd+b^2d)}{a(b^2-4ac)\sqrt{a+bx^2+cx^4}}$$

$\downarrow$ 1576

$$\frac{1}{2}\int \frac{gx^2+e}{(cx^4+bx^2+a)^{3/2}}dx^2 - \frac{\frac{\sqrt[4]{a}(\sqrt{a}+\sqrt{c}x^2)\sqrt{\frac{a+bx^2+cx^4}{(\sqrt{a}+\sqrt{c}x^2)^2}}(\sqrt{c}(bd-2af)+\sqrt{a}(2cd-bf))\,\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2\sqrt[4]{c}\sqrt{a+bx^2+cx^4}} - \sqrt{c}(bd-2af)\left(\frac{\sqrt[4]{a}(\sqrt{a}+\sqrt{c}\ldots}{\ldots}\right.}{a(b^2-4ac)}$$

$$\frac{x(cx^2(bd-2af)-abf-2acd+b^2d)}{a(b^2-4ac)\sqrt{a+bx^2+cx^4}}$$

$\downarrow$ 1158

$$-\frac{\frac{\sqrt[4]{a}(\sqrt{a}+\sqrt{c}x^2)\sqrt{\frac{a+bx^2+cx^4}{(\sqrt{a}+\sqrt{c}x^2)^2}}(\sqrt{c}(bd-2af)+\sqrt{a}(2cd-bf))\,\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right),\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{2\sqrt[4]{c}\sqrt{a+bx^2+cx^4}} - \sqrt{c}(bd-2af)\left(\frac{\sqrt[4]{a}(\sqrt{a}+\ldots}{\ldots}\right.}{a(b^2-4ac)}$$

$$\frac{x(cx^2(bd-2af)-abf-2acd+b^2d)}{a(b^2-4ac)\sqrt{a+bx^2+cx^4}} - \frac{-2ag+x^2(2ce-bg)+be}{(b^2-4ac)\sqrt{a+bx^2+cx^4}}$$

input
```
Int[(d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4)^(3/2),x]
```

3.106. $\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{3/2}}\,dx$

output
```
(x*(b^2*d - 2*a*c*d - a*b*f + c*(b*d - 2*a*f)*x^2))/(a*(b^2 - 4*a*c)*Sqrt[
a + b*x^2 + c*x^4]) - (b*e - 2*a*g + (2*c*e - b*g)*x^2)/((b^2 - 4*a*c)*Sqr
t[a + b*x^2 + c*x^4]) - (-(Sqrt[c]*(b*d - 2*a*f)*(-((x*Sqrt[a + b*x^2 + c*
x^4])/(Sqrt[a] + Sqrt[c]*x^2)) + (a^(1/4)*(Sqrt[a] + Sqrt[c]*x^2)*Sqrt[(a
+ b*x^2 + c*x^4)/(Sqrt[a] + Sqrt[c]*x^2)^2]*EllipticE[2*ArcTan[(c^(1/4)*x)
/a^(1/4)], (2 - b/(Sqrt[a]*Sqrt[c]))/4])/(c^(1/4)*Sqrt[a + b*x^2 + c*x^4])
)) + (a^(1/4)*(Sqrt[c]*(b*d - 2*a*f) + Sqrt[a]*(2*c*d - b*f))*(Sqrt[a] + S
qrt[c]*x^2)*Sqrt[(a + b*x^2 + c*x^4)/(Sqrt[a] + Sqrt[c]*x^2)^2]*EllipticF[
2*ArcTan[(c^(1/4)*x)/a^(1/4)], (2 - b/(Sqrt[a]*Sqrt[c]))/4])/(2*c^(1/4)*Sq
rt[a + b*x^2 + c*x^4]))/(a*(b^2 - 4*a*c))
```

3.106.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 1158
```
Int[((d_.) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(3/2), x_Symbo
l] :> Simp[-2*((b*d - 2*a*e + (2*c*d - b*e)*x)/((b^2 - 4*a*c)*Sqrt[a + b*x
+ c*x^2])), x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1416
```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4], x_Symbol] :> With[{q = Rt[c
/a, 4]}, Simp[(1 + q^2*x^2)*(Sqrt[(a + b*x^2 + c*x^4)/(a*(1 + q^2*x^2)^2)]/
(2*q*Sqrt[a + b*x^2 + c*x^4]))*EllipticF[2*ArcTan[q*x], 1/2 - b*(q^2/(4*c))
], x]] /; FreeQ[{a, b, c}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[c/a]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 1509
```
Int[((d_) + (e_.)*(x_)^2)/Sqrt[(a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4], x_Symbo
l] :> With[{q = Rt[c/a, 4]}, Simp[(-d)*x*(Sqrt[a + b*x^2 + c*x^4]/(a*(1 + q
^2*x^2))), x] + Simp[d*(1 + q^2*x^2)*(Sqrt[(a + b*x^2 + c*x^4)/(a*(1 + q^2*
x^2)^2)]/(q*Sqrt[a + b*x^2 + c*x^4]))*EllipticE[2*ArcTan[q*x], 1/2 - b*(q^2
/(4*c))], x] /; EqQ[e + d*q^2, 0]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2
- 4*a*c, 0] && PosQ[c/a]
```

rule 1511
```
Int[((d_) + (e_.)*(x_)^2)/Sqrt[(a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4], x_Symbo
l] :> With[{q = Rt[c/a, 2]}, Simp[(e + d*q)/q   Int[1/Sqrt[a + b*x^2 + c*x^
4], x], x] - Simp[e/q   Int[(1 - q*x^2)/Sqrt[a + b*x^2 + c*x^4], x], x] /;
NeQ[e + d*q, 0]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && Pos
Q[c/a]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.106.4 Maple [A] (verified)

Time = 2.11 (sec) , antiderivative size = 565, normalized size of antiderivative = 1.26

method	result
elliptic	$-\frac{2c\left(-\frac{(2af-bd)x^3}{2a\left(4ac-b^2\right)}+\frac{(bg-2ec)x^2}{2c\left(4ac-b^2\right)}-\frac{\left(abf+2acd-b^2d\right)x}{2ac\left(4ac-b^2\right)}+\frac{2ag-be}{2\left(4ac-b^2\right)c}\right)}{\sqrt{\left(x^4+\frac{b\,x^2}{c}+\frac{a}{c}\right)c}}+\frac{\left(\frac{d}{a}-\frac{abf+2acd-b^2d}{a\left(4ac-b^2\right)}\right)\sqrt{2}\sqrt{4-\frac{2\left(-b+\sqrt{-4ac+b^2}\right)x^2}{a}}\sqrt{4+\cdots}}{4\sqrt{\frac{-b+\sqrt{\cdots}}{\cdots}}}$
default	Expression too large to display

input
```
int((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^(3/2),x,method=_RETURNVERBOSE)
```

output
```
-2*c*(-1/2*(2*a*f-b*d)/a/(4*a*c-b^2)*x^3+1/2*(b*g-2*c*e)/c/(4*a*c-b^2)*x^2
-1/2*(a*b*f+2*a*c*d-b^2*d)/a/c/(4*a*c-b^2)*x+1/2*(2*a*g-b*e)/(4*a*c-b^2)/c
)/((x^4+1/c*b*x^2+a/c)*c)^(1/2)+1/4*(d/a-(a*b*f+2*a*c*d-b^2*d)/a/(4*a*c-b^
2))*2^(1/2)/((-b+(-4*a*c+b^2)^(1/2))/a)^(1/2)*(4-2*(-b+(-4*a*c+b^2)^(1/2))
/a*x^2)^(1/2)*(4+2*(b+(-4*a*c+b^2)^(1/2))/a*x^2)^(1/2)/(c*x^4+b*x^2+a)^(1/
2)*EllipticF(1/2*x*2^(1/2)*((-b+(-4*a*c+b^2)^(1/2))/a)^(1/2),1/2*(-4+2*b*(
b+(-4*a*c+b^2)^(1/2))/a/c)^(1/2))+1/2*c*(2*a*f-b*d)/(4*a*c-b^2)*2^(1/2)/((
-b+(-4*a*c+b^2)^(1/2))/a)^(1/2)*(4-2*(-b+(-4*a*c+b^2)^(1/2))/a*x^2)^(1/2)*
(4+2*(b+(-4*a*c+b^2)^(1/2))/a*x^2)^(1/2)/(c*x^4+b*x^2+a)^(1/2)/(b+(-4*a*c+
b^2)^(1/2))*(EllipticF(1/2*x*2^(1/2)*((-b+(-4*a*c+b^2)^(1/2))/a)^(1/2),1/2
*(-4+2*b*(b+(-4*a*c+b^2)^(1/2))/a/c)^(1/2))-EllipticE(1/2*x*2^(1/2)*((-b+(
-4*a*c+b^2)^(1/2))/a)^(1/2),1/2*(-4+2*b*(b+(-4*a*c+b^2)^(1/2))/a/c)^(1/2))
)
```

3.106.5 Fricas [A] (verification not implemented)

Time = 0.11 (sec) , antiderivative size = 723, normalized size of antiderivative = 1.62

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{3/2}}\,dx =$$

$$-\frac{\sqrt{\frac{1}{2}}\Big(ab^2cd-2\,a^2bcf+(b^2c^2d-2\,abc^2f)x^4+(b^3cd-2\,ab^2cf)x^2-(a^2bcd-2\,a^3cf+(abc^2d-2\,a^2c^2f)x^4}{}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^(3/2),x, algorithm="fricas")
```

output
```
-1/2*(sqrt(1/2)*(a*b^2*c*d - 2*a^2*b*c*f + (b^2*c^2*d - 2*a*b*c^2*f)*x^4 +
 (b^3*c*d - 2*a*b^2*c*f)*x^2 - (a^2*b*c*d - 2*a^3*c*f + (a*b*c^2*d - 2*a^2
*c^2*f)*x^4 + (a*b^2*c*d - 2*a^2*b*c*f)*x^2)*sqrt((b^2 - 4*a*c)/a^2))*sqrt
(a)*sqrt((a*sqrt((b^2 - 4*a*c)/a^2) - b)/a)*elliptic_e(arcsin(sqrt(1/2)*x*
sqrt((a*sqrt((b^2 - 4*a*c)/a^2) - b)/a)), 1/2*(a*b*sqrt((b^2 - 4*a*c)/a^2)
 + b^2 - 2*a*c)/(a*c)) - sqrt(1/2)*(((2*a*b + b^2)*c^2*d - (a*b^2*c + 2*a*
b*c^2)*f)*x^4 + (2*a^2*b + a*b^2)*c*d + ((2*a*b^2 + b^3)*c*d - (a*b^3 + 2*
a*b^2*c)*f)*x^2 - (a^2*b^2 + 2*a^2*b*c)*f + (((2*a^2 - a*b)*c^2*d - (a^2*b
*c - 2*a^2*c^2)*f)*x^4 + (2*a^3 - a^2*b)*c*d + ((2*a^2*b - a*b^2)*c*d - (a
^2*b^2 - 2*a^2*b*c)*f)*x^2 - (a^3*b - 2*a^3*c)*f)*sqrt((b^2 - 4*a*c)/a^2))
*sqrt(a)*sqrt((a*sqrt((b^2 - 4*a*c)/a^2) - b)/a)*elliptic_f(arcsin(sqrt(1/
2)*x*sqrt((a*sqrt((b^2 - 4*a*c)/a^2) - b)/a)), 1/2*(a*b*sqrt((b^2 - 4*a*c)
/a^2) + b^2 - 2*a*c)/(a*c)) + 2*(a^2*b*c*e - 2*a^3*c*g - (a*b*c^2*d - 2*a^
2*c^2*f)*x^3 + (2*a^2*c^2*e - a^2*b*c*g)*x^2 + (a^2*b*c*f - (a*b^2*c - 2*a
^2*c^2)*d)*x)*sqrt(c*x^4 + b*x^2 + a))/(a^3*b^2*c - 4*a^4*c^2 + (a^2*b^2*c
^2 - 4*a^3*c^3)*x^4 + (a^2*b^3*c - 4*a^3*b*c^2)*x^2)
```

3.106.6 Sympy [F]

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{3/2}}\,dx = \int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{\frac{3}{2}}}\,dx$$

input
```
integrate((g*x**3+f*x**2+e*x+d)/(c*x**4+b*x**2+a)**(3/2),x)
```

output
```
Integral((d + e*x + f*x**2 + g*x**3)/(a + b*x**2 + c*x**4)**(3/2), x)
```

3.106.7 Maxima [F]

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{3/2}}\,dx = \int \frac{gx^3+fx^2+ex+d}{(cx^4+bx^2+a)^{\frac{3}{2}}}\,dx$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^(3/2),x, algorithm="maxima")
```

output
```
integrate((g*x^3 + f*x^2 + e*x + d)/(c*x^4 + b*x^2 + a)^(3/2), x)
```

3.106.8 Giac [F]

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{3/2}}\,dx=\int \frac{gx^3+fx^2+ex+d}{(cx^4+bx^2+a)^{\frac{3}{2}}}\,dx$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^(3/2),x, algorithm="giac")
```

output
```
integrate((g*x^3 + f*x^2 + e*x + d)/(c*x^4 + b*x^2 + a)^(3/2), x)
```

3.106.9 Mupad [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{3/2}}\,dx=\int \frac{g\,x^3+f\,x^2+e\,x+d}{(c\,x^4+b\,x^2+a)^{3/2}}\,dx$$

input
```
int((d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4)^(3/2),x)
```

output
```
int((d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4)^(3/2), x)
```

3.107 $\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{5/2}}\,dx$

3.107.1 Optimal result

Integrand size = 32, antiderivative size = 680

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{5/2}}\,dx = \frac{x(b^2d-2acd-abf+c(bd-2af)x^2)}{3a(b^2-4ac)(a+bx^2+cx^4)^{3/2}} - \frac{be-2ag+(2ce-bg)x^2}{3(b^2-4ac)(a+bx^2+cx^4)^{3/2}} + \frac{4(2ce-bg)(b+2cx^2)}{3(b^2-4ac)^2\sqrt{a+bx^2+cx^4}} + \frac{x(2b^4d-17ab^2cd+20a^2c^2d+ab^3f+4a^2bcf+c(2b^3d-16abcd+ab^2f+12a^2cf)x^2)}{3a^2(b^2-4ac)^2\sqrt{a+bx^2+cx^4}} - \frac{\sqrt{c}(2b^3d-16abcd+ab^2f+12a^2cf)x\sqrt{a+bx^2+cx^4}}{3a^2(b^2-4ac)^2(\sqrt{a}+\sqrt{c}x^2)} + \frac{\sqrt[4]{c}(2b^3d-16abcd+ab^2f+12a^2cf)(\sqrt{a}+\sqrt{c}x^2)\sqrt{\frac{a+bx^2+cx^4}{(\sqrt{a}+\sqrt{c}x^2)^2}}E\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right)|\frac{1}{4}\left(2-\frac{b}{\sqrt{a}\sqrt{c}}\right)\right)}{3a^{7/4}(b^2-4ac)^2\sqrt{a+bx^2+cx^4}} - \frac{\sqrt[4]{c}(2b^2d-3\sqrt{a}b\sqrt{c}d-10acd+abf+6a^{3/2}\sqrt{c}f)(\sqrt{a}+\sqrt{c}x^2)\sqrt{\frac{a+bx^2+cx^4}{(\sqrt{a}+\sqrt{c}x^2)^2}}\text{EllipticF}\left(2\arctan\left(\frac{\sqrt[4]{c}x}{\sqrt[4]{a}}\right.\right.}{6a^{7/4}(b-2\sqrt{a}\sqrt{c})(b^2-4ac)\sqrt{a+bx^2+cx^4}}$$

output
```
1/3*x*(b^2*d-2*a*c*d-a*b*f+c*(-2*a*f+b*d)*x^2)/a/(-4*a*c+b^2)/(c*x^4+b*x^2
+a)^(3/2)+1/3*(-b*e+2*a*g-(-b*g+2*c*e)*x^2)/(-4*a*c+b^2)/(c*x^4+b*x^2+a)^(
3/2)+4/3*(-b*g+2*c*e)*(2*c*x^2+b)/(-4*a*c+b^2)^2/(c*x^4+b*x^2+a)^(1/2)+1/3
*x*(2*b^4*d-17*a*b^2*c*d+20*a^2*c^2*d+a*b^3*f+4*a^2*b*c*f+c*(12*a^2*c*f+a*
b^2*f-16*a*b*c*d+2*b^3*d)*x^2)/a^2/(-4*a*c+b^2)^2/(c*x^4+b*x^2+a)^(1/2)-1/
3*(12*a^2*c*f+a*b^2*f-16*a*b*c*d+2*b^3*d)*x*c^(1/2)*(c*x^4+b*x^2+a)^(1/2)/
a^2/(-4*a*c+b^2)^2/(a^(1/2)+x^2*c^(1/2))+1/3*c^(1/4)*(12*a^2*c*f+a*b^2*f-1
6*a*b*c*d+2*b^3*d)*(cos(2*arctan(c^(1/4)*x/a^(1/4)))^2)^(1/2)/cos(2*arctan
(c^(1/4)*x/a^(1/4)))*EllipticE(sin(2*arctan(c^(1/4)*x/a^(1/4))),1/2*(2-b/a
^(1/2)/c^(1/2))^(1/2))*(a^(1/2)+x^2*c^(1/2))*((c*x^4+b*x^2+a)/(a^(1/2)+x^2
*c^(1/2))^2)^(1/2)/a^(7/4)/(-4*a*c+b^2)^2/(c*x^4+b*x^2+a)^(1/2)-1/6*c^(1/4
)*(cos(2*arctan(c^(1/4)*x/a^(1/4)))^2)^(1/2)/cos(2*arctan(c^(1/4)*x/a^(1/4
)))*EllipticF(sin(2*arctan(c^(1/4)*x/a^(1/4))),1/2*(2-b/a^(1/2)/c^(1/2))^(
1/2))*(a^(1/2)+x^2*c^(1/2))*(2*b^2*d-10*a*c*d+a*b*f+6*a^(3/2)*f*c^(1/2)-3*
b*d*a^(1/2)*c^(1/2))*((c*x^4+b*x^2+a)/(a^(1/2)+x^2*c^(1/2))^2)^(1/2)/a^(7/
4)/(-4*a*c+b^2)/(b-2*a^(1/2)*c^(1/2))/(c*x^4+b*x^2+a)^(1/2)
```

3.107.2 Mathematica [C] (verified)

Result contains complex when optimal does not.

Time = 11.99 (sec) , antiderivative size = 598, normalized size of antiderivative = 0.88

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{5/2}}\,dx = \frac{-4a(b^2-4ac)\left(-2a^2g-bdx(b+cx^2)+2acx(d+x(e+fx))+ab(e+x(f-g\right.}{}$$

input
```
Integrate[(d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4)^(5/2),x]
```

output
```
(-4*a*(b^2 - 4*a*c)*(-2*a^2*g - b*d*x*(b + c*x^2) + 2*a*c*x*(d + x*(e + f*
x)) + a*b*(e + x*(f - g*x))) + 4*(a + b*x^2 + c*x^4)*(2*b^3*d*x*(b + c*x^2
) + a*b*x*(-17*b*c*d + b^2*f - 16*c^2*d*x^2 + b*c*f*x^2) + 4*a^2*(-(b^2*g)
 + c^2*x*(5*d + x*(4*e + 3*f*x)) + b*c*(2*e + x*(f - 2*g*x)))) + (I*Sqrt[2
]*Sqrt[(b + Sqrt[b^2 - 4*a*c] + 2*c*x^2)/(b + Sqrt[b^2 - 4*a*c])]*Sqrt[1 +
 (2*c*x^2)/(b - Sqrt[b^2 - 4*a*c])]*(a + b*x^2 + c*x^4)*(-((-b + Sqrt[b^2
- 4*a*c])*(2*b^3*d - 16*a*b*c*d + a*b^2*f + 12*a^2*c*f)*EllipticE[I*ArcSin
h[Sqrt[2]*Sqrt[c/(b + Sqrt[b^2 - 4*a*c])]*x], (b + Sqrt[b^2 - 4*a*c])/(b -
 Sqrt[b^2 - 4*a*c])]) + (-2*b^4*d + b^3*(2*Sqrt[b^2 - 4*a*c]*d - a*f) + 4*
a*b*c*(-4*Sqrt[b^2 - 4*a*c]*d + a*f) + a*b^2*(18*c*d + Sqrt[b^2 - 4*a*c]*f
) + 4*a^2*c*(-10*c*d + 3*Sqrt[b^2 - 4*a*c]*f))*EllipticF[I*ArcSinh[Sqrt[2]
*Sqrt[c/(b + Sqrt[b^2 - 4*a*c])]*x], (b + Sqrt[b^2 - 4*a*c])/(b - Sqrt[b^2
 - 4*a*c])]))/Sqrt[c/(b + Sqrt[b^2 - 4*a*c])])/(12*a^2*(b^2 - 4*a*c)^2*(a
+ b*x^2 + c*x^4)^(3/2))
```

3.107.3 Rubi [A] (verified)

Time = 1.04 (sec) , antiderivative size = 650, normalized size of antiderivative = 0.96, number of steps used = 13, number of rules used = 12, $\frac{\text{number of rules}}{\text{integrand size}} = 0.375$, Rules used = {2202, 1492, 25, 1492, 27, 1511, 27, 1416, 1509, 1576, 1159, 1088}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{5/2}}\,dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{fx^2+d}{(cx^4+bx^2+a)^{5/2}}dx+\int \frac{x(gx^2+e)}{(cx^4+bx^2+a)^{5/2}}dx$$

$$\Big\downarrow \text{1492}$$

$$-\frac{\int -\frac{2db^2+afb+3c(bd-2af)x^2-10acd}{(cx^4+bx^2+a)^{3/2}}dx}{3a(b^2-4ac)}+\int \frac{x(gx^2+e)}{(cx^4+bx^2+a)^{5/2}}dx+\frac{x(cx^2(bd-2af)-abf-2acd+b^2d)}{3a(b^2-4ac)(a+bx^2+cx^4)^{3/2}}$$

$$\Big\downarrow \text{25}$$

$$\frac{\int \frac{2db^2+afb+3c(bd-2af)x^2-10acd}{(cx^4+bx^2+a)^{3/2}}dx}{3a(b^2-4ac)}+\int \frac{x(gx^2+e)}{(cx^4+bx^2+a)^{5/2}}dx+\frac{x(cx^2(bd-2af)-abf-2acd+b^2d)}{3a(b^2-4ac)(a+bx^2+cx^4)^{3/2}}$$

$\Big\downarrow$ 1492

$$\frac{\frac{x\left(cx^2\left(12a^2cf+ab^2f-16abcd+2b^3d\right)+4a^2bcf+20a^2c^2d+ab^3f-17ab^2cd+2b^4d\right)}{a(b^2-4ac)\sqrt{a+bx^2+cx^4}}-\frac{\int \frac{c\left(\left(2db^3+afb^2-16acdb+12a^2cf\right)x^2+a\left(db^2+8afb-20acd\right)\right)}{\sqrt{cx^4+bx^2+a}}dx}{a(b^2-4ac)}}{3a\left(b^2-4ac\right)}+\int \frac{x\left(gx^2+e\right)}{\left(cx^4+bx^2+a\right)^{5/2}}dx+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{3a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^{3/2}}$$

$\Big\downarrow$ 27

$$\frac{\frac{x\left(cx^2\left(12a^2cf+ab^2f-16abcd+2b^3d\right)+4a^2bcf+20a^2c^2d+ab^3f-17ab^2cd+2b^4d\right)}{a(b^2-4ac)\sqrt{a+bx^2+cx^4}}-\frac{c\int \frac{\left(2db^3+afb^2-16acdb+12a^2cf\right)x^2+a\left(db^2+8afb-20acd\right)}{\sqrt{cx^4+bx^2+a}}dx}{a(b^2-4ac)}}{3a\left(b^2-4ac\right)}+\int \frac{x\left(gx^2+e\right)}{\left(cx^4+bx^2+a\right)^{5/2}}dx+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{3a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^{3/2}}$$

$\Big\downarrow$ 1511

$$\frac{\frac{x\left(cx^2\left(12a^2cf+ab^2f-16abcd+2b^3d\right)+4a^2bcf+20a^2c^2d+ab^3f-17ab^2cd+2b^4d\right)}{a(b^2-4ac)\sqrt{a+bx^2+cx^4}}-\frac{c\left(\frac{\sqrt{a}\left(12a^2cf+\sqrt{a}\sqrt{c}\left(8abf-20acd+b^2d\right)+ab^2f-16abcd+2b^3d\right)\int \cdots}{\sqrt{c}}\cdots\right.}{a\cdots}}{3a\left(b^2-4ac\right)}$$

$$\int \frac{x\left(gx^2+e\right)}{\left(cx^4+bx^2+a\right)^{5/2}}dx+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{3a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^{3/2}}$$

$\Big\downarrow$ 27

$$\frac{\frac{x\left(cx^2\left(12a^2cf+ab^2f-16abcd+2b^3d\right)+4a^2bcf+20a^2c^2d+ab^3f-17ab^2cd+2b^4d\right)}{a(b^2-4ac)\sqrt{a+bx^2+cx^4}}-\frac{c\left(\frac{\sqrt{a}\left(12a^2cf+\sqrt{a}\sqrt{c}\left(8abf-20acd+b^2d\right)+ab^2f-16abcd+2b^3d\right)\int \cdots}{\sqrt{c}}\cdots\right.}{a(b^2\cdots}}{3a\left(b^2-4ac\right)}$$

$$\int \frac{x\left(gx^2+e\right)}{\left(cx^4+bx^2+a\right)^{5/2}}dx+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{3a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^{3/2}}$$

$\Big\downarrow$ 1416

3.107. $\int \frac{d+ex+fx^2+gx^3}{\left(a+bx^2+cx^4\right)^{5/2}}\,dx$

$$\frac{\frac{x\left(cx^2\left(12a^2cf+ab^2f-16abcd+2b^3d\right)+4a^2bcf+20a^2c^2d+ab^3f-17ab^2cd+2b^4d\right)}{a(b^2-4ac)\sqrt{a+bx^2+cx^4}}-\frac{c\left(\frac{\sqrt[4]{a}\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}\left(12a^2cf+\sqrt{a}\sqrt{c}\left(8abf-20acd\cdots\right.\right.}{2c^{3/\cdots}}\cdots\right)}{\cdots}}{3a\left(b^2-4ac\right)}$$

$$\int \frac{x\left(gx^2+e\right)}{\left(cx^4+bx^2+a\right)^{5/2}}dx+\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{3a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^{3/2}}$$

$\downarrow$ 1509

$$\int \frac{x\left(gx^2+e\right)}{\left(cx^4+bx^2+a\right)^{5/2}}dx+$$

$$\frac{\frac{x\left(cx^2\left(12a^2cf+ab^2f-16abcd+2b^3d\right)+4a^2bcf+20a^2c^2d+ab^3f-17ab^2cd+2b^4d\right)}{a(b^2-4ac)\sqrt{a+bx^2+cx^4}}-\frac{c\left(\frac{\sqrt[4]{a}\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}\left(12a^2cf+\sqrt{a}\sqrt{c}\left(8abf-20acd\cdots\right.\right.}{2c^{3/\cdots}}\cdots\right)}{\cdots}}{\cdots}$$

$$\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{3a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^{3/2}}$$

$\downarrow$ 1576

$$\frac{1}{2}\int \frac{gx^2+e}{\left(cx^4+bx^2+a\right)^{5/2}}dx^2+$$

$$\frac{\frac{x\left(cx^2\left(12a^2cf+ab^2f-16abcd+2b^3d\right)+4a^2bcf+20a^2c^2d+ab^3f-17ab^2cd+2b^4d\right)}{a(b^2-4ac)\sqrt{a+bx^2+cx^4}}-\frac{c\left(\frac{\sqrt[4]{a}\left(\sqrt{a}+\sqrt{c}x^2\right)\sqrt{\frac{a+bx^2+cx^4}{\left(\sqrt{a}+\sqrt{c}x^2\right)^2}}\left(12a^2cf+\sqrt{a}\sqrt{c}\left(8abf-20acd\cdots\right.\right.}{2c^{3/\cdots}}\cdots\right)}{\cdots}}{\cdots}$$

$$\frac{x\left(cx^2(bd-2af)-abf-2acd+b^2d\right)}{3a\left(b^2-4ac\right)\left(a+bx^2+cx^4\right)^{3/2}}$$

$\downarrow$ 1159

3.107. $\int \frac{d+ex+fx^2+gx^3}{\left(a+bx^2+cx^4\right)^{5/2}}\,dx$

$$\frac{1}{2}\left(-\frac{4(2ce-bg)\int \frac{1}{(cx^4+bx^2+a)^{3/2}}dx^2}{3(b^2-4ac)}-\frac{2(-2ag+x^2(2ce-bg)+be)}{3(b^2-4ac)(a+bx^2+cx^4)^{3/2}}\right)+$$

$$\frac{\frac{x(cx^2(12a^2cf+ab^2f-16abcd+2b^3d)+4a^2bcf+20a^2c^2d+ab^3f-17ab^2cd+2b^4d)}{a(b^2-4ac)\sqrt{a+bx^2+cx^4}}-\frac{c\left(\frac{\sqrt[4]{a}(\sqrt{a}+\sqrt{c}x^2)\sqrt{\frac{a+bx^2+cx^4}{(\sqrt{a}+\sqrt{c}x^2)^2}}(12a^2cf+\sqrt{a}\sqrt{c}(8abf-20acd\ldots}{2c^{3/\ldots}}\right.}{\ldots}}{\ldots}$$

$$\frac{x(cx^2(bd-2af)-abf-2acd+b^2d)}{3a(b^2-4ac)(a+bx^2+cx^4)^{3/2}}$$

$\downarrow$ 1088

$$\frac{\frac{x(cx^2(12a^2cf+ab^2f-16abcd+2b^3d)+4a^2bcf+20a^2c^2d+ab^3f-17ab^2cd+2b^4d)}{a(b^2-4ac)\sqrt{a+bx^2+cx^4}}-\frac{c\left(\frac{\sqrt[4]{a}(\sqrt{a}+\sqrt{c}x^2)\sqrt{\frac{a+bx^2+cx^4}{(\sqrt{a}+\sqrt{c}x^2)^2}}(12a^2cf+\sqrt{a}\sqrt{c}(8abf-20acd\ldots}{2c^{3/\ldots}}\right.}{\ldots}}{\ldots}$$

$$\frac{x(cx^2(bd-2af)-abf-2acd+b^2d)}{3a(b^2-4ac)(a+bx^2+cx^4)^{3/2}}+$$

$$\frac{1}{2}\left(\frac{8(b+2cx^2)(2ce-bg)}{3(b^2-4ac)^2\sqrt{a+bx^2+cx^4}}-\frac{2(-2ag+x^2(2ce-bg)+be)}{3(b^2-4ac)(a+bx^2+cx^4)^{3/2}}\right)$$

input
```
Int[(d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4)^(5/2),x]
```

3.107. $\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{5/2}}\,dx$

output
```
(x*(b^2*d - 2*a*c*d - a*b*f + c*(b*d - 2*a*f)*x^2))/(3*a*(b^2 - 4*a*c)*(a
+ b*x^2 + c*x^4)^(3/2)) + ((-2*(b*e - 2*a*g + (2*c*e - b*g)*x^2))/(3*(b^2
- 4*a*c)*(a + b*x^2 + c*x^4)^(3/2)) + (8*(2*c*e - b*g)*(b + 2*c*x^2))/(3*(
b^2 - 4*a*c)^2*Sqrt[a + b*x^2 + c*x^4]))/2 + ((x*(2*b^4*d - 17*a*b^2*c*d +
 20*a^2*c^2*d + a*b^3*f + 4*a^2*b*c*f + c*(2*b^3*d - 16*a*b*c*d + a*b^2*f
+ 12*a^2*c*f)*x^2))/(a*(b^2 - 4*a*c)*Sqrt[a + b*x^2 + c*x^4]) - (c*(-(((2*
b^3*d - 16*a*b*c*d + a*b^2*f + 12*a^2*c*f)*(-((x*Sqrt[a + b*x^2 + c*x^4])/
(Sqrt[a] + Sqrt[c]*x^2)) + (a^(1/4)*(Sqrt[a] + Sqrt[c]*x^2)*Sqrt[(a + b*x^
2 + c*x^4)/(Sqrt[a] + Sqrt[c]*x^2)^2]*EllipticE[2*ArcTan[(c^(1/4)*x)/a^(1/
4)], (2 - b/(Sqrt[a]*Sqrt[c]))/4])/(c^(1/4)*Sqrt[a + b*x^2 + c*x^4])))/Sqr
t[c]) + (a^(1/4)*(2*b^3*d - 16*a*b*c*d + a*b^2*f + 12*a^2*c*f + Sqrt[a]*Sq
rt[c]*(b^2*d - 20*a*c*d + 8*a*b*f))*(Sqrt[a] + Sqrt[c]*x^2)*Sqrt[(a + b*x^
2 + c*x^4)/(Sqrt[a] + Sqrt[c]*x^2)^2]*EllipticF[2*ArcTan[(c^(1/4)*x)/a^(1/
4)], (2 - b/(Sqrt[a]*Sqrt[c]))/4])/(2*c^(3/4)*Sqrt[a + b*x^2 + c*x^4])))/(
a*(b^2 - 4*a*c)))/(3*a*(b^2 - 4*a*c))
```

3.107.3.1 Defintions of rubi rules used

rule 25
```
Int[-(Fx_), x_Symbol] :> Simp[Identity[-1]   Int[Fx, x], x]
```

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 1088
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-3/2), x_Symbol] :> Simp[-2*((b +
2*c*x)/((b^2 - 4*a*c)*Sqrt[a + b*x + c*x^2])), x] /; FreeQ[{a, b, c}, x] &&
 NeQ[b^2 - 4*a*c, 0]
```

rule 1159
```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[((b*d - 2*a*e + (2*c*d - b*e)*x)/((p + 1)*(b^2 - 4*a*c)))*(a + b*
x + c*x^2)^(p + 1), x] - Simp[(2*p + 3)*((2*c*d - b*e)/((p + 1)*(b^2 - 4*a*
c)))   Int[(a + b*x + c*x^2)^(p + 1), x], x] /; FreeQ[{a, b, c, d, e}, x] &
& LtQ[p, -1] && NeQ[p, -3/2]
```

rule 1416
```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4], x_Symbol] :> With[{q = Rt[c
/a, 4]}, Simp[(1 + q^2*x^2)*(Sqrt[(a + b*x^2 + c*x^4)/(a*(1 + q^2*x^2)^2)]/
(2*q*Sqrt[a + b*x^2 + c*x^4]))*EllipticF[2*ArcTan[q*x], 1/2 - b*(q^2/(4*c))
], x]] /; FreeQ[{a, b, c}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[c/a]
```

rule 1492
```
Int[((d_) + (e_.)*(x_)^2)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symb
ol] :> Simp[x*(a*b*e - d*(b^2 - 2*a*c) - c*(b*d - 2*a*e)*x^2)*((a + b*x^2 +
 c*x^4)^(p + 1)/(2*a*(p + 1)*(b^2 - 4*a*c))), x] + Simp[1/(2*a*(p + 1)*(b^2
 - 4*a*c))   Int[Simp[(2*p + 3)*d*b^2 - a*b*e - 2*a*c*d*(4*p + 5) + (4*p +
7)*(d*b - 2*a*e)*c*x^2, x]*(a + b*x^2 + c*x^4)^(p + 1), x], x] /; FreeQ[{a,
 b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] &&
 LtQ[p, -1] && IntegerQ[2*p]
```

rule 1509
```
Int[((d_) + (e_.)*(x_)^2)/Sqrt[(a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4], x_Symbo
l] :> With[{q = Rt[c/a, 4]}, Simp[(-d)*x*(Sqrt[a + b*x^2 + c*x^4]/(a*(1 + q
^2*x^2))), x] + Simp[d*(1 + q^2*x^2)*(Sqrt[(a + b*x^2 + c*x^4)/(a*(1 + q^2*
x^2)^2)]/(q*Sqrt[a + b*x^2 + c*x^4]))*EllipticE[2*ArcTan[q*x], 1/2 - b*(q^2
/(4*c))], x] /; EqQ[e + d*q^2, 0]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2
- 4*a*c, 0] && PosQ[c/a]
```

rule 1511
```
Int[((d_) + (e_.)*(x_)^2)/Sqrt[(a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4], x_Symbo
l] :> With[{q = Rt[c/a, 2]}, Simp[(e + d*q)/q   Int[1/Sqrt[a + b*x^2 + c*x^
4], x], x] - Simp[e/q   Int[(1 - q*x^2)/Sqrt[a + b*x^2 + c*x^4], x], x] /;
NeQ[e + d*q, 0]] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && Pos
Q[c/a]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.107. $\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{5/2}} dx$

3.107.4 Maple [A] (verified)

Time = 2.32 (sec) , antiderivative size = 829, normalized size of antiderivative = 1.22

method	result
elliptic	$\frac{\left(\frac{(2af-bd)x^3}{3ca\left(4ac-b^2\right)}-\frac{(bg-2ec)x^2}{3\left(4ac-b^2\right)c^2}+\frac{\left(abf+2acd-b^2d\right)x}{3a\left(4ac-b^2\right)c^2}-\frac{2ag-be}{3\left(4ac-b^2\right)c^2}\right)\sqrt{c\,x^4+b\,x^2+a}}{\left(x^4+\frac{b\,x^2}{c}+\frac{a}{c}\right)^2}-\frac{2c\left(-\frac{\left(12a^2cf+a\,b^2f-16abcd+2b^3d\right)x^3}{6a^2\left(4ac-b^2\right)^2}+\frac{4(bg-2ec}{3\left(4ac-b\right.}\right.}{}$
default	Expression too large to display

input
```
int((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^(5/2),x,method=_RETURNVERBOSE)
```

output
```
(1/3/c*(2*a*f-b*d)/a/(4*a*c-b^2)*x^3-1/3*(b*g-2*c*e)/(4*a*c-b^2)/c^2*x^2+1
/3*(a*b*f+2*a*c*d-b^2*d)/a/(4*a*c-b^2)/c^2*x-1/3*(2*a*g-b*e)/(4*a*c-b^2)/c
^2)*(c*x^4+b*x^2+a)^(1/2)/(x^4+1/c*b*x^2+a/c)^2-2*c*(-1/6*(12*a^2*c*f+a*b^
2*f-16*a*b*c*d+2*b^3*d)/a^2/(4*a*c-b^2)^2*x^3+4/3*(b*g-2*c*e)/(4*a*c-b^2)^
2*x^2-1/6*(4*a^2*b*c*f+20*a^2*c^2*d+a*b^3*f-17*a*b^2*c*d+2*b^4*d)/a^2/(4*a
*c-b^2)^2/c*x+2/3*b*(b*g-2*c*e)/(4*a*c-b^2)^2/c)/((x^4+1/c*b*x^2+a/c)*c)^(
1/2)+1/4*(-1/3/(4*a*c-b^2)*(a*b*f-10*a*c*d+2*b^2*d)/a^2-1/3*(4*a^2*b*c*f+2
0*a^2*c^2*d+a*b^3*f-17*a*b^2*c*d+2*b^4*d)/a^2/(4*a*c-b^2)^2)*2^(1/2)/((-b+
(-4*a*c+b^2)^(1/2))/a)^(1/2)*(4-2*(-b+(-4*a*c+b^2)^(1/2))/a*x^2)^(1/2)*(4+
2*(b+(-4*a*c+b^2)^(1/2))/a*x^2)^(1/2)/(c*x^4+b*x^2+a)^(1/2)*EllipticF(1/2*
x*2^(1/2)*((-b+(-4*a*c+b^2)^(1/2))/a)^(1/2),1/2*(-4+2*b*(b+(-4*a*c+b^2)^(1
/2))/a/c)^(1/2))+1/6*c*(12*a^2*c*f+a*b^2*f-16*a*b*c*d+2*b^3*d)/(4*a*c-b^2)
^2/a*2^(1/2)/((-b+(-4*a*c+b^2)^(1/2))/a)^(1/2)*(4-2*(-b+(-4*a*c+b^2)^(1/2)
)/a*x^2)^(1/2)*(4+2*(b+(-4*a*c+b^2)^(1/2))/a*x^2)^(1/2)/(c*x^4+b*x^2+a)^(1
/2)/(b+(-4*a*c+b^2)^(1/2))*(EllipticF(1/2*x*2^(1/2)*((-b+(-4*a*c+b^2)^(1/2
))/a)^(1/2),1/2*(-4+2*b*(b+(-4*a*c+b^2)^(1/2))/a/c)^(1/2))-EllipticE(1/2*x
*2^(1/2)*((-b+(-4*a*c+b^2)^(1/2))/a)^(1/2),1/2*(-4+2*b*(b+(-4*a*c+b^2)^(1/
2))/a/c)^(1/2)))
```

3.107.5 Fricas [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 1948 vs. 2(652) = 1304.

Time = 0.14 (sec) , antiderivative size = 1948, normalized size of antiderivative = 2.86

$$\int \frac{d+ex+fx^2+gx^3}{\left(a+bx^2+cx^4\right)^{5/2}}\,dx = \text{Too large to display}$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^(5/2),x, algorithm="fricas")
```

output
```
-1/6*(sqrt(1/2)*((2*(b^4*c^2 - 8*a*b^2*c^3)*d + (a*b^3*c^2 + 12*a^2*b*c^3)
*f)*x^8 + 2*(2*(b^5*c - 8*a*b^3*c^2)*d + (a*b^4*c + 12*a^2*b^2*c^2)*f)*x^6
 + (2*(b^6 - 6*a*b^4*c - 16*a^2*b^2*c^2)*d + (a*b^5 + 14*a^2*b^3*c + 24*a^
3*b*c^2)*f)*x^4 + 2*(2*(a*b^5 - 8*a^2*b^3*c)*d + (a^2*b^4 + 12*a^3*b^2*c)*
f)*x^2 + 2*(a^2*b^4 - 8*a^3*b^2*c)*d + (a^3*b^3 + 12*a^4*b*c)*f - ((2*(a*b
^3*c^2 - 8*a^2*b*c^3)*d + (a^2*b^2*c^2 + 12*a^3*c^3)*f)*x^8 + 2*(2*(a*b^4*
c - 8*a^2*b^2*c^2)*d + (a^2*b^3*c + 12*a^3*b*c^2)*f)*x^6 + (2*(a*b^5 - 6*a
^2*b^3*c - 16*a^3*b*c^2)*d + (a^2*b^4 + 14*a^3*b^2*c + 24*a^4*c^2)*f)*x^4
+ 2*(2*(a^2*b^4 - 8*a^3*b^2*c)*d + (a^3*b^3 + 12*a^4*b*c)*f)*x^2 + 2*(a^3*
b^3 - 8*a^4*b*c)*d + (a^4*b^2 + 12*a^5*c)*f)*sqrt((b^2 - 4*a*c)/a^2))*sqrt
(a)*sqrt((a*sqrt((b^2 - 4*a*c)/a^2) - b)/a)*elliptic_e(arcsin(sqrt(1/2)*x*
sqrt((a*sqrt((b^2 - 4*a*c)/a^2) - b)/a)), 1/2*(a*b*sqrt((b^2 - 4*a*c)/a^2)
 + b^2 - 2*a*c)/(a*c)) + sqrt(1/2)*(((4*(5*a^2*b + 4*a*b^2)*c^3 - (a*b^3 +
 2*b^4)*c^2)*d - (12*a^2*b*c^3 + (8*a^2*b^2 + a*b^3)*c^2)*f)*x^8 + 2*((4*(
5*a^2*b^2 + 4*a*b^3)*c^2 - (a*b^4 + 2*b^5)*c)*d - (12*a^2*b^2*c^2 + (8*a^2
*b^3 + a*b^4)*c)*f)*x^6 - ((a*b^5 + 2*b^6 - 8*(5*a^3*b + 4*a^2*b^2)*c^2 -
6*(3*a^2*b^3 + 2*a*b^4)*c)*d + (8*a^2*b^4 + a*b^5 + 24*a^3*b*c^2 + 2*(8*a^
3*b^2 + 7*a^2*b^3)*c)*f)*x^4 - 2*((a^2*b^4 + 2*a*b^5 - 4*(5*a^3*b^2 + 4*a^
2*b^3)*c)*d + (8*a^3*b^3 + a^2*b^4 + 12*a^3*b^2*c)*f)*x^2 - (a^3*b^3 + 2*a
^2*b^4 - 4*(5*a^4*b + 4*a^3*b^2)*c)*d - (8*a^4*b^2 + a^3*b^3 + 12*a^4*b...
```

3.107.6 Sympy [F(-1)]

Timed out.

$$\int \frac{d + ex + fx^2 + gx^3}{(a + bx^2 + cx^4)^{5/2}} \, dx = \text{Timed out}$$

input
```
integrate((g*x**3+f*x**2+e*x+d)/(c*x**4+b*x**2+a)**(5/2),x)
```

output
```
Timed out
```

3.107.7 Maxima [F]

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{5/2}}\,dx = \int \frac{gx^3+fx^2+ex+d}{(cx^4+bx^2+a)^{\frac{5}{2}}}\,dx$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^(5/2),x, algorithm="maxima")
```

output
```
integrate((g*x^3 + f*x^2 + e*x + d)/(c*x^4 + b*x^2 + a)^(5/2), x)
```

3.107.8 Giac [F]

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{5/2}}\,dx = \int \frac{gx^3+fx^2+ex+d}{(cx^4+bx^2+a)^{\frac{5}{2}}}\,dx$$

input
```
integrate((g*x^3+f*x^2+e*x+d)/(c*x^4+b*x^2+a)^(5/2),x, algorithm="giac")
```

output
```
integrate((g*x^3 + f*x^2 + e*x + d)/(c*x^4 + b*x^2 + a)^(5/2), x)
```

3.107.9 Mupad [F(-1)]

Timed out.

$$\int \frac{d+ex+fx^2+gx^3}{(a+bx^2+cx^4)^{5/2}}\,dx = \int \frac{g\,x^3+f\,x^2+e\,x+d}{(c\,x^4+b\,x^2+a)^{5/2}}\,dx$$

input
```
int((d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4)^(5/2),x)
```

output
```
int((d + e*x + f*x^2 + g*x^3)/(a + b*x^2 + c*x^4)^(5/2), x)
```

3.108 $\int \frac{ag-cgx^4}{(a+bx^2+cx^4)^{3/2}}\,dx$

3.108.1 Optimal result

Integrand size = 28, antiderivative size = 19

$$\int \frac{ag-cgx^4}{(a+bx^2+cx^4)^{3/2}}\,dx=\frac{gx}{\sqrt{a+bx^2+cx^4}}$$

output
```
g*x/(c*x^4+b*x^2+a)^(1/2)
```

3.108.2 Mathematica [A] (verified)

Time = 10.09 (sec) , antiderivative size = 19, normalized size of antiderivative = 1.00

$$\int \frac{ag-cgx^4}{(a+bx^2+cx^4)^{3/2}}\,dx=\frac{gx}{\sqrt{a+bx^2+cx^4}}$$

input
```
Integrate[(a*g - c*g*x^4)/(a + b*x^2 + c*x^4)^(3/2),x]
```

output
```
(g*x)/Sqrt[a + b*x^2 + c*x^4]
```

3.108.3 Rubi [A] (verified)

Time = 0.16 (sec) , antiderivative size = 19, normalized size of antiderivative = 1.00, number of steps used = 1, number of rules used = 1, $\frac{\text{number of rules}}{\text{integrand size}} = 0.036$, Rules used = {2021}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{ag - cgx^4}{(a + bx^2 + cx^4)^{3/2}} \, dx$$

$$\Big\downarrow \ 2021$$

$$\frac{gx}{\sqrt{a + bx^2 + cx^4}}$$

input
```
Int[(a*g - c*g*x^4)/(a + b*x^2 + c*x^4)^(3/2),x]
```

output
```
(g*x)/Sqrt[a + b*x^2 + c*x^4]
```

3.108.3.1 Defintions of rubi rules used

rule 2021
```
Int[(Pp_)*(Qq_)^(m_.), x_Symbol] :> With[{p = Expon[Pp, x], q = Expon[Qq, x
]}, Simp[Coeff[Pp, x, p]*x^(p - q + 1)*(Qq^(m + 1)/((p + m*q + 1)*Coeff[Qq,
 x, q])), x] /; NeQ[p + m*q + 1, 0] && EqQ[(p + m*q + 1)*Coeff[Qq, x, q]*Pp
, Coeff[Pp, x, p]*x^(p - q)*((p - q + 1)*Qq + (m + 1)*x*D[Qq, x])]] /; Free
Q[m, x] && PolyQ[Pp, x] && PolyQ[Qq, x] && NeQ[m, -1]
```

3.108.4 Maple [A] (verified)

Time = 1.10 (sec) , antiderivative size = 18, normalized size of antiderivative = 0.95

method	result	size
gosper	$\frac{gx}{\sqrt{c\,x^{4}+b\,x^{2}+a}}$	18
default	$\frac{gx}{\sqrt{c\,x^{4}+b\,x^{2}+a}}$	18
trager	$\frac{gx}{\sqrt{c\,x^{4}+b\,x^{2}+a}}$	18
elliptic	$\frac{gx}{\sqrt{c\,x^{4}+b\,x^{2}+a}}$	18
pseudoelliptic	$\frac{gx}{\sqrt{c\,x^{4}+b\,x^{2}+a}}$	18

input
```
int((-c*g*x^4+a*g)/(c*x^4+b*x^2+a)^(3/2),x,method=_RETURNVERBOSE)
```

output
```
g*x/(c*x^4+b*x^2+a)^(1/2)
```

3.108.5 Fricas [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 17, normalized size of antiderivative = 0.89

$$\int \frac{ag - cgx^4}{(a + bx^2 + cx^4)^{3/2}} \, dx = \frac{gx}{\sqrt{cx^4 + bx^2 + a}}$$

input
```
integrate((-c*g*x^4+a*g)/(c*x^4+b*x^2+a)^(3/2),x, algorithm="fricas")
```

output
```
g*x/sqrt(c*x^4 + b*x^2 + a)
```

3.108.6 Sympy [F]

$$\begin{aligned}\int \frac{ag - cgx^4}{(a + bx^2 + cx^4)^{3/2}} \, dx = &\\ -g\Bigg(\int \left(-\frac{a}{a\sqrt{a + bx^2 + cx^4} + bx^2\sqrt{a + bx^2 + cx^4} + cx^4\sqrt{a + bx^2 + cx^4}}\right) dx &\\ + \int \frac{cx^4}{a\sqrt{a + bx^2 + cx^4} + bx^2\sqrt{a + bx^2 + cx^4} + cx^4\sqrt{a + bx^2 + cx^4}} \, dx\Bigg)&\end{aligned}$$

input
```
integrate((-c*g*x**4+a*g)/(c*x**4+b*x**2+a)**(3/2),x)
```

output
```
-g*(Integral(-a/(a*sqrt(a + b*x**2 + c*x**4) + b*x**2*sqrt(a + b*x**2 + c*
x**4) + c*x**4*sqrt(a + b*x**2 + c*x**4)), x) + Integral(c*x**4/(a*sqrt(a
+ b*x**2 + c*x**4) + b*x**2*sqrt(a + b*x**2 + c*x**4) + c*x**4*sqrt(a + b*
x**2 + c*x**4)), x))
```

3.108.7 Maxima [A] (verification not implemented)

Time = 0.22 (sec) , antiderivative size = 17, normalized size of antiderivative = 0.89

$$\int \frac{ag - cgx^4}{(a + bx^2 + cx^4)^{3/2}}\, dx = \frac{gx}{\sqrt{cx^4 + bx^2 + a}}$$

input
```
integrate((-c*g*x^4+a*g)/(c*x^4+b*x^2+a)^(3/2),x, algorithm="maxima")
```

output
```
g*x/sqrt(c*x^4 + b*x^2 + a)
```

3.108.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 60 vs. 2(17) = 34.

Time = 0.67 (sec) , antiderivative size = 60, normalized size of antiderivative = 3.16

$$\int \frac{ag - cgx^4}{(a + bx^2 + cx^4)^{3/2}}\, dx = \frac{(b^4g - 8\, ab^2cg + 16\, a^2c^2g)x}{\sqrt{cx^4 + bx^2 + a}(b^4 - 8\, ab^2c + 16\, a^2c^2)}$$

input
```
integrate((-c*g*x^4+a*g)/(c*x^4+b*x^2+a)^(3/2),x, algorithm="giac")
```

output
```
(b^4*g - 8*a*b^2*c*g + 16*a^2*c^2*g)*x/(sqrt(c*x^4 + b*x^2 + a)*(b^4 - 8*a
*b^2*c + 16*a^2*c^2))
```

3.108.9 Mupad [B] (verification not implemented)

Time = 8.01 (sec) , antiderivative size = 17, normalized size of antiderivative = 0.89

$$\int \frac{ag - cgx^4}{(a + bx^2 + cx^4)^{3/2}}\, dx = \frac{g\, x}{\sqrt{c\, x^4 + b\, x^2 + a}}$$

input
```
int((a*g - c*g*x^4)/(a + b*x^2 + c*x^4)^(3/2),x)
```

output
```
(g*x)/(a + b*x^2 + c*x^4)^(1/2)
```

3.109 $\int \frac{ag+ex-cgx^4}{(a+bx^2+cx^4)^{3/2}} \, dx$

3.109.1 Optimal result

Integrand size = 31, antiderivative size = 57

$$\int \frac{ag+ex-cgx^4}{(a+bx^2+cx^4)^{3/2}} \, dx = \frac{gx}{\sqrt{a+bx^2+cx^4}} - \frac{e(b+2cx^2)}{(b^2-4ac)\sqrt{a+bx^2+cx^4}}$$

output
```
g*x/(c*x^4+b*x^2+a)^(1/2)-e*(2*c*x^2+b)/(-4*a*c+b^2)/(c*x^4+b*x^2+a)^(1/2)
```

3.109.2 Mathematica [A] (verified)

Time = 10.13 (sec) , antiderivative size = 51, normalized size of antiderivative = 0.89

$$\int \frac{ag+ex-cgx^4}{(a+bx^2+cx^4)^{3/2}} \, dx = \frac{-be+b^2gx-4acgx-2cex^2}{(b^2-4ac)\sqrt{a+bx^2+cx^4}}$$

input
```
Integrate[(a*g + e*x - c*g*x^4)/(a + b*x^2 + c*x^4)^(3/2),x]
```

output
```
(-(b*e) + b^2*g*x - 4*a*c*g*x - 2*c*e*x^2)/((b^2 - 4*a*c)*Sqrt[a + b*x^2 +
 c*x^4])
```

3.109.3 Rubi [A] (verified)

Time = 0.27 (sec) , antiderivative size = 57, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.161, Rules used = {2202, 27, 1432, 1088, 2021}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{ag-cgx^4+ex}{(a+bx^2+cx^4)^{3/2}}dx$$

$$\Big\downarrow \text{ 2202}$$

$$\int \frac{ex}{(cx^4+bx^2+a)^{3/2}}dx+\int \frac{ag-cgx^4}{(cx^4+bx^2+a)^{3/2}}dx$$

$$\Big\downarrow \text{ 27}$$

$$e\int \frac{x}{(cx^4+bx^2+a)^{3/2}}dx+\int \frac{ag-cgx^4}{(cx^4+bx^2+a)^{3/2}}dx$$

$$\Big\downarrow \text{ 1432}$$

$$\frac{1}{2}e\int \frac{1}{(cx^4+bx^2+a)^{3/2}}dx^2+\int \frac{ag-cgx^4}{(cx^4+bx^2+a)^{3/2}}dx$$

$$\Big\downarrow \text{ 1088}$$

$$\int \frac{ag-cgx^4}{(cx^4+bx^2+a)^{3/2}}dx-\frac{e(b+2cx^2)}{(b^2-4ac)\sqrt{a+bx^2+cx^4}}$$

$$\Big\downarrow \text{ 2021}$$

$$\frac{gx}{\sqrt{a+bx^2+cx^4}}-\frac{e(b+2cx^2)}{(b^2-4ac)\sqrt{a+bx^2+cx^4}}$$

input
```
Int[(a*g + e*x - c*g*x^4)/(a + b*x^2 + c*x^4)^(3/2),x]
```

output
```
(g*x)/Sqrt[a + b*x^2 + c*x^4] - (e*(b + 2*c*x^2))/((b^2 - 4*a*c)*Sqrt[a +
b*x^2 + c*x^4])
```

3.109.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 1088
```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-3/2), x_Symbol] :> Simp[-2*((b +
2*c*x)/((b^2 - 4*a*c)*Sqrt[a + b*x + c*x^2])), x] /; FreeQ[{a, b, c}, x] &&
 NeQ[b^2 - 4*a*c, 0]
```

rule 1432
```
Int[(x_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Simp[1/2
 Subst[Int[(a + b*x + c*x^2)^p, x], x, x^2], x] /; FreeQ[{a, b, c, p}, x]
```

rule 2021
```
Int[(Pp_)*(Qq_)^(m_.), x_Symbol] :> With[{p = Expon[Pp, x], q = Expon[Qq, x
]}, Simp[Coeff[Pp, x, p]*x^(p - q + 1)*(Qq^(m + 1)/((p + m*q + 1)*Coeff[Qq,
 x, q])), x] /; NeQ[p + m*q + 1, 0] && EqQ[(p + m*q + 1)*Coeff[Qq, x, q]*Pp
, Coeff[Pp, x, p]*x^(p - q)*((p - q + 1)*Qq + (m + 1)*x*D[Qq, x])]] /; Free
Q[m, x] && PolyQ[Pp, x] && PolyQ[Qq, x] && NeQ[m, -1]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.109.4 Maple [A] (verified)

Time = 1.60 (sec) , antiderivative size = 52, normalized size of antiderivative = 0.91

method	result
gosper	$\frac{4acgx-b^2gx+2c\,x^2e+be}{\sqrt{c\,x^4+b\,x^2+a}\,(4ac-b^2)}$
trager	$\frac{4acgx-b^2gx+2c\,x^2e+be}{\sqrt{c\,x^4+b\,x^2+a}\,(4ac-b^2)}$
elliptic	$\frac{e(2c\,x^2+b)}{\sqrt{c\,x^4+b\,x^2+a}\,(4ac-b^2)}+\frac{gx}{\sqrt{c\,x^4+b\,x^2+a}}$
default	$ag\left(-\frac{2c\left(\frac{b\,x^3}{2a(4ac-b^2)}-\frac{(2ac-b^2)x}{2a(4ac-b^2)c}\right)}{\sqrt{\left(x^4+\frac{b\,x^2}{c}+\frac{a}{c}\right)c}}+\frac{\left(\frac{1}{a}-\frac{2ac-b^2}{a(4ac-b^2)}\right)\sqrt{2}\sqrt{4-\frac{2(-b+\sqrt{-4ac+b^2})x^2}{a}}\sqrt{4+\frac{2(b+\sqrt{-4ac+b^2})x^2}{a}}\,F\left(\frac{x\sqrt{2}\sqrt{-b}}{}\right.}{4\sqrt{\frac{-b+\sqrt{-4ac+b^2}}{a}}\sqrt{c\,x^4+b\,x^2+a}}\right.$

input
```
int((-c*g*x^4+a*g+e*x)/(c*x^4+b*x^2+a)^(3/2),x,method=_RETURNVERBOSE)
```

output
```
(4*a*c*g*x-b^2*g*x+2*c*e*x^2+b*e)/(c*x^4+b*x^2+a)^(1/2)/(4*a*c-b^2)
```

3.109.5 Fricas [A] (verification not implemented)

Time = 0.27 (sec) , antiderivative size = 82, normalized size of antiderivative = 1.44

$$\int \frac{ag+ex-cgx^4}{\left(a+bx^2+cx^4\right)^{3/2}}\,dx=-\frac{\sqrt{cx^4+bx^2+a}(2\,cex^2-(b^2-4\,ac)gx+be)}{(b^2c-4\,ac^2)x^4+ab^2-4\,a^2c+(b^3-4\,abc)x^2}$$

input
```
integrate((-c*g*x^4+a*g+e*x)/(c*x^4+b*x^2+a)^(3/2),x, algorithm="fricas")
```

output
```
-sqrt(c*x^4 + b*x^2 + a)*(2*c*e*x^2 - (b^2 - 4*a*c)*g*x + b*e)/((b^2*c - 4
*a*c^2)*x^4 + a*b^2 - 4*a^2*c + (b^3 - 4*a*b*c)*x^2)
```

3.109.6 Sympy [F]

$$\int \frac{ag + ex - cgx^4}{(a + bx^2 + cx^4)^{3/2}}\, dx =$$
$$- \int \left(-\frac{ag}{a\sqrt{a + bx^2 + cx^4} + bx^2\sqrt{a + bx^2 + cx^4} + cx^4\sqrt{a + bx^2 + cx^4}}\right) dx$$
$$- \int \left(-\frac{ex}{a\sqrt{a + bx^2 + cx^4} + bx^2\sqrt{a + bx^2 + cx^4} + cx^4\sqrt{a + bx^2 + cx^4}}\right) dx$$
$$- \int \frac{cgx^4}{a\sqrt{a + bx^2 + cx^4} + bx^2\sqrt{a + bx^2 + cx^4} + cx^4\sqrt{a + bx^2 + cx^4}}\, dx$$

input
```
integrate((-c*g*x**4+a*g+e*x)/(c*x**4+b*x**2+a)**(3/2),x)
```

output
```
-Integral(-a*g/(a*sqrt(a + b*x**2 + c*x**4) + b*x**2*sqrt(a + b*x**2 + c*x
**4) + c*x**4*sqrt(a + b*x**2 + c*x**4)), x) - Integral(-e*x/(a*sqrt(a + b
*x**2 + c*x**4) + b*x**2*sqrt(a + b*x**2 + c*x**4) + c*x**4*sqrt(a + b*x**
2 + c*x**4)), x) - Integral(c*g*x**4/(a*sqrt(a + b*x**2 + c*x**4) + b*x**2
*sqrt(a + b*x**2 + c*x**4) + c*x**4*sqrt(a + b*x**2 + c*x**4)), x)
```

3.109.7 Maxima [A] (verification not implemented)

Time = 0.24 (sec) , antiderivative size = 51, normalized size of antiderivative = 0.89

$$\int \frac{ag + ex - cgx^4}{(a + bx^2 + cx^4)^{3/2}}\, dx = -\frac{2\, cex^2 + be - (b^2g - 4\, acg)x}{\sqrt{cx^4 + bx^2 + a}(b^2 - 4\, ac)}$$

input
```
integrate((-c*g*x^4+a*g+e*x)/(c*x^4+b*x^2+a)^(3/2),x, algorithm="maxima")
```

output
```
-(2*c*e*x^2 + b*e - (b^2*g - 4*a*c*g)*x)/(sqrt(c*x^4 + b*x^2 + a)*(b^2 - 4
*a*c))
```

3.109.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 138 vs. 2(53) = 106.

Time = 0.66 (sec) , antiderivative size = 138, normalized size of antiderivative = 2.42

$$\int \frac{ag+ex-cgx^4}{(a+bx^2+cx^4)^{3/2}}\,dx = -\frac{\left(\frac{2\,(b^2ce-4\,ac^2e)x}{b^4-8\,ab^2c+16\,a^2c^2} - \frac{b^4g-8\,ab^2cg+16\,a^2c^2g}{b^4-8\,ab^2c+16\,a^2c^2}\right)x + \frac{b^3e-4\,abce}{b^4-8\,ab^2c+16\,a^2c^2}}{\sqrt{cx^4+bx^2+a}}$$

input
```
integrate((-c*g*x^4+a*g+e*x)/(c*x^4+b*x^2+a)^(3/2),x, algorithm="giac")
```

output
```
-((2*(b^2*c*e - 4*a*c^2*e)*x/(b^4 - 8*a*b^2*c + 16*a^2*c^2) - (b^4*g - 8*a
*b^2*c*g + 16*a^2*c^2*g)/(b^4 - 8*a*b^2*c + 16*a^2*c^2))*x + (b^3*e - 4*a*
b*c*e)/(b^4 - 8*a*b^2*c + 16*a^2*c^2))/sqrt(c*x^4 + b*x^2 + a)
```

3.109.9 Mupad [B] (verification not implemented)

Time = 7.92 (sec) , antiderivative size = 51, normalized size of antiderivative = 0.89

$$\int \frac{ag+ex-cgx^4}{(a+bx^2+cx^4)^{3/2}}\,dx = \frac{-g\,b^2\,x+e\,b+2\,c\,e\,x^2+4\,a\,c\,g\,x}{(4\,a\,c-b^2)\,\sqrt{c\,x^4+b\,x^2+a}}$$

input
```
int((a*g + e*x - c*g*x^4)/(a + b*x^2 + c*x^4)^(3/2),x)
```

output
```
(b*e + 2*c*e*x^2 - b^2*g*x + 4*a*c*g*x)/((4*a*c - b^2)*(a + b*x^2 + c*x^4)
^(1/2))
```

3.110 $\int \frac{ag+fx^3-cgx^4}{(a+bx^2+cx^4)^{3/2}}\,dx$

3.110.1 Optimal result

Integrand size = 33, antiderivative size = 57

$$\int \frac{ag+fx^3-cgx^4}{(a+bx^2+cx^4)^{3/2}}\,dx=\frac{gx}{\sqrt{a+bx^2+cx^4}}+\frac{f(2a+bx^2)}{(b^2-4ac)\sqrt{a+bx^2+cx^4}}$$

output
```
g*x/(c*x^4+b*x^2+a)^(1/2)+f*(b*x^2+2*a)/(-4*a*c+b^2)/(c*x^4+b*x^2+a)^(1/2)
```

3.110.2 Mathematica [A] (verified)

Time = 10.08 (sec) , antiderivative size = 48, normalized size of antiderivative = 0.84

$$\int \frac{ag+fx^3-cgx^4}{(a+bx^2+cx^4)^{3/2}}\,dx=\frac{bx(bg+fx)+2a(f-2cgx)}{(b^2-4ac)\sqrt{a+bx^2+cx^4}}$$

input
```
Integrate[(a*g + f*x^3 - c*g*x^4)/(a + b*x^2 + c*x^4)^(3/2),x]
```

output
```
(b*x*(b*g + f*x) + 2*a*(f - 2*c*g*x))/((b^2 - 4*a*c)*Sqrt[a + b*x^2 + c*x^
4])
```

3.110.3 Rubi [A] (verified)

Time = 0.28 (sec) , antiderivative size = 57, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.152, Rules used = {2202, 27, 1434, 1158, 2021}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{ag-cgx^4+fx^3}{\left(a+bx^2+cx^4\right)^{3/2}}dx$$

$$\Big\downarrow \text{ 2202}$$

$$\int \frac{fx^3}{\left(cx^4+bx^2+a\right)^{3/2}}dx+\int \frac{ag-cgx^4}{\left(cx^4+bx^2+a\right)^{3/2}}dx$$

$$\Big\downarrow \text{ 27}$$

$$f\int \frac{x^3}{\left(cx^4+bx^2+a\right)^{3/2}}dx+\int \frac{ag-cgx^4}{\left(cx^4+bx^2+a\right)^{3/2}}dx$$

$$\Big\downarrow \text{ 1434}$$

$$\frac{1}{2}f\int \frac{x^2}{\left(cx^4+bx^2+a\right)^{3/2}}dx^2+\int \frac{ag-cgx^4}{\left(cx^4+bx^2+a\right)^{3/2}}dx$$

$$\Big\downarrow \text{ 1158}$$

$$\int \frac{ag-cgx^4}{\left(cx^4+bx^2+a\right)^{3/2}}dx+\frac{f\left(2a+bx^2\right)}{\left(b^2-4ac\right)\sqrt{a+bx^2+cx^4}}$$

$$\Big\downarrow \text{ 2021}$$

$$\frac{f\left(2a+bx^2\right)}{\left(b^2-4ac\right)\sqrt{a+bx^2+cx^4}}+\frac{gx}{\sqrt{a+bx^2+cx^4}}$$

input
```
Int[(a*g + f*x^3 - c*g*x^4)/(a + b*x^2 + c*x^4)^(3/2),x]
```

output
```
(g*x)/Sqrt[a + b*x^2 + c*x^4] + (f*(2*a + b*x^2))/((b^2 - 4*a*c)*Sqrt[a +
b*x^2 + c*x^4])
```

3.110.3.1 Defintions of rubi rules used

rule 27
```
Int[(a_)*(Fx_), x_Symbol] :> Simp[a   Int[Fx, x], x] /; FreeQ[a, x] &&  !Ma
tchQ[Fx, (b_)*(Gx_) /; FreeQ[b, x]]
```

rule 1158
```
Int[((d_.) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(3/2), x_Symbo
l] :> Simp[-2*((b*d - 2*a*e + (2*c*d - b*e)*x)/((b^2 - 4*a*c)*Sqrt[a + b*x
+ c*x^2])), x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1434
```
Int[(x_)^(m_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_.), x_Symbol] :> Simp
[1/2   Subst[Int[x^((m - 1)/2)*(a + b*x + c*x^2)^p, x], x, x^2], x] /; Free
Q[{a, b, c, p}, x] && IntegerQ[(m - 1)/2]
```

rule 2021
```
Int[(Pp_)*(Qq_)^(m_.), x_Symbol] :> With[{p = Expon[Pp, x], q = Expon[Qq, x
]}, Simp[Coeff[Pp, x, p]*x^(p - q + 1)*(Qq^(m + 1)/((p + m*q + 1)*Coeff[Qq,
 x, q])), x] /; NeQ[p + m*q + 1, 0] && EqQ[(p + m*q + 1)*Coeff[Qq, x, q]*Pp
, Coeff[Pp, x, p]*x^(p - q)*((p - q + 1)*Qq + (m + 1)*x*D[Qq, x])]] /; Free
Q[m, x] && PolyQ[Pp, x] && PolyQ[Qq, x] && NeQ[m, -1]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.110.4 Maple [A] (verified)

Time = 1.60 (sec) , antiderivative size = 53, normalized size of antiderivative = 0.93

method	result
gosper	$\frac{4acgx-b^2gx-bf\,x^2-2af}{\sqrt{c\,x^4+b\,x^2+a}\,(4ac-b^2)}$
trager	$\frac{4acgx-b^2gx-bf\,x^2-2af}{\sqrt{c\,x^4+b\,x^2+a}\,(4ac-b^2)}$
elliptic	$-\frac{f(b\,x^2+2a)}{\sqrt{c\,x^4+b\,x^2+a}\,(4ac-b^2)}+\frac{gx}{\sqrt{c\,x^4+b\,x^2+a}}$
default	$-\frac{f(b\,x^2+2a)}{\sqrt{c\,x^4+b\,x^2+a}\,(4ac-b^2)}+ag\left(-\frac{2c\left(\frac{b\,x^3}{2a\left(4ac-b^2\right)}-\frac{\left(2ac-b^2\right)x}{2a\left(4ac-b^2\right)c}\right)}{\sqrt{\left(x^4+\frac{b\,x^2}{c}+\frac{a}{c}\right)c}}+\frac{\left(\frac{1}{a}-\frac{2ac-b^2}{a\left(4ac-b^2\right)}\right)\sqrt{2}\sqrt{4-\frac{2\left(-b+\sqrt{-4ac+b^2}\right)x^2}{a}}\sqrt{4+\ldots}}{4\sqrt{\frac{-b+\ldots}{}}}\right.$

input
```
int((-c*g*x^4+f*x^3+a*g)/(c*x^4+b*x^2+a)^(3/2),x,method=_RETURNVERBOSE)
```

output
```
(4*a*c*g*x-b^2*g*x-b*f*x^2-2*a*f)/(c*x^4+b*x^2+a)^(1/2)/(4*a*c-b^2)
```

3.110.5 Fricas [A] (verification not implemented)

Time = 0.28 (sec) , antiderivative size = 80, normalized size of antiderivative = 1.40

$$\int \frac{ag+fx^3-cgx^4}{(a+bx^2+cx^4)^{3/2}}\,dx=\frac{\sqrt{cx^4+bx^2+a}(bfx^2+(b^2-4\,ac)gx+2\,af)}{(b^2c-4\,ac^2)x^4+ab^2-4\,a^2c+(b^3-4\,abc)x^2}$$

input
```
integrate((-c*g*x^4+f*x^3+a*g)/(c*x^4+b*x^2+a)^(3/2),x, algorithm="fricas"
)
```

output
```
sqrt(c*x^4 + b*x^2 + a)*(b*f*x^2 + (b^2 - 4*a*c)*g*x + 2*a*f)/((b^2*c - 4*
a*c^2)*x^4 + a*b^2 - 4*a^2*c + (b^3 - 4*a*b*c)*x^2)
```

3.110.6 Sympy [F]

$$\int \frac{ag + fx^3 - cgx^4}{(a + bx^2 + cx^4)^{3/2}}\,dx = -\int \left(-\frac{ag}{a\sqrt{a + bx^2 + cx^4} + bx^2\sqrt{a + bx^2 + cx^4} + cx^4\sqrt{a + bx^2 + cx^4}}\right)dx - \int \left(-\frac{fx^3}{a\sqrt{a + bx^2 + cx^4} + bx^2\sqrt{a + bx^2 + cx^4} + cx^4\sqrt{a + bx^2 + cx^4}}\right)dx - \int \frac{cgx^4}{a\sqrt{a + bx^2 + cx^4} + bx^2\sqrt{a + bx^2 + cx^4} + cx^4\sqrt{a + bx^2 + cx^4}}\,dx$$

input
```
integrate((-c*g*x**4+f*x**3+a*g)/(c*x**4+b*x**2+a)**(3/2),x)
```

output
```
-Integral(-a*g/(a*sqrt(a + b*x**2 + c*x**4) + b*x**2*sqrt(a + b*x**2 + c*x
**4) + c*x**4*sqrt(a + b*x**2 + c*x**4)), x) - Integral(-f*x**3/(a*sqrt(a
+ b*x**2 + c*x**4) + b*x**2*sqrt(a + b*x**2 + c*x**4) + c*x**4*sqrt(a + b*
x**2 + c*x**4)), x) - Integral(c*g*x**4/(a*sqrt(a + b*x**2 + c*x**4) + b*x
**2*sqrt(a + b*x**2 + c*x**4) + c*x**4*sqrt(a + b*x**2 + c*x**4)), x)
```

3.110.7 Maxima [A] (verification not implemented)

Time = 0.23 (sec) , antiderivative size = 49, normalized size of antiderivative = 0.86

$$\int \frac{ag + fx^3 - cgx^4}{(a + bx^2 + cx^4)^{3/2}}\,dx = \frac{bfx^2 + 2\,af + (b^2g - 4\,acg)x}{\sqrt{cx^4 + bx^2 + a}(b^2 - 4\,ac)}$$

input
```
integrate((-c*g*x^4+f*x^3+a*g)/(c*x^4+b*x^2+a)^(3/2),x, algorithm="maxima"
)
```

output
```
(b*f*x^2 + 2*a*f + (b^2*g - 4*a*c*g)*x)/(sqrt(c*x^4 + b*x^2 + a)*(b^2 - 4*
a*c))
```

3.110.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 136 vs. $2(53) = 106$.

Time = 0.62 (sec) , antiderivative size = 136, normalized size of antiderivative = 2.39

$$\int \frac{ag + fx^3 - cgx^4}{(a + bx^2 + cx^4)^{3/2}}\,dx = \frac{\left(\frac{(b^3f - 4\,abcf)x}{b^4 - 8\,ab^2c + 16\,a^2c^2} + \frac{b^4g - 8\,ab^2cg + 16\,a^2c^2g}{b^4 - 8\,ab^2c + 16\,a^2c^2}\right)x + \frac{2\,(ab^2f - 4\,a^2cf)}{b^4 - 8\,ab^2c + 16\,a^2c^2}}{\sqrt{cx^4 + bx^2 + a}}$$

input
```
integrate((-c*g*x^4+f*x^3+a*g)/(c*x^4+b*x^2+a)^(3/2),x, algorithm="giac")
```

output
```
(((b^3*f - 4*a*b*c*f)*x/(b^4 - 8*a*b^2*c + 16*a^2*c^2) + (b^4*g - 8*a*b^2*
c*g + 16*a^2*c^2*g)/(b^4 - 8*a*b^2*c + 16*a^2*c^2))*x + 2*(a*b^2*f - 4*a^2
*c*f)/(b^4 - 8*a*b^2*c + 16*a^2*c^2))/sqrt(c*x^4 + b*x^2 + a)
```

3.110.9 Mupad [B] (verification not implemented)

Time = 7.94 (sec) , antiderivative size = 51, normalized size of antiderivative = 0.89

$$\int \frac{ag + fx^3 - cgx^4}{(a + bx^2 + cx^4)^{3/2}}\,dx = -\frac{g\,b^2\,x + f\,b\,x^2 - 4\,a\,c\,g\,x + 2\,a\,f}{(4\,a\,c - b^2)\,\sqrt{c\,x^4 + b\,x^2 + a}}$$

input
```
int((a*g + f*x^3 - c*g*x^4)/(a + b*x^2 + c*x^4)^(3/2),x)
```

output
```
-(2*a*f + b*f*x^2 + b^2*g*x - 4*a*c*g*x)/((4*a*c - b^2)*(a + b*x^2 + c*x^4
)^(1/2))
```

3.111 $\int \frac{ag+ex+fx^3-cgx^4}{(a+bx^2+cx^4)^{3/2}}\,dx$

3.111.1 Optimal result

Integrand size = 36, antiderivative size = 69

$$\int \frac{ag+ex+fx^3-cgx^4}{(a+bx^2+cx^4)^{3/2}}\,dx=\frac{gx}{\sqrt{a+bx^2+cx^4}}-\frac{be-2af+(2ce-bf)x^2}{(b^2-4ac)\sqrt{a+bx^2+cx^4}}$$

output
```
g*x/(c*x^4+b*x^2+a)^(1/2)+(-b*e+2*a*f-(-b*f+2*c*e)*x^2)/(-4*a*c+b^2)/(c*x^
4+b*x^2+a)^(1/2)
```

3.111.2 Mathematica [A] (verified)

Time = 10.16 (sec) , antiderivative size = 61, normalized size of antiderivative = 0.88

$$\int \frac{ag+ex+fx^3-cgx^4}{(a+bx^2+cx^4)^{3/2}}\,dx=\frac{-be+2af+b^2gx-4acgx-2cex^2+bfx^2}{(b^2-4ac)\sqrt{a+bx^2+cx^4}}$$

input
```
Integrate[(a*g + e*x + f*x^3 - c*g*x^4)/(a + b*x^2 + c*x^4)^(3/2),x]
```

output
```
(-(b*e) + 2*a*f + b^2*g*x - 4*a*c*g*x - 2*c*e*x^2 + b*f*x^2)/((b^2 - 4*a*c
)*Sqrt[a + b*x^2 + c*x^4])
```

3.111.3 Rubi [A] (verified)

Time = 0.30 (sec) , antiderivative size = 69, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.111, Rules used = {2202, 1576, 1158, 2021}

Below are the steps used by Rubi to obtain the solution. The rule number used for the transformation is given above next to the arrow. The rules definitions used are listed below.

$$\int \frac{ag - cgx^4 + ex + fx^3}{(a + bx^2 + cx^4)^{3/2}} \, dx$$

$$\Big\downarrow \text{2202}$$

$$\int \frac{x(fx^2 + e)}{(cx^4 + bx^2 + a)^{3/2}} dx + \int \frac{ag - cgx^4}{(cx^4 + bx^2 + a)^{3/2}} dx$$

$$\Big\downarrow \text{1576}$$

$$\frac{1}{2} \int \frac{fx^2 + e}{(cx^4 + bx^2 + a)^{3/2}} dx^2 + \int \frac{ag - cgx^4}{(cx^4 + bx^2 + a)^{3/2}} dx$$

$$\Big\downarrow \text{1158}$$

$$\int \frac{ag - cgx^4}{(cx^4 + bx^2 + a)^{3/2}} dx - \frac{-2af + x^2(2ce - bf) + be}{(b^2 - 4ac)\sqrt{a + bx^2 + cx^4}}$$

$$\Big\downarrow \text{2021}$$

$$\frac{gx}{\sqrt{a + bx^2 + cx^4}} - \frac{-2af + x^2(2ce - bf) + be}{(b^2 - 4ac)\sqrt{a + bx^2 + cx^4}}$$

input
```
Int[(a*g + e*x + f*x^3 - c*g*x^4)/(a + b*x^2 + c*x^4)^(3/2),x]
```

output
```
(g*x)/Sqrt[a + b*x^2 + c*x^4] - (b*e - 2*a*f + (2*c*e - b*f)*x^2)/((b^2 -
4*a*c)*Sqrt[a + b*x^2 + c*x^4])
```

3.111.3.1 Defintions of rubi rules used

rule 1158
```
Int[((d_.) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(3/2), x_Symbo
l] :> Simp[-2*((b*d - 2*a*e + (2*c*d - b*e)*x)/((b^2 - 4*a*c)*Sqrt[a + b*x
+ c*x^2])), x] /; FreeQ[{a, b, c, d, e}, x]
```

rule 1576
```
Int[(x_)*((d_) + (e_.)*(x_)^2)^(q_.)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(
p_.), x_Symbol] :> Simp[1/2   Subst[Int[(d + e*x)^q*(a + b*x + c*x^2)^p, x]
, x, x^2], x] /; FreeQ[{a, b, c, d, e, p, q}, x]
```

rule 2021
```
Int[(Pp_)*(Qq_)^(m_.), x_Symbol] :> With[{p = Expon[Pp, x], q = Expon[Qq, x
]}, Simp[Coeff[Pp, x, p]*x^(p - q + 1)*(Qq^(m + 1)/((p + m*q + 1)*Coeff[Qq,
 x, q])), x] /; NeQ[p + m*q + 1, 0] && EqQ[(p + m*q + 1)*Coeff[Qq, x, q]*Pp
, Coeff[Pp, x, p]*x^(p - q)*((p - q + 1)*Qq + (m + 1)*x*D[Qq, x])]] /; Free
Q[m, x] && PolyQ[Pp, x] && PolyQ[Qq, x] && NeQ[m, -1]
```

rule 2202
```
Int[(Pn_)*((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4)^(p_), x_Symbol] :> Module[{n
 = Expon[Pn, x], k}, Int[Sum[Coeff[Pn, x, 2*k]*x^(2*k), {k, 0, n/2}]*(a + b
*x^2 + c*x^4)^p, x] + Int[x*Sum[Coeff[Pn, x, 2*k + 1]*x^(2*k), {k, 0, (n -
1)/2}]*(a + b*x^2 + c*x^4)^p, x]] /; FreeQ[{a, b, c, p}, x] && PolyQ[Pn, x]
 &&  !PolyQ[Pn, x^2]
```

3.111.4 Maple [A] (verified)

Time = 1.79 (sec) , antiderivative size = 63, normalized size of antiderivative = 0.91

method	result	size
gosper	$\frac{4acgx-b^2gx-bf\,x^2+2c\,x^2e-2af+be}{\sqrt{c\,x^4+b\,x^2+a}\,(4ac-b^2)}$	63
trager	$\frac{4acgx-b^2gx-bf\,x^2+2c\,x^2e-2af+be}{\sqrt{c\,x^4+b\,x^2+a}\,(4ac-b^2)}$	63
elliptic	$-\frac{bf\,x^2-2c\,x^2e+2af-be}{\sqrt{c\,x^4+b\,x^2+a}\,(4ac-b^2)}+\frac{gx}{\sqrt{c\,x^4+b\,x^2+a}}$	69
default	Expression too large to display	1012

input
```
int((-c*g*x^4+f*x^3+a*g+e*x)/(c*x^4+b*x^2+a)^(3/2),x,method=_RETURNVERBOSE
)
```

output
```
(4*a*c*g*x-b^2*g*x-b*f*x^2+2*c*e*x^2-2*a*f+b*e)/(c*x^4+b*x^2+a)^(1/2)/(4*a
*c-b^2)
```

3.111.5 Fricas [A] (verification not implemented)

Time = 0.29 (sec) , antiderivative size = 92, normalized size of antiderivative = 1.33

$$\int \frac{ag+ex+fx^3-cgx^4}{(a+bx^2+cx^4)^{3/2}}\,dx = \frac{\sqrt{cx^4+bx^2+a}((b^2-4\,ac)gx-(2\,ce-bf)x^2-be+2\,af)}{(b^2c-4\,ac^2)x^4+ab^2-4\,a^2c+(b^3-4\,abc)x^2}$$

input
```
integrate((-c*g*x^4+f*x^3+a*g+e*x)/(c*x^4+b*x^2+a)^(3/2),x, algorithm="fri
cas")
```

output
```
sqrt(c*x^4 + b*x^2 + a)*((b^2 - 4*a*c)*g*x - (2*c*e - b*f)*x^2 - b*e + 2*a
*f)/((b^2*c - 4*a*c^2)*x^4 + a*b^2 - 4*a^2*c + (b^3 - 4*a*b*c)*x^2)
```

3.111.6 Sympy [F]

$$\int \frac{ag+ex+fx^3-cgx^4}{(a+bx^2+cx^4)^{3/2}}\,dx = -\int \left(-\frac{ag}{a\sqrt{a+bx^2+cx^4}+bx^2\sqrt{a+bx^2+cx^4}+cx^4\sqrt{a+bx^2+cx^4}}\right)dx - \int \left(-\frac{ex}{a\sqrt{a+bx^2+cx^4}+bx^2\sqrt{a+bx^2+cx^4}+cx^4\sqrt{a+bx^2+cx^4}}\right)dx - \int \left(-\frac{fx^3}{a\sqrt{a+bx^2+cx^4}+bx^2\sqrt{a+bx^2+cx^4}+cx^4\sqrt{a+bx^2+cx^4}}\right)dx - \int \frac{cgx^4}{a\sqrt{a+bx^2+cx^4}+bx^2\sqrt{a+bx^2+cx^4}+cx^4\sqrt{a+bx^2+cx^4}}\,dx$$

input
```
integrate((-c*g*x**4+f*x**3+a*g+e*x)/(c*x**4+b*x**2+a)**(3/2),x)
```

output
```
-Integral(-a*g/(a*sqrt(a + b*x**2 + c*x**4) + b*x**2*sqrt(a + b*x**2 + c*x
**4) + c*x**4*sqrt(a + b*x**2 + c*x**4)), x) - Integral(-e*x/(a*sqrt(a + b
*x**2 + c*x**4) + b*x**2*sqrt(a + b*x**2 + c*x**4) + c*x**4*sqrt(a + b*x**
2 + c*x**4)), x) - Integral(-f*x**3/(a*sqrt(a + b*x**2 + c*x**4) + b*x**2*
sqrt(a + b*x**2 + c*x**4) + c*x**4*sqrt(a + b*x**2 + c*x**4)), x) - Integr
al(c*g*x**4/(a*sqrt(a + b*x**2 + c*x**4) + b*x**2*sqrt(a + b*x**2 + c*x**4
) + c*x**4*sqrt(a + b*x**2 + c*x**4)), x)
```

3.111.7 Maxima [A] (verification not implemented)

Time = 0.23 (sec) , antiderivative size = 94, normalized size of antiderivative = 1.36

$$\int \frac{ag + ex + fx^3 - cgx^4}{(a + bx^2 + cx^4)^{3/2}}\, dx = -\frac{\sqrt{cx^4 + bx^2 + a}((2\,ce - bf)x^2 + be - 2\,af - (b^2g - 4\,acg)x)}{(b^2c - 4\,ac^2)x^4 + ab^2 - 4\,a^2c + (b^3 - 4\,abc)x^2}$$

input
```
integrate((-c*g*x^4+f*x^3+a*g+e*x)/(c*x^4+b*x^2+a)^(3/2),x, algorithm="max
ima")
```

output
```
-sqrt(c*x^4 + b*x^2 + a)*((2*c*e - b*f)*x^2 + b*e - 2*a*f - (b^2*g - 4*a*c
*g)*x)/((b^2*c - 4*a*c^2)*x^4 + a*b^2 - 4*a^2*c + (b^3 - 4*a*b*c)*x^2)
```

3.111.8 Giac [B] (verification not implemented)

Leaf count of result is larger than twice the leaf count of optimal. 164 vs. 2(65) = 130.

Time = 0.66 (sec) , antiderivative size = 164, normalized size of antiderivative = 2.38

$$\int \frac{ag + ex + fx^3 - cgx^4}{(a + bx^2 + cx^4)^{3/2}}\, dx = -\frac{\left(\frac{(2\,b^2ce - 8\,ac^2e - b^3f + 4\,abcf)x}{b^4 - 8\,ab^2c + 16\,a^2c^2} - \frac{b^4g - 8\,ab^2cg + 16\,a^2c^2g}{b^4 - 8\,ab^2c + 16\,a^2c^2}\right)x + \frac{b^3e - 4\,abce - 2\,ab^2f + 8\,a^2cf}{b^4 - 8\,ab^2c + 16\,a^2c^2}}{\sqrt{cx^4 + bx^2 + a}}$$

input
```
integrate((-c*g*x^4+f*x^3+a*g+e*x)/(c*x^4+b*x^2+a)^(3/2),x, algorithm="gia
c")
```

output
```
-(((2*b^2*c*e - 8*a*c^2*e - b^3*f + 4*a*b*c*f)*x/(b^4 - 8*a*b^2*c + 16*a^2
*c^2) - (b^4*g - 8*a*b^2*c*g + 16*a^2*c^2*g)/(b^4 - 8*a*b^2*c + 16*a^2*c^2
))*x + (b^3*e - 4*a*b*c*e - 2*a*b^2*f + 8*a^2*c*f)/(b^4 - 8*a*b^2*c + 16*a
^2*c^2))/sqrt(c*x^4 + b*x^2 + a)
```

3.111.9 Mupad [B] (verification not implemented)

Time = 8.00 (sec) , antiderivative size = 62, normalized size of antiderivative = 0.90

$$\int \frac{ag+ex+fx^3-cgx^4}{(a+bx^2+cx^4)^{3/2}}\,dx = -\frac{g\,b^2\,x+f\,b\,x^2-e\,b-2\,c\,e\,x^2-4\,a\,c\,g\,x+2\,a\,f}{(4\,a\,c-b^2)\,\sqrt{c\,x^4+b\,x^2+a}}$$

input
```
int((a*g + e*x + f*x^3 - c*g*x^4)/(a + b*x^2 + c*x^4)^(3/2),x)
```

output
```
-(2*a*f - b*e + b*f*x^2 - 2*c*e*x^2 + b^2*g*x - 4*a*c*g*x)/((4*a*c - b^2)*
(a + b*x^2 + c*x^4)^(1/2))
```

CHAPTER 4

Appendix

4.1 Listing of Grading functions

The following are the current version of the grading functions used for grading the quality of the antiderivative with reference to the optimal antiderivative included in the test suite.

There is a version for Maple and for Mathematica/Rubi. There is a version for grading Sympy and version for use with Sagemath.

The following are links to the current source code.

The following are the listings of source code of the grading functions.

4.1.1 Mathematica and Rubi grading function

```
(* Original version thanks to Albert Rich emailed on 03/21/2017 *)
(* ::Package:: *)

(* Nasser: April 7,2022. add second output which gives reason for the grade *)
(*                       Small rewrite of logic in main function to make it*)
(*                       match Maple's logic. No change in functionality otherwise*)

(* ::Subsection:: *)
(*GradeAntiderivative[result,optimal]*)


(* ::Text:: *)
(*If result and optimal are mathematical expressions, *)
(*         GradeAntiderivative[result,optimal] returns*)
```

```
(*  "F" if the result fails to integrate an expression that*)
(*      is integrable*)
(*  "C" if result involves higher level functions than necessary*)
(*  "B" if result is more than twice the size of the optimal*)
(*      antiderivative*)
(*  "A" if result can be considered optimal*)


GradeAntiderivative[result_,optimal_] := Module[{expnResult,expnOptimal,leafCountResult,leafC
    expnResult  = ExpnType[result];
    expnOptimal = ExpnType[optimal];
    leafCountResult = LeafCount[result];
    leafCountOptimal = LeafCount[optimal];

    (*Print["expnResult=",expnResult," expnOptimal=",expnOptimal];*)
    If[expnResult<=expnOptimal,
        If[Not[FreeQ[result,Complex]], (*result contains complex*)
            If[Not[FreeQ[optimal,Complex]], (*optimal contains complex*)
                If[leafCountResult<=2*leafCountOptimal,
                    finalresult={"A"," "}
                    ,(*ELSE*)
                    finalresult={"B","Both result and optimal contain complex but leaf count
                ]
            ,(*ELSE*)
                finalresult={"C","Result contains complex when optimal does not."}
            ]
        ,(*ELSE*)(*result does not contains complex*)
            If[leafCountResult<=2*leafCountOptimal,
               finalresult={"A"," "}
            ,(*ELSE*)
               finalresult={"B","Leaf count is larger than twice the leaf count of optimal. $
              ]
        ]
    ,(*ELSE*) (*expnResult>expnOptimal*)
        If[FreeQ[result,Integrate] && FreeQ[result,Int],
            finalresult={"C","Result contains higher order function than in optimal. Order "<
            ,
            finalresult={"F","Contains unresolved integral."}
        ]
    ];

    finalresult
]
```

```
(* ::Text:: *)
(*The following summarizes the type number assigned an *)
(*expression based on the functions it involves*)
(*1 = rational function*)
(*2 = algebraic function*)
(*3 = elementary function*)
(*4 = special function*)
(*5 = hyperpergeometric function*)
(*6 = appell function*)
(*7 = rootsum function*)
(*8 = integrate function*)
(*9 = unknown function*)


ExpnType[expn_] :=
  If[AtomQ[expn],
    1,
  If[ListQ[expn],
    Max[Map[ExpnType,expn]],
  If[Head[expn]===Power,
    If[IntegerQ[expn[[2]]],
      ExpnType[expn[[1]]],
    If[Head[expn[[2]]]===Rational,
      If[IntegerQ[expn[[1]]] || Head[expn[[1]]]===Rational,
        1,
      Max[ExpnType[expn[[1]]],2]],
    Max[ExpnType[expn[[1]]],ExpnType[expn[[2]]],3]]],
  If[Head[expn]===Plus || Head[expn]===Times,
    Max[ExpnType[First[expn]],ExpnType[Rest[expn]]],
  If[ElementaryFunctionQ[Head[expn]],
    Max[3,ExpnType[expn[[1]]]],
  If[SpecialFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],4]],
  If[HypergeometricFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],5]],
  If[AppellFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],6]],
  If[Head[expn]===RootSum,
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],7]],
  If[Head[expn]===Integrate || Head[expn]===Int,
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],8]],
  9]]]]]]]]]]
```

```
ElementaryFunctionQ[func_] :=
  MemberQ[{
  Exp,Log,
  Sin,Cos,Tan,Cot,Sec,Csc,
  ArcSin,ArcCos,ArcTan,ArcCot,ArcSec,ArcCsc,
  Sinh,Cosh,Tanh,Coth,Sech,Csch,
  ArcSinh,ArcCosh,ArcTanh,ArcCoth,ArcSech,ArcCsch
},func]


SpecialFunctionQ[func_] :=
  MemberQ[{
  Erf, Erfc, Erfi,
  FresnelS, FresnelC,
  ExpIntegralE, ExpIntegralEi, LogIntegral,
  SinIntegral, CosIntegral, SinhIntegral, CoshIntegral,
  Gamma, LogGamma, PolyGamma,
  Zeta, PolyLog, ProductLog,
  EllipticF, EllipticE, EllipticPi
},func]


HypergeometricFunctionQ[func_] :=
  MemberQ[{Hypergeometric1F1,Hypergeometric2F1,HypergeometricPFQ},func]


AppellFunctionQ[func_] :=
  MemberQ[{AppellF1},func]
```

4.1.2 Maple grading function

```
# File: GradeAntiderivative.mpl
# Original version thanks to Albert Rich emailed on 03/21/2017

#Nasser 03/22/2017  Use Maple leaf count instead since buildin
#Nasser 03/23/2017  missing 'ln' for ElementaryFunctionQ added
#Nasser 03/24/2017  corrected the check for complex result
#Nasser 10/27/2017  check for leafsize and do not call ExpnType()
#                   if leaf size is "too large". Set at 500,000
```

```
#Nasser 12/22/2019  Added debug flag, added 'dilog' to special functions
#                   see problem 156, file Apostol_Problems
#Nasser 4/07/2022   add second output which gives reason for the grade

GradeAntiderivative := proc(result,optimal)
local leaf_count_result,
        leaf_count_optimal,
        ExpnType_result,
        ExpnType_optimal,
        debug:=false;

        leaf_count_result:=leafcount(result);
        #do NOT call ExpnType() if leaf size is too large. Recursion problem
        if leaf_count_result > 500000 then
            return "B","result has leaf size over 500,000. Avoiding possible recursion issues
        fi;

        leaf_count_optimal := leafcount(optimal);
        ExpnType_result  := ExpnType(result);
        ExpnType_optimal := ExpnType(optimal);

        if debug then
            print("ExpnType_result",ExpnType_result," ExpnType_optimal=",ExpnType_optimal);
        fi;

# If result and optimal are mathematical expressions,
#  GradeAntiderivative[result,optimal] returns
#   "F" if the result fails to integrate an expression that
#       is integrable
#   "C" if result involves higher level functions than necessary
#   "B" if result is more than twice the size of the optimal
#       antiderivative
#   "A" if result can be considered optimal

    #This check below actually is not needed, since I only
    #call this grading only for passed integrals. i.e. I check
    #for "F" before calling this. But no harm of keeping it here.
    #just in case.


    if not type(result,freeof('int')) then
        return "F","Result contains unresolved integral";
    fi;
```

```
if ExpnType_result<=ExpnType_optimal then
    if debug then
         print("ExpnType_result<=ExpnType_optimal");
    fi;
    if is_contains_complex(result) then
        if is_contains_complex(optimal) then
            if debug then
                    print("both result and optimal complex");
            fi;
            if leaf_count_result<=2*leaf_count_optimal then
                return "A"," ";
            else
               return "B",cat("Both result and optimal contain complex but leaf count of
                               convert(leaf_count_result,string)," vs. $2 (",
                               convert(leaf_count_optimal,string)," ) = ",convert(2*leaf_
            end if
        else #result contains complex but optimal is not
            if debug then
                    print("result contains complex but optimal is not");
            fi;
            return "C","Result contains complex when optimal does not.";
        fi;
    else # result do not contain complex
         # this assumes optimal do not as well. No check is needed here.
        if debug then
               print("result do not contain complex, this assumes optimal do not as well"
        fi;
        if leaf_count_result<=2*leaf_count_optimal then
            if debug then
                print("leaf_count_result<=2*leaf_count_optimal");
            fi;
            return "A"," ";
        else
            if debug then
                print("leaf_count_result>2*leaf_count_optimal");
            fi;
            return "B",cat("Leaf count of result is larger than twice the leaf count of o
                               convert(leaf_count_result,string),"$ vs. $2(",
                               convert(leaf_count_optimal,string),")=",convert(2*leaf_cou
        fi;
    fi;
```

```
    else #ExpnType(result) > ExpnType(optimal)
        if debug then
            print("ExpnType(result) > ExpnType(optimal)");
        fi;
        return "C",cat("Result contains higher order function than in optimal. Order ",
                       convert(ExpnType_result,string)," vs. order ",
                       convert(ExpnType_optimal,string),".");
    fi;

end proc:

#
# is_contains_complex(result)
# takes expressions and returns true if it contains "I" else false
#
#Nasser 032417
is_contains_complex:= proc(expression)
  return (has(expression,I));
end proc:

# The following summarizes the type number assigned an expression
# based on the functions it involves
# 1 = rational function
# 2 = algebraic function
# 3 = elementary function
# 4 = special function
# 5 = hyperpergeometric function
# 6 = appell function
# 7 = rootsum function
# 8 = integrate function
# 9 = unknown function

ExpnType := proc(expn)
  if type(expn,'atomic') then
    1
  elif type(expn,'list') then
    apply(max,map(ExpnType,expn))
  elif type(expn,'sqrt') then
    if type(op(1,expn),'rational') then
       1
    else
       max(2,ExpnType(op(1,expn)))
    end if
```

```
  elif type(expn,'`^`') then
    if type(op(2,expn),'integer') then
      ExpnType(op(1,expn))
    elif type(op(2,expn),'rational') then
      if type(op(1,expn),'rational') then
         1
      else
         max(2,ExpnType(op(1,expn)))
      end if
    else
         max(3,ExpnType(op(1,expn)),ExpnType(op(2,expn)))
    end if
  elif type(expn,'`+`') or type(expn,'`*`') then
    max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
  elif ElementaryFunctionQ(op(0,expn)) then
    max(3,ExpnType(op(1,expn)))
  elif SpecialFunctionQ(op(0,expn)) then
    max(4,apply(max,map(ExpnType,[op(expn)])))
  elif HypergeometricFunctionQ(op(0,expn)) then
    max(5,apply(max,map(ExpnType,[op(expn)])))
  elif AppellFunctionQ(op(0,expn)) then
    max(6,apply(max,map(ExpnType,[op(expn)])))
  elif op(0,expn)='int' then
    max(8,apply(max,map(ExpnType,[op(expn)]))) else
  9
  end if
end proc:


ElementaryFunctionQ := proc(func)
  member(func,[
        exp,log,ln,
        sin,cos,tan,cot,sec,csc,
        arcsin,arccos,arctan,arccot,arcsec,arccsc,
        sinh,cosh,tanh,coth,sech,csch,
        arcsinh,arccosh,arctanh,arccoth,arcsech,arccsch])
end proc:

SpecialFunctionQ := proc(func)
  member(func,[
        erf,erfc,erfi,
        FresnelS,FresnelC,
        Ei,Ei,Li,Si,Ci,Shi,Chi,
```

```
        GAMMA,lnGAMMA,Psi,Zeta,polylog,dilog,LambertW,
        EllipticF,EllipticE,EllipticPi])
end proc:

HypergeometricFunctionQ := proc(func)
  member(func,[Hypergeometric1F1,hypergeom,HypergeometricPFQ])
end proc:

AppellFunctionQ := proc(func)
  member(func,[AppellF1])
end proc:

# u is a sum or product.  rest(u) returns all but the
# first term or factor of u.
rest := proc(u) local v;
  if nops(u)=2 then
     op(2,u)
  else
     apply(op(0,u),op(2..nops(u),u))
  end if
end proc:

#leafcount(u) returns the number of nodes in u.
#Nasser 3/23/17  Replaced by build-in leafCount from package in Maple
leafcount := proc(u)
   MmaTranslator[Mma][LeafCount](u);
end proc:
```

4.1.3 Sympy grading function

```
#Dec 24, 2019. Nasser M. Abbasi:
#              Port of original Maple grading function by
#              Albert Rich to use with Sympy/Python
#Dec 27, 2019  Nasser. Added `RootSum`. See problem 177, Timofeev file
#              added 'exp_polar'
from sympy import *

def leaf_count(expr):
   #sympy do not have leaf count function. This is approximation
   return round(1.7*count_ops(expr))

def is_sqrt(expr):
```

```
    if isinstance(expr,Pow):
        if expr.args[1] == Rational(1,2):
            return True
        else:
            return False
    else:
        return False

def is_elementary_function(func):
    return func in [exp,log,ln,sin,cos,tan,cot,sec,csc,
            asin,acos,atan,acot,asec,acsc,sinh,cosh,tanh,coth,sech,csch,
            asinh,acosh,atanh,acoth,asech,acsch
        ]

def is_special_function(func):
    return func in [ erf,erfc,erfi,
             fresnels,fresnelc,Ei,Ei,Li,Si,Ci,Shi,Chi,
             gamma,loggamma,digamma,zeta,polylog,LambertW,
             elliptic_f,elliptic_e,elliptic_pi,exp_polar
         ]

def is_hypergeometric_function(func):
    return func in [hyper]

def is_appell_function(func):
    return func in [appellf1]

def is_atom(expn):
    try:
        if expn.isAtom or isinstance(expn,int) or isinstance(expn,float):
          return True
        else:
          return False

    except AttributeError as error:
        return False

def expnType(expn):
    debug=False
    if debug:
        print("expn=",expn,"type(expn)=",type(expn))

    if is_atom(expn):
```

```
        return 1
    elif isinstance(expn,list):
        return max(map(expnType, expn))    #apply(max,map(ExpnType,expn))
    elif  is_sqrt(expn):
        if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
            return 1
        else:
            return max(2,expnType(expn.args[0]))   #max(2,ExpnType(op(1,expn)))
    elif isinstance(expn,Pow):    #type(expn,'`^`')
        if isinstance(expn.args[1],Integer):  #type(op(2,expn),'integer')
            return expnType(expn.args[0])   #ExpnType(op(1,expn))
        elif isinstance(expn.args[1],Rational):  #type(op(2,expn),'rational')
            if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
                return 1
            else:
                return max(2,expnType(expn.args[0]))   #max(2,ExpnType(op(1,expn)))
        else:
            return max(3,expnType(expn.args[0]),expnType(expn.args[1])) #max(3,ExpnType(op(1,expn)),ExpnT
    elif isinstance(expn,Add) or isinstance(expn,Mul): #type(expn,'`+`') or type(expn,'`*`')
        m1 = expnType(expn.args[0])
        m2 = expnType(list(expn.args[1:]))
        return max(m1,m2)  #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
    elif is_elementary_function(expn.func):  #ElementaryFunctionQ(op(0,expn))
        return max(3,expnType(expn.args[0]))  #max(3,ExpnType(op(1,expn)))
    elif is_special_function(expn.func): #SpecialFunctionQ(op(0,expn))
        m1 = max(map(expnType, list(expn.args)))
        return max(4,m1)    #max(4,apply(max,map(ExpnType,[op(expn)])))
    elif is_hypergeometric_function(expn.func): #HypergeometricFunctionQ(op(0,expn))
        m1 = max(map(expnType, list(expn.args)))
        return max(5,m1)    #max(5,apply(max,map(ExpnType,[op(expn)])))
    elif is_appell_function(expn.func):
        m1 = max(map(expnType, list(expn.args)))
        return max(6,m1)    #max(5,apply(max,map(ExpnType,[op(expn)])))
    elif isinstance(expn,RootSum):
        m1 = max(map(expnType, list(expn.args))) #Apply[Max,Append[Map[ExpnType,Apply[List,expn]],7]],
        return max(7,m1)
    elif str(expn).find("Integral") != -1:
        m1 = max(map(expnType, list(expn.args)))
        return max(8,m1)    #max(5,apply(max,map(ExpnType,[op(expn)])))
    else:
        return 9

#main function
```

```
def grade_antiderivative(result,optimal):

    #print ("Enter grade_antiderivative for sagemath")
    #print("Enter grade_antiderivative, result=",result," optimal=",optimal)

    leaf_count_result  = leaf_count(result)
    leaf_count_optimal = leaf_count(optimal)

    #print("leaf_count_result=",leaf_count_result)
    #print("leaf_count_optimal=",leaf_count_optimal)

    expnType_result  = expnType(result)
    expnType_optimal = expnType(optimal)

    if str(result).find("Integral") != -1:
        grade = "F"
        grade_annotation =""
    else:
        if expnType_result <= expnType_optimal:
            if result.has(I):
                if optimal.has(I): #both result and optimal complex
                    if leaf_count_result <= 2*leaf_count_optimal:
                        grade = "A"
                        grade_annotation =""
                    else:
                        grade = "B"
                        grade_annotation ="Both result and optimal contain complex but leaf count of result is large
                else: #result contains complex but optimal is not
                    grade = "C"
                    grade_annotation ="Result contains complex when optimal does not."
            else: # result do not contain complex, this assumes optimal do not as well
                if leaf_count_result <= 2*leaf_count_optimal:
                    grade = "A"
                    grade_annotation =""
                else:
                    grade = "B"
                    grade_annotation ="Leaf count of result is larger than twice the leaf count of optimal. "+str(le
        else:
            grade = "C"
            grade_annotation ="Result contains higher order function than in optimal. Order "+str(ExpnType_


    #print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)
```

```
    return grade, grade_annotation
```

4.1.4 SageMath grading function

```
#Dec 24, 2019. Nasser: Ported original Maple grading function by
#             Albert Rich to use with Sagemath. This is used to
#             grade Fricas, Giac and Maxima results.
#Dec 24, 2019. Nasser: Added 'exp_integral_e' and 'sng', 'sin_integral'
#             'arctan2','floor','abs','log_integral'
#June 4, 2022  Made default grade_annotation "none" instead of "" due
#             issue later when reading the file.
#July 14, 2022. Added ellipticF. This is until they fix sagemath, then remove it.

from sage.all import *
from sage.symbolic.operators import add_vararg, mul_vararg

debug=False;

def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    #print("Enter tree_size, expr is ",expr)

    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)

def is_sqrt(expr):
    if expr.operator() == operator.pow:   #isinstance(expr,Pow):
        if expr.operands()[1]==1/2: #expr.args[1] == Rational(1,2):
            if debug: print ("expr is sqrt")
            return True
        else:
```

```
            return False
    else:
        return False

def is_elementary_function(func):
    #debug=False
    m = func.name() in ['exp','log','ln',
            'sin','cos','tan','cot','sec','csc',
            'arcsin','arccos','arctan','arccot','arcsec','arccsc',
            'sinh','cosh','tanh','coth','sech','csch',
            'arcsinh','arccosh','arctanh','arccoth','arcsech','arccsch','sgn',
        'arctan2','floor','abs'
        ]
    if debug:
        if m:
            print ("func ", func , " is elementary_function")
        else:
            print ("func ", func , " is NOT elementary_function")


    return m

def is_special_function(func):
    #debug=False
    if debug:
        print ("type(func)=", type(func))

    m= func.name() in ['erf','erfc','erfi','fresnel_sin','fresnel_cos','Ei',
            'Ei','Li','Si','sin_integral','Ci','cos_integral','Shi','sinh_integral'
            'Chi','cosh_integral','gamma','log_gamma','psi,zeta',
            'polylog','lambert_w','elliptic_f','elliptic_e','ellipticF',
            'elliptic_pi','exp_integral_e','log_integral']

    if debug:
        print ("m=",m)
        if m:
            print ("func ", func ," is special_function")
        else:
            print ("func ", func ," is NOT special_function")


    return m
```

```
def is_hypergeometric_function(func):
    return func.name() in ['hypergeometric','hypergeometric_M','hypergeometric_U']

def is_appell_function(func):
    return func.name() in ['hypergeometric']   #[appellf1] can't find this in sagemath

def is_atom(expn):

    #debug=False
    if debug:
         print ("Enter is_atom, expn=",expn)

    if not hasattr(expn, 'parent'):
        return False


    #thanks to answer at https://ask.sagemath.org/question/49179/what-is-sagemath-equivalent-to-atomic-
    try:
        if expn.parent() is SR:
            return expn.operator() is None
        if expn.parent() in (ZZ, QQ, AA, QQbar):
            return expn in expn.parent() # Should always return True
        if hasattr(expn.parent(),"base_ring") and hasattr(expn.parent(),"gens"):
            return expn in expn.parent().base_ring() or expn in expn.parent().gens()

        return False

    except AttributeError as error:
        print("Exception,AttributeError in  is_atom")
        print ("cought exception" , type(error).__name__ )
        return False


def expnType(expn):

    if debug:
        print (">>>>>Enter expnType, expn=", expn)
        print (">>>>>is_atom(expn)=", is_atom(expn))

    if is_atom(expn):
        return 1
    elif type(expn)==list:   #isinstance(expn,list):
```

```
        return max(map(expnType, expn))    #apply(max,map(ExpnType,expn))
    elif  is_sqrt(expn):
        if  type(expn.operands()[0])==Rational: #type(isinstance(expn.args[0],Rational):
            return 1
        else:
            return max(2,expnType(expn.operands()[0]))   #max(2,expnType(expn.args[0]))
    elif expn.operator() == operator.pow:   #isinstance(expn,Pow)
        if type(expn.operands()[1])==Integer:   #isinstance(expn.args[1],Integer)
            return expnType(expn.operands()[0])   #expnType(expn.args[0])
        elif type(expn.operands()[1])==Rational:   #isinstance(expn.args[1],Rational)
            if type(expn.operands()[0])==Rational: #isinstance(expn.args[0],Rational)
                return 1
            else:
                return max(2,expnType(expn.operands()[0]))   #max(2,expnType(expn.args[0]))
        else:
            return max(3,expnType(expn.operands()[0]),expnType(expn.operands()[1])) #max(3,expnType(expn.
    elif expn.operator() == add_vararg or expn.operator() == mul_vararg: #isinstance(expn,Add) or isinst
        m1 = expnType(expn.operands()[0]) #expnType(expn.args[0])
        m2 = expnType(expn.operands()[1:]) #expnType(list(expn.args[1:]))
        return max(m1,m2)   #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
    elif is_elementary_function(expn.operator()):   #is_elementary_function(expn.func)
        return max(3,expnType(expn.operands()[0]))
    elif is_special_function(expn.operator()): #is_special_function(expn.func)
        m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
        return max(4,m1)   #max(4,m1)
    elif is_hypergeometric_function(expn.operator()): #is_hypergeometric_function(expn.func)
        m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
        return max(5,m1)   #max(5,m1)
    elif is_appell_function(expn.operator()):
        m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
        return max(6,m1)   #max(6,m1)
    elif str(expn).find("Integral") != -1: #this will never happen, since it
            #is checked before calling the grading function that is passed.
            #but kept it here.
        m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
        return max(8,m1)   #max(5,apply(max,map(ExpnType,[op(expn)])))
    else:
        return 9

#main function
def grade_antiderivative(result,optimal):
```

```
    if debug:
        print ("Enter grade_antiderivative for sagemath")
        print("Enter grade_antiderivative, result=",result)
        print("Enter grade_antiderivative, optimal=",optimal)
        print("type(anti)=",type(result))
        print("type(optimal)=",type(optimal))

    leaf_count_result  = tree_size(result) #leaf_count(result)
    leaf_count_optimal = tree_size(optimal) #leaf_count(optimal)

    #if debug: print ("leaf_count_result=", leaf_count_result, "leaf_count_optimal=",leaf_count_optimal)


    expnType_result  = expnType(result)
    expnType_optimal = expnType(optimal)

    if debug: print ("expnType_result=", expnType_result, "expnType_optimal=",expnType_optimal)

    if expnType_result <= expnType_optimal:
        if result.has(I):
            if optimal.has(I): #both result and optimal complex
                if leaf_count_result <= 2*leaf_count_optimal:
                    grade = "A"
                    grade_annotation =" "
                else:
                    grade = "B"
                    grade_annotation ="Both result and optimal contain complex but leaf count of result is larger t
            else: #result contains complex but optimal is not
                grade = "C"
                grade_annotation ="Result contains complex when optimal does not."
        else: # result do not contain complex, this assumes optimal do not as well
            if leaf_count_result <= 2*leaf_count_optimal:
                grade = "A"
                grade_annotation =" "
            else:
                grade = "B"
                grade_annotation ="Leaf count of result is larger than twice the leaf count of optimal. "+str(leaf_
    else:
        grade = "C"
        grade_annotation ="Result contains higher order function than in optimal. Order "+str(expnType_resu


    print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)
```

```
    return grade, grade_annotation
```